CONDITIONAL INTEGRAL TRANSFORMS AND CONVOLUTIONS OF BOUNDED FUNCTIONS ON AN ANALOGUE OF WIENER SPACE

DONG HYUN CHO*

ABSTRACT. Let $C[0, t]$ denote the function space of all real-valued continuous paths on $[0, t]$. Define $X_n : C[0, t] \rightarrow \mathbb{R}^{n+1}$ and $X_{n+1} : C[0, t] \rightarrow \mathbb{R}^{n+2}$ by $X_n(x) = (x(t_0), x(t_1), \dots, x(t_n))$ and $X_{n+1}(x) = (x(t_0), x(t_1), \dots, x(t_n), x(t_{n+1}))$, where $0 = t_0 < t_1 < \dots < t_n < t_{n+1} = t$. In the present paper, using simple formulas for the conditional expectations with the conditioning functions X_n and X_{n+1} , we evaluate the $L_p(1 \leq p \leq \infty)$ -analytic conditional Fourier-Feynman transforms and the conditional convolution products of the functions which have the form

$$\int_{L_2[0, t]} \exp\{i(v, x)\} d\sigma(v) \int_{\mathbb{R}^r} \exp\left\{i \sum_{j=1}^r z_j(v_j, x)\right\} d\rho(z_1, \dots, z_r)$$

for $x \in C[0, t]$, where $\{v_1, \dots, v_r\}$ is an orthonormal subset of $L_2[0, t]$ and σ and ρ are the complex Borel measures of bounded variations on $L_2[0, t]$ and $\mathbb{R}^r$, respectively. We then investigate the inverse transforms of the function with their relationships and finally prove that the analytic conditional Fourier-Feynman transforms of the conditional convolution products for the functions, can be expressed in terms of the products of the conditional Fourier-Feynman transforms of each function.

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1. Introduction and preliminaries

Let $C_0[0, t]$ denote the Wiener space, that is, the space of real-valued continuous functions x on the closed interval $[0, t]$ with $x(0) = 0$. Chang and Skoug ([3]) introduced the concepts of conditional Fourier-Feynman transform and conditional convolution product on the Wiener space $C_0[0, t]$. In that paper, they examined the effects that drift has on the conditional Fourier-Feynman transform, the conditional convolution product, and various relationships that occur between them. Further works were studied by the author and his coauthors ([2, 9]). In fact, they ([2]) introduced the L_1 -analytic conditional Fourier-Feynman transform and the conditional convolution product over Wiener paths in abstract Wiener space and then, established their relationships between them of certain cylinder type functions. The author ([9]) extended the relationships between the conditional convolution product and the L_p ($1 \leq p \leq 2$)-analytic conditional Fourier-Feynman transform of the same kind of cylinder functions. Moreover, on $C[0, t]$, the space of the real-valued continuous paths on $[0, t]$, Kim ([12]) extended the relationships between the conditional convolution product and the L_p ($1 \leq p \leq \infty$)-analytic conditional Fourier-Feynman transform of the functions in a Banach algebra which corresponds to the Cameron-Storvick's Banach algebra $\mathcal{S}$ ([1]). The author ([4, 5, 6]) established several relationships between the L_p ($1 \leq p \leq \infty$)-analytic conditional Fourier-Feynman transforms and the conditional convolution products of the cylinder functions on $C[0, t]$. In particular, he ([4, 5]) derived an evaluation formula for the L_p -analytic conditional Fourier-Feynman transforms and the conditional convolution products of the same cylinder functions with the conditioning functions $X_n : C[0, t] \rightarrow \mathbb{R}^{n+1}$ and $X_{n+1} : C[0, t] \rightarrow \mathbb{R}^{n+2}$ given by $X_n(x) = (x(t_0), x(t_1), \dots, x(t_n))$ and $X_{n+1}(x) = (x(t_0), x(t_1), \dots, x(t_n), x(t_{n+1}))$, where $0 = t_0 < t_1 < \dots < t_n < t_{n+1} = t$ is a partition of $[0, t]$, and then, derived their relationships. Note that X_n is independent of the present time t while X_{n+1} is wholly dependent on the present time. In this paper, we further develop the relationships in ([4, 5, 12]) on the more generalized space $(C[0, t], w_\varphi)$, an analogue of the Wiener space associated with the probability measure φ on the Borel class $\mathcal{B}(\mathbb{R})$ of $\mathbb{R}$ ([11, 13, 14]). For the conditioning functions X_n and X_{n+1} , we proceed to study the relationships between the conditional convolution products and the analytic conditional Fourier-Feynman transforms of bounded functions defined on $C[0, t]$. In fact, using simple formulas for the conditional w_φ -integrals given X_n and X_{n+1} ,

we evaluate the $L_p(1 \leq p \leq \infty)$ -analytic conditional Fourier-Feynman transforms and the conditional convolution products for the functions of the form

$$(1.1) \quad \int_{L_2[0,t]} \exp\{i(v, x)\} d\sigma(v) \int_{\mathbb{R}^r} \exp\left\{i \sum_{j=1}^r z_j(v_j, x)\right\} d\rho(z_1, \dots, z_r)$$

for w_φ -a.e. $x \in C[0, t]$, where $\{v_1, \dots, v_r\}$ is an orthonormal set in $L_2[0, t]$, σ is a complex Borel measure of bounded variation on $L_2[0, t]$ and ρ is a bounded complex Borel measure on $\mathbb{R}^r$. We then investigate various relationships between the conditional Fourier-Feynman transforms and the conditional convolution products of the functions given by (1.1). Finally, we show that the L_p -analytic conditional Fourier-Feynman transform of the conditional convolution product for the functions Ψ_1 and Ψ_2 given by (1.1), can be expressed by the formula

$$\begin{aligned} & T_q^{(p)}[(\Psi_1 * \Psi_2)_q | X_n](\cdot, \vec{\xi}_n) | X_n(y, \vec{\zeta}_n) \\ &= \left[T_q^{(p)}[\Psi_1 | X_n] \left(\frac{1}{\sqrt{2}} y, \frac{1}{\sqrt{2}} (\vec{\zeta}_n + \vec{\xi}_n) \right) \right] \\ & \quad \times \left[T_q^{(p)}[\Psi_2 | X_n] \left(\frac{1}{\sqrt{2}} y, \frac{1}{\sqrt{2}} (\vec{\zeta}_n - \vec{\xi}_n) \right) \right] \end{aligned}$$

for a nonzero real q , w_φ -a.e. $y \in C[0, t]$ and P_{X_n} -a.e. $\vec{\xi}_n, \vec{\zeta}_n \in \mathbb{R}^{n+1}$. Thus the analytic conditional Fourier-Feynman transform of the conditional convolution product for the functions, can be interpreted as the product of the analytic conditional Fourier-Feynman transform of each function.

Let $\mathbb{C}$, $\mathbb{C}_+$ and $\mathbb{C}_+^\sim$ denote the set of complex numbers, the set of complex numbers with positive real parts and the set of nonzero complex numbers with nonnegative real parts, respectively.

Let $C = C[0, t]$ be the space of all real-valued continuous functions on the closed interval $[0, t]$ with its Borel class $\mathcal{B}(C[0, t])$. For a probability measure φ on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$, let w_φ be an analogue of the Wiener measure on $\mathcal{B}(C[0, t])$ associated with φ ([11, 13, 14]). Let $F : C[0, t] \rightarrow \mathbb{C}$ be integrable and X be a random vector on $C[0, t]$ assuming that the value space of X is a normed space equipped with the Borel σ -algebra. Then, we have the conditional expectation $E[F|X]$ of F given X from a well known probability theory. Furthermore, there exists a P_X -integrable $\mathbb{C}$ -valued function ψ on the value space of X such that $E[F|X](x) = (\psi \circ X)(x)$ for w_φ -a.e. $x \in C[0, t]$, where P_X is the probability distribution

of X . The function ψ is called the conditional w_φ -integral of F given X and it is also denoted by $E[F|X]$.

Throughout this paper, let $0 = t_0 < t_1 < \cdots < t_n < t_{n+1} = t$ be a partition of $[0, t]$, where n is a positive integer. For any x in $C[0, t]$, define the polygonal function $[x]$ of x by

$$(1.2) \quad [x](s) = \sum_{j=1}^{n+1} \chi_{(t_{j-1}, t_j]}(s) \left(\frac{t_j - s}{t_j - t_{j-1}} x(t_{j-1}) + \frac{s - t_{j-1}}{t_j - t_{j-1}} x(t_j) \right) + \chi_{\{t_0\}}(s) x(t_0)$$

for $s \in [0, t]$, where $\chi_{(t_{j-1}, t_j]}$ and $\chi_{\{t_0\}}$ denote the indicator functions. Similarly, for $\vec{\xi}_{n+1} = (\xi_0, \xi_1, \dots, \xi_{n+1}) \in \mathbb{R}^{n+2}$, define the polygonal function $[\vec{\xi}_{n+1}]$ of $\vec{\xi}_{n+1}$ by (1.2), where $x(t_j)$ is replaced by ξ_j for $j = 0, 1, \dots, n+1$. Let $X_{n+1} : C[0, t] \rightarrow \mathbb{R}^{n+2}$ and $X_n : C[0, t] \rightarrow \mathbb{R}^{n+1}$ be given by

$$(1.3) \quad X_{n+1}(x) = (x(t_0), x(t_1), \dots, x(t_n), x(t_{n+1}))$$

and

$$(1.4) \quad X_n(x) = (x(t_0), x(t_1), \dots, x(t_n)),$$

respectively. For a function $F : C[0, t] \rightarrow \mathbb{C}$ and $\lambda > 0$, let $F^\lambda(x) = F(\lambda^{-\frac{1}{2}}x)$, $X_{n+1}^\lambda(x) = X_{n+1}(\lambda^{-\frac{1}{2}}x)$ and $X_n^\lambda(x) = X_n(\lambda^{-\frac{1}{2}}x)$, where X_{n+1} and X_n are given by (1.3) and (1.4), respectively. Suppose that $E[F^\lambda]$ exists for each $\lambda > 0$. By the definition of the conditional w_φ -integral and the equation (6) of Theorem 2.9 in [8],

$$E[F^\lambda | X_{n+1}^\lambda](\vec{\xi}_{n+1}) = E[F(\lambda^{-\frac{1}{2}}(x - [x]) + [\vec{\xi}_{n+1}])]$$

for $P_{X_{n+1}^\lambda}$ -a.e. $\vec{\xi}_{n+1} \in \mathbb{R}^{n+2}$, where $P_{X_{n+1}^\lambda}$ is the probability distribution of X_{n+1}^λ on $(\mathbb{R}^{n+2}, \mathcal{B}(\mathbb{R}^{n+2}))$. Throughout this paper, for $y \in C[0, t]$ let

$$I_F^\lambda(y, \vec{\xi}_{n+1}) = E[F(y + \lambda^{-\frac{1}{2}}(x - [x]) + [\vec{\xi}_{n+1}])]$$

unless otherwise specified, where the expectation is taken over the variable x . Moreover, we can obtain from (2.6) of Theorem 2.5 in [7]

$$(1.5) \quad E[F^\lambda | X_n^\lambda](\vec{\xi}_n) = \left[\frac{\lambda}{2\pi(t - t_n)} \right]^{\frac{1}{2}} \int_{\mathbb{R}} I_F^\lambda(0, \vec{\xi}_{n+1}) \exp \left\{ -\frac{\lambda}{2} \frac{(\xi_{n+1} - \xi_n)^2}{t - t_n} \right\} d\xi_{n+1}$$

for $P_{X_n^\lambda}$ -a.e. $\vec{\xi}_n = (\xi_0, \xi_1, \dots, \xi_n) \in \mathbb{R}^{n+1}$, where $\vec{\xi}_{n+1} = (\xi_0, \xi_1, \dots, \xi_n, \xi_{n+1})$ for $\xi_{n+1} \in \mathbb{R}$ and $P_{X_n^\lambda}$ is the probability distribution of X_n^λ on

$(\mathbb{R}^{n+1}, \mathcal{B}(\mathbb{R}^{n+1}))$. For $y \in C[0, t]$, let $K_F^\lambda(y, \vec{\xi}_n)$ be given by (1.5) where 0 is replaced by y . If $I_F^\lambda(0, \vec{\xi}_{n+1})$ has the analytic extension $J_\lambda^*(F)(\vec{\xi}_{n+1})$ on $\mathbb{C}_+$ as a function of λ , then it is called the conditional analytic Wiener w_φ -integral of F given X_{n+1} with parameter λ and denoted by $E^{anw\lambda}[F|X_{n+1}](\vec{\xi}_{n+1}) = J_\lambda^*(F)(\vec{\xi}_{n+1})$ for $\vec{\xi}_{n+1} \in \mathbb{R}^{n+2}$. Moreover, if for a nonzero real q , $E^{anw\lambda}[F|X_{n+1}](\vec{\xi}_{n+1})$ has a limit as λ approaches to $-iq$ through $\mathbb{C}_+$, then it is called the conditional analytic Feynman w_φ -integral of F given X_{n+1} with parameter q and denoted by $E^{anf_q}[F|X_{n+1}](\vec{\xi}_{n+1}) = \lim_{\lambda \rightarrow -iq} E^{anw\lambda}[F|X_{n+1}](\vec{\xi}_{n+1})$. Similarly, the definitions of $E^{anw\lambda}[F|X_n](\vec{\xi}_n)$ and $E^{anf_q}[F|X_n](\vec{\xi}_n)$ are understood with $K_F^\lambda(0, \vec{\xi}_n)$ if X_{n+1} is replaced by X_n .

For a given extended real number p with $1 < p \leq \infty$, suppose that p and p' are related by $\frac{1}{p} + \frac{1}{p'} = 1$ (possibly $p' = 1$ if $p = \infty$). Let F_n and F be measurable functions such that $\lim_{n \rightarrow \infty} \int_C |F_n(x) - F(x)|^{p'} dw_\varphi(x) = 0$. Then we write $\text{l.i.m.}_{n \rightarrow \infty} (w^{p'})(F_n) = F$ and call F the limit in the mean of order p' . A similar definition is understood when n is replaced by a continuously varying parameter.

Let F and G be defined on $C[0, t]$ and let X_{n+1} be given by (1.3). For $\lambda \in \mathbb{C}_+$ and w_φ -a.e. $y \in C[0, t]$, let

$$T_\lambda[F|X_{n+1}](y, \vec{\xi}_{n+1}) = E^{anw\lambda}[F(y + \cdot)|X_{n+1}](\vec{\xi}_{n+1})$$

for $P_{X_{n+1}}$ -a.e. $\vec{\xi}_{n+1} \in \mathbb{R}^{n+2}$ if it exists. For a nonzero real q and w_φ -a.e. $y \in C[0, t]$, define the L_1 -analytic conditional Fourier-Feynman transform $T_q^{(1)}[F|X_{n+1}]$ of F given X_{n+1} by the formula

$$T_q^{(1)}[F|X_{n+1}](y, \vec{\xi}_{n+1}) = E^{anf_q}[F(y + \cdot)|X_{n+1}](\vec{\xi}_{n+1})$$

for $P_{X_{n+1}}$ -a.e. $\vec{\xi}_{n+1} \in \mathbb{R}^{n+2}$ if it exists. For $1 < p \leq \infty$, define the L_p -analytic conditional Fourier-Feynman transform $T_q^{(p)}[F|X_{n+1}]$ of F given X_{n+1} by the formula

$$T_q^{(p)}[F|X_{n+1}](\cdot, \vec{\xi}_{n+1}) = \text{l.i.m.}_{\lambda \rightarrow -iq} (w^{p'})(T_\lambda[F|X_{n+1}](\cdot, \vec{\xi}_{n+1}))$$

for $P_{X_{n+1}}$ -a.e. $\vec{\xi}_{n+1} \in \mathbb{R}^{n+2}$, where λ approaches to $-iq$ through $\mathbb{C}_+$. We also define the conditional convolution product $[(F * G)_\lambda|X_{n+1}]$ of F and G given X_{n+1} by the formula, for w_φ -a.e. $y \in C[0, t]$

$$\begin{aligned}
& [(F * G)_\lambda | X_{n+1}](y, \vec{\xi}_{n+1}) \\
&= \begin{cases} E^{anw_\lambda} \left[F \left(\frac{y + \cdot}{\sqrt{2}} \right) G \left(\frac{y - \cdot}{\sqrt{2}} \right) \Big| X_{n+1} \right] (\vec{\xi}_{n+1}), & \lambda \in \mathbb{C}_+; \\ E^{anf_q} \left[F \left(\frac{y + \cdot}{\sqrt{2}} \right) G \left(\frac{y - \cdot}{\sqrt{2}} \right) \Big| X_{n+1} \right] (\vec{\xi}_{n+1}), & \lambda = -iq \end{cases}
\end{aligned}$$

if they exist for $P_{X_{n+1}}$ -a.e. $\vec{\xi}_{n+1} \in \mathbb{R}^{n+2}$. If $\lambda = -iq$, we replace $[(F * G)_\lambda | X_{n+1}]$ by $[(F * G)_q | X_{n+1}]$. Similar definitions and notations are understood with $\vec{\xi}_n \in \mathbb{R}^{n+1}$ if X_{n+1} is replaced by X_n which is given by (1.4).

2. Conditional Fourier-Feynman transform with final time conditioning function

For v in $L_2[0, t]$ and x in $C[0, t]$, let (v, x) denote the Paley-Wiener-Zygmund integral of v according to x ([11]). Note that $\langle \cdot, \cdot \rangle$ and $\langle \cdot, \cdot \rangle_{\mathbb{R}^r}$ denote the inner product over $L_2[0, t]$ and the dot product on the r -dimensional Euclidean space $\mathbb{R}^r$, respectively.

For each $j = 1, \dots, n+1$, let $\alpha_j = (t_j - t_{j-1})^{-\frac{1}{2}} \chi_{(t_{j-1}, t_j]}$ on $[0, t]$. Let V be the subspace of $L_2[0, t]$ generated by $\{\alpha_1, \dots, \alpha_{n+1}\}$ and $V^\perp$ denote the orthogonal complement of V . Let $\mathcal{P}$ and $\mathcal{P}^\perp$ be the orthogonal projections from $L_2[0, t]$ to V and $V^\perp$, respectively. Throughout this paper, let $\{v_1, v_2, \dots, v_r\}$ be an orthonormal subset of $L_2[0, t]$ such that $\{\mathcal{P}^\perp v_1, \dots, \mathcal{P}^\perp v_r\}$ is an independent set. Let $\{e_1, \dots, e_r\}$ be the orthonormal set obtained from $\{\mathcal{P}^\perp v_1, \dots, \mathcal{P}^\perp v_r\}$ by the Gram-Schmidt orthonormalization process. Now, for $l = 1, \dots, r$, let $\mathcal{P}^\perp v_l = \sum_{j=1}^r \beta_{lj} e_j$ be the linear combinations of the e_j s and let $A = [\beta_{lj}]_{r \times r}$ be the coefficient matrix of the combinations. We can regard A as the linear transformation $T_A : \mathbb{R}^r \rightarrow \mathbb{R}^r$ given by $T_A \vec{z} = \vec{z}A$, where $\vec{z}$ is any row-vector in $\mathbb{R}^r$. Note that A is invertible so that T_A is an isomorphism. For $v \in L_2[0, t]$, let

$$(2.1) \quad c_j(v) = \langle v, e_j \rangle$$

for $j = 1, \dots, r$ and let

$$(2.2) \quad (\vec{v}, x) = ((v_1, x), \dots, (v_r, x))$$

for $x \in C[0, t]$. Furthermore, for $\vec{z} \in \mathbb{R}^r$ and $\vec{\xi}_{n+1} \in \mathbb{R}^{n+2}$ let

$$(2.3) \quad H_1(x, \vec{\xi}_{n+1}, v, \vec{z}) = \exp\{i[(v, x + [\vec{\xi}_{n+1}]) + \langle (\vec{v}, x + [\vec{\xi}_{n+1}]), \vec{z} \rangle_{\mathbb{R}^r}]\}$$

and for $\lambda \in \mathbb{C}_+^\sim$ let

$$(2.4) \quad H_2(\lambda, v, \vec{z}) = \exp \left\{ -\frac{1}{2\lambda} [\|\mathcal{P}^\perp v\|_2^2 + 2\langle \vec{c}(\mathcal{P}^\perp v), T_A \vec{z} \rangle_{\mathbb{R}^r} + \|T_A \vec{z}\|_{\mathbb{R}^r}^2] \right\},$$

where $\vec{c}(\mathcal{P}^\perp v) = (c_1(\mathcal{P}^\perp v), \dots, c_r(\mathcal{P}^\perp v))$ and the c_j s are given by (2.1). Note that by the Bessel's inequality,

$$(2.5) \quad \begin{aligned} & |H_2(\lambda, v, \vec{z})| \\ &= \exp \left\{ -\frac{\operatorname{Re} \lambda}{2|\lambda|^2} [\|\mathcal{P}^\perp v\|_2^2 - \|\vec{c}(\mathcal{P}^\perp v)\|_{\mathbb{R}^r}^2 + \|\vec{c}(\mathcal{P}^\perp v) + T_A \vec{z}\|_{\mathbb{R}^r}^2] \right\} \\ &\leq 1. \end{aligned}$$

Using the same method as used in the proof of Theorem 2.6 in [10], we can prove the following lemma.

LEMMA 2.1. For $x \in C[0, t]$, $\lambda > 0$, $v \in L_2[0, t]$ and $\vec{z} \in \mathbb{R}^r$, let

$$(2.6) \quad H_3(\lambda, v, \vec{z}, x) = \exp \{ i\lambda^{-\frac{1}{2}} [(v, x - [x]) + \langle (\vec{v}, x - [x]), \vec{z} \rangle_{\mathbb{R}^r}] \}.$$

Then

$$\int_C H_3(\lambda, v, \vec{z}, x) dw_\varphi(x) = H_2(\lambda, v, \vec{z}),$$

where H_2 is given by (2.4).

Let $\hat{M}(\mathbb{R}^r)$ be the set of all functions ϕ on $\mathbb{R}^r$ defined by

$$(2.7) \quad \phi(\vec{u}) = \int_{\mathbb{R}^r} \exp \{ i \langle \vec{u}, \vec{z} \rangle_{\mathbb{R}^r} \} d\rho(\vec{z}),$$

where ρ is a complex Borel measure of bounded variation over $\mathbb{R}^r$, and let Φ be given by

$$(2.8) \quad \Phi(x) = \phi(\vec{v}, x)$$

for w_φ -a.e. $x \in C[0, t]$, where $(\vec{v}, x)$ and $\phi \in \hat{M}(\mathbb{R}^r)$ are given by (2.2) and (2.7), respectively. Let $\mathcal{M} = \mathcal{M}(L_2[0, t])$ be the class of all $\mathbb{C}$ -valued Borel measures of bounded variation over $L_2[0, t]$ and let $\mathcal{S}_{w_\varphi}$ be the space of all functions F which for $\sigma \in \mathcal{M}$ have the form

$$(2.9) \quad F(x) = \int_{L_2[0, t]} \exp \{ i(v, x) \} d\sigma(v)$$

for w_φ -a.e. $x \in C[0, t]$. Note that $\mathcal{S}_{w_\varphi}$ is a Banach algebra which is equivalent to $\mathcal{M}$ with the norm $\|F\| = \|\sigma\|$, the total variation of σ [11].

Now we have the following theorem.

THEOREM 2.2. Let $1 \leq p \leq \infty$ and X_{n+1} be given by (1.3). For w_φ -a.e. $x \in C[0, t]$, let $\Psi(x) = F(x)\Phi(x)$, where Φ and F are given by (2.8) and (2.9), respectively. Then for a nonzero real q , w_φ -a.e. $y \in C[0, t]$ and $P_{X_{n+1}}$ -a.e. $\vec{\xi}_{n+1} \in \mathbb{R}^{n+2}$,

$$(2.10) \quad \begin{aligned} T_q^{(p)}[\Psi|X_{n+1}](y, \vec{\xi}_{n+1}) \\ = \int_{\mathbb{R}^r} \int_{L_2[0, t]} H_1(y, \vec{\xi}_{n+1}, v, \vec{z}) H_2(-iq, v, \vec{z}) d\sigma(v) d\rho(\vec{z}), \end{aligned}$$

where H_1 and H_2 are given by (2.3) and (2.4), respectively.

Proof. For $\lambda > 0$, $y \in C[0, t]$ and $\vec{\xi}_{n+1} \in \mathbb{R}^{n+2}$,

$$\begin{aligned} I_\Psi^\lambda(y, \vec{\xi}_{n+1}) \\ = \int_{\mathbb{R}^r} \int_{L_2[0, t]} \int_C H_1(y, \vec{\xi}_{n+1}, v, \vec{z}) H_3(\lambda, v, \vec{z}, x) dw_\varphi(x) d\sigma(v) d\rho(\vec{z}), \end{aligned}$$

where H_3 is given by (2.6). By Lemma 2.1, we obtain that

$$I_\Psi^\lambda(y, \vec{\xi}_{n+1}) = \int_{\mathbb{R}^r} \int_{L_2[0, t]} H_1(y, \vec{\xi}_{n+1}, v, \vec{z}) H_2(\lambda, v, \vec{z}) d\sigma(v) d\rho(\vec{z}).$$

By (2.5), the Moreras theorem and the dominated convergence theorem, we have the existence of $T_\lambda[\Psi|X_{n+1}](y, \vec{\xi}_{n+1})$ as the analytic extension of $I_\Psi^\lambda(y, \vec{\xi}_{n+1})$ on $\mathbb{C}_+$. Let $T_q^{(p)}[\Psi|X_{n+1}](y, \vec{\xi}_{n+1})$ be given by (2.10) and $\frac{1}{p} + \frac{1}{p'} = 1$. Then

$$\begin{aligned} & \|T_\lambda[\Psi|X_{n+1}](\cdot, \vec{\xi}_{n+1}) - T_q^{(p)}[\Psi|X_{n+1}](\cdot, \vec{\xi}_{n+1})\|_{p'} \\ & \leq \int_{\mathbb{R}^r} \int_{L_2[0, t]} |H_2(\lambda, v, \vec{z}) - H_2(-iq, v, \vec{z})| d|\sigma|(v) d|\rho|(\vec{z}) \end{aligned}$$

which converges to 0 as λ approaches to $-iq$ through $\mathbb{C}_+$ by the dominated convergence theorem. Now, the proof is completed. $\square$

THEOREM 2.3. Let ϕ_1, ϕ_2 and ρ_1, ρ_2 be related by (2.7), respectively, and let F_1, F_2 and σ_1, σ_2 be related by (2.9), respectively. Let $\Psi_1(x) = F_1(x)\phi_1(\vec{v}, x)$ and $\Psi_2(x) = F_2(x)\phi_2(\vec{v}, x)$ for w_φ -a.e. $x \in C[0, t]$. Furthermore, let X_{n+1} be given by (1.3). Then for a nonzero real q , w_φ -a.e.

$y \in C[0, t]$ and $P_{X_{n+1}}$ -a.e. $\vec{\xi}_{n+1} \in \mathbb{R}^{n+2}$,

$$\begin{aligned} & [(\Psi_1 * \Psi_2)_q | X_{n+1}](y, \vec{\xi}_{n+1}) \\ &= \int_{\mathbb{R}^r} \int_{\mathbb{R}^r} \int_{L_2[0, t]} \int_{L_2[0, t]} H_1 \left(y, \vec{\xi}_{n+1}, \frac{1}{\sqrt{2}} v_1, \frac{1}{\sqrt{2}} \vec{z}_1 \right) H_1 \left(y, \right. \\ & \quad \left. - \vec{\xi}_{n+1}, \frac{1}{\sqrt{2}} v_2, \frac{1}{\sqrt{2}} \vec{z}_2 \right) H_2 \left(-iq, \frac{1}{\sqrt{2}}(v_1 - v_2), \frac{1}{\sqrt{2}}(\vec{z}_1 - \vec{z}_2) \right) \\ & \quad d\sigma_1(v_1) d\sigma_2(v_2) d\rho_1(\vec{z}_1) d\rho_2(\vec{z}_2), \end{aligned}$$

where H_1 and H_2 are given by (2.3) and (2.4), respectively.

Proof. For $\lambda > 0$, w_φ -a.e. $y \in C[0, t]$ and $P_{X_{n+1}}$ -a.e. $\vec{\xi}_{n+1} \in \mathbb{R}^{n+2}$,

$$\begin{aligned} & [(\Psi_1 * \Psi_2)_\lambda | X_{n+1}](y, \vec{\xi}_{n+1}) \\ &= \int_{\mathbb{R}^r} \int_{\mathbb{R}^r} \int_{L_2[0, t]} \int_{L_2[0, t]} \int_C \exp \left\{ \frac{i}{\sqrt{2}} [(v_1, y + [\vec{\xi}_{n+1}]) + \langle (\vec{v}, y + \right. \\ & \quad \left. [\vec{\xi}_{n+1}]), \vec{z}_1 \rangle_{\mathbb{R}^r} + (v_2, y - [\vec{\xi}_{n+1}]) + \langle (\vec{v}, y - [\vec{\xi}_{n+1}]), \vec{z}_2 \rangle_{\mathbb{R}^r}] + \right. \\ & \quad \left. \frac{i}{\sqrt{2}\lambda} [(v_1 - v_2, x - [x]) + \langle (\vec{v}, x - [x]), \vec{z}_1 - \vec{z}_2 \rangle_{\mathbb{R}^r}] \right\} dw_\varphi(x) \\ & \quad d\sigma_1(v_1) d\sigma_2(v_2) d\rho_1(\vec{z}_1) d\rho_2(\vec{z}_2) \\ &= \int_{\mathbb{R}^r} \int_{\mathbb{R}^r} \int_{L_2[0, t]} \int_{L_2[0, t]} \int_C H_1 \left(y, \vec{\xi}_{n+1}, \frac{1}{\sqrt{2}} v_1, \frac{1}{\sqrt{2}} \vec{z}_1 \right) H_1 \left(y, \right. \\ & \quad \left. - \vec{\xi}_{n+1}, \frac{1}{\sqrt{2}} v_2, \frac{1}{\sqrt{2}} \vec{z}_2 \right) H_3 \left(\lambda, \frac{1}{\sqrt{2}}(v_1 - v_2), \frac{1}{\sqrt{2}}(\vec{z}_1 - \vec{z}_2), x \right) \\ & \quad dw_\varphi(x) d\sigma_1(v_1) d\sigma_2(v_2) d\rho_1(\vec{z}_1) d\rho_2(\vec{z}_2), \end{aligned}$$

where H_3 is given by (2.6). By Lemma 2.1, we obtain that

$$\begin{aligned} & [(\Psi_1 * \Psi_2)_\lambda | X_{n+1}](y, \vec{\xi}_{n+1}) \\ &= \int_{\mathbb{R}^r} \int_{\mathbb{R}^r} \int_{L_2[0, t]} \int_{L_2[0, t]} H_1 \left(y, \vec{\xi}_{n+1}, \frac{1}{\sqrt{2}} v_1, \frac{1}{\sqrt{2}} \vec{z}_1 \right) H_1 \left(y, \right. \\ & \quad \left. - \vec{\xi}_{n+1}, \frac{1}{\sqrt{2}} v_2, \frac{1}{\sqrt{2}} \vec{z}_2 \right) H_2 \left(\lambda, \frac{1}{\sqrt{2}}(v_1 - v_2), \frac{1}{\sqrt{2}}(\vec{z}_1 - \vec{z}_2) \right) \\ & \quad d\sigma_1(v_1) d\sigma_2(v_2) d\rho_1(\vec{z}_1) d\rho_2(\vec{z}_2). \end{aligned}$$

By (2.5), the Morera's theorem and the dominated convergence theorem, we have the result. $\square$

3. Conditional Fourier-Feynman transform without final time conditioning function

In this section, we evaluate time-independent conditional Fourier-Feynman transform and conditional convolution product of the functions as given in the previous section.

LEMMA 3.1. For $\vec{\xi}_n = (\xi_0, \xi_1, \dots, \xi_n) \in \mathbb{R}^{n+1}$, let $\vec{\xi}_{n+1} = (\xi_0, \xi_1, \dots, \xi_n, \xi_{n+1})$, where $\xi_{n+1} \in \mathbb{R}$. Furthermore, for $y \in C[0, t]$, $v \in L_2[0, t]$, $\vec{z} = (z_1, \dots, z_r) \in \mathbb{R}^r$, let $H_1(y, \vec{\xi}_{n+1}, v, \vec{z})$ be given by (2.3), let

$$(3.1) \quad H_4(\vec{\xi}_n, v, \vec{z}) = \exp \left\{ i \sum_{j=1}^n (\xi_j - \xi_{j-1}) \left[(\mathcal{P}v)(t_j) + \sum_{l=1}^r z_l (\mathcal{P}v_l)(t_j) \right] \right\}$$

and for $\lambda \in \mathbb{C}_+^\sim$, let

$$(3.2) \quad H_5(\lambda, v, \vec{z}) = \exp \left\{ -\frac{1}{2\lambda} \left[(\mathcal{P}v)(t) + \sum_{l=1}^r z_l (\mathcal{P}v_l)(t) \right]^2 \right\}.$$

Then for $\lambda > 0$,

$$(3.3) \quad \left[\frac{\lambda}{2\pi(t-t_n)} \right]^{\frac{1}{2}} \int_{\mathbb{R}} H_1(y, \vec{\xi}_{n+1}, v, \vec{z}) \exp \left\{ -\frac{\lambda(\xi_{n+1} - \xi_n)^2}{2(t-t_n)} \right\} d\xi_{n+1} \\ = H_1(y, 0, v, \vec{z}) H_4(\vec{\xi}_n, v, \vec{z}) H_5(\lambda, v, \vec{z}).$$

Proof. By the definition of the Paley-Wiener-Zygmund integral, it is not difficult to show that for $v \in L_2[0, t]$

$$(v, [\vec{\xi}_{n+1}]) = \sum_{j=1}^n (\mathcal{P}v)(t_j) (\xi_j - \xi_{j-1}) + (\mathcal{P}v)(t) (\xi_{n+1} - \xi_n).$$

Now, let I_λ be the left hand side of (3.3). Then

$$I_\lambda = H_1(y, 0, v, \vec{z}) \left[\frac{\lambda}{2\pi(t-t_n)} \right]^{\frac{1}{2}} \int_{\mathbb{R}} H_4(\vec{\xi}_n, v, \vec{z}) \exp \left\{ i \left[(\mathcal{P}v)(t) + \sum_{l=1}^r z_l (\mathcal{P}v_l)(t) \right] (\xi_{n+1} - \xi_n) - \frac{\lambda(\xi_{n+1} - \xi_n)^2}{2(t-t_n)} \right\} d\xi_{n+1} \\ = H_1(y, 0, v, \vec{z}) H_4(\vec{\xi}_n, v, \vec{z}) H_5(\lambda, v, \vec{z})$$

by the following well-known integration formula

$$\int_{\mathbb{R}} \exp \{-au^2 + ibu\} du = \left(\frac{\pi}{a} \right)^{\frac{1}{2}} \exp \left\{ -\frac{b^2}{4a} \right\}$$

for $a \in \mathbb{C}_+$ and $b \in \mathbb{R}$. Now, the proof is completed. $\square$

THEOREM 3.2. *Let $1 \leq p \leq \infty$ and X_n be given by (1.4). For w_φ -a.e. $x \in C[0, t]$, let $\Psi(x) = F(x)\Phi(x)$, where Φ and F are given by (2.8) and (2.9), respectively. Then for a nonzero real q , w_φ -a.e. $y \in C[0, t]$ and P_{X_n} -a.e. $\vec{\xi}_n \in \mathbb{R}^{n+1}$,*

$$\begin{aligned} T_q^{(p)}[\Psi|X_n](y, \vec{\xi}_n) &= \int_{\mathbb{R}^r} \int_{L_2[0, t]} H_1(y, 0, v, \vec{z}) H_4(\vec{\xi}_n, v, \vec{z}) \\ &\quad \times H_2(-iq, v, \vec{z}) H_5(-iq, v, \vec{z}) d\sigma(v) d\rho(\vec{z}), \end{aligned}$$

where H_1 , H_2 , H_4 , and H_5 are given by (2.3), (2.4), (3.1) and (3.2), respectively.

Proof. For $\vec{\xi}_n = (\xi_0, \xi_1, \dots, \xi_n) \in \mathbb{R}^{n+1}$, let $\vec{\xi}_{n+1} = (\xi_0, \xi_1, \dots, \xi_n, \xi_{n+1})$, where $\xi_{n+1} \in \mathbb{R}$. For $\lambda > 0$, $y \in C[0, t]$ and $\vec{\xi}_n \in \mathbb{R}^{n+1}$,

$$\begin{aligned} K_\Psi^\lambda(y, \vec{\xi}_n) &= \left[\frac{\lambda}{2\pi(t - t_n)} \right]^{\frac{1}{2}} \int_{\mathbb{R}} I_\Psi^\lambda(y, \vec{\xi}_{n+1}) \exp \left\{ -\frac{\lambda(\xi_{n+1} - \xi_n)^2}{2(t - t_n)} \right\} d\xi_{n+1} \\ &= \left[\frac{\lambda}{2\pi(t - t_n)} \right]^{\frac{1}{2}} \int_{\mathbb{R}^r} \int_{L_2[0, t]} H_2(\lambda, v, \vec{z}) \int_{\mathbb{R}} H_1(y, \vec{\xi}_{n+1}, v, \vec{z}) \\ &\quad \times \exp \left\{ -\frac{\lambda(\xi_{n+1} - \xi_n)^2}{2(t - t_n)} \right\} d\xi_{n+1} d\sigma(v) d\rho(\vec{z}) \end{aligned}$$

by Theorem 2.2. By Lemma 3.1, we obtain that

$$\begin{aligned} K_\Psi^\lambda(y, \vec{\xi}_n) &= \int_{\mathbb{R}^r} \int_{L_2[0, t]} H_1(y, 0, v, \vec{z}) H_2(\lambda, v, \vec{z}) H_4(\vec{\xi}_n, v, \vec{z}) \\ &\quad \times H_5(\lambda, v, \vec{z}) d\sigma(v) d\rho(\vec{z}). \end{aligned}$$

By (2.5), the Morera's theorem and the dominated convergence theorem, we have the existence of $T_\lambda[\Psi|X_n](y, \vec{\xi}_n)$ as the analytic extension of $K_\Psi^\lambda(y, \vec{\xi}_n)$ on $\mathbb{C}_+$. Let $\frac{1}{p} + \frac{1}{p'} = 1$. Then for $\lambda \in \mathbb{C}_+$,

$$\begin{aligned} &\|T_\lambda[\Psi|X_n](\cdot, \vec{\xi}_n) - T_q^{(p)}[\Psi|X_n](\cdot, \vec{\xi}_n)\|_{p'} \\ &\leq \int_{\mathbb{R}^r} \int_{L_2[0, t]} |H_2(\lambda, v, \vec{z}) H_5(\lambda, v, \vec{z}) - H_2(-iq, v, \vec{z}) H_5(-iq, v, \vec{z})| d|\sigma|(v) d|\rho|(\vec{z}) \end{aligned}$$

which converges to 0 as λ approaches to $-iq$ through $\mathbb{C}_+$. Now, the proof is completed. $\square$

THEOREM 3.3. Let Ψ_1 and Ψ_2 be as given in Theorem 2.3, and let X_n be given by (1.4). Then for a nonzero real q , w_φ -a.e. $y \in C[0, t]$ and P_{X_n} -a.e. $\vec{\xi}_n \in \mathbb{R}^{n+1}$, $[(\Psi_1 * \Psi_2)_q | X_n](y, \vec{\xi}_n)$ is given by

$$\begin{aligned} & [(\Psi_1 * \Psi_2)_q | X_n](y, \vec{\xi}_n) \\ &= \int_{\mathbb{R}^r} \int_{\mathbb{R}^r} \int_{L_2[0, t]} \int_{L_2[0, t]} H_1 \left(y, 0, \frac{1}{\sqrt{2}}(v_1 + v_2), \frac{1}{\sqrt{2}}(\vec{z}_1 + \vec{z}_2) \right) H_4 \left(\vec{\xi}_n, \right. \\ & \quad \left. \frac{1}{\sqrt{2}}(v_1 - v_2), \frac{1}{\sqrt{2}}(\vec{z}_1 - \vec{z}_2) \right) H_2 \left(-iq, \frac{1}{\sqrt{2}}(v_1 - v_2), \frac{1}{\sqrt{2}}(\vec{z}_1 - \vec{z}_2) \right) \\ & \quad \times H_5 \left(-iq, \frac{1}{\sqrt{2}}(v_1 - v_2), \frac{1}{\sqrt{2}}(\vec{z}_1 - \vec{z}_2) \right) d\sigma_1(v_1) d\sigma_2(v_2) d\rho_1(\vec{z}_1) d\rho_2(\vec{z}_2), \end{aligned}$$

where H_1 , H_2 , H_4 and H_5 are given by (2.3), (2.4), (3.1) and (3.2), respectively.

Proof. For $\vec{\xi}_n = (\xi_0, \xi_1, \dots, \xi_n) \in \mathbb{R}^{n+1}$, let $\vec{\xi}_{n+1} = (\xi_0, \xi_1, \dots, \xi_n, \xi_{n+1})$, where $\xi_{n+1} \in \mathbb{R}$. Note that for $y \in C[0, t]$, for $v_1, v_2 \in L_2[0, t]$ and for $\vec{z}_1, \vec{z}_2 \in \mathbb{R}^r$

$$\begin{aligned} & H_1 \left(y, \vec{\xi}_{n+1}, \frac{1}{\sqrt{2}}v_1, \frac{1}{\sqrt{2}}\vec{z}_1 \right) H_1 \left(y, -\vec{\xi}_{n+1}, \frac{1}{\sqrt{2}}v_2, \frac{1}{\sqrt{2}}\vec{z}_2 \right) \\ &= H_1 \left(y, 0, \frac{1}{\sqrt{2}}(v_1 + v_2), \frac{1}{\sqrt{2}}(\vec{z}_1 + \vec{z}_2) \right) \\ & \quad \times H_1 \left(0, \vec{\xi}_{n+1}, \frac{1}{\sqrt{2}}(v_1 - v_2), \frac{1}{\sqrt{2}}(\vec{z}_1 - \vec{z}_2) \right) \end{aligned}$$

so that we have for $\lambda > 0$

$$\begin{aligned} I_\lambda(\vec{\xi}_n) &\equiv \left[\frac{\lambda}{2\pi(t - t_n)} \right]^{\frac{1}{2}} \int_{\mathbb{R}} H_1 \left(y, \vec{\xi}_{n+1}, \frac{1}{\sqrt{2}}v_1, \frac{1}{\sqrt{2}}\vec{z}_1 \right) H_1 \left(y, -\vec{\xi}_{n+1}, \right. \\ & \quad \left. \frac{1}{\sqrt{2}}v_2, \frac{1}{\sqrt{2}}\vec{z}_2 \right) \exp \left\{ -\frac{\lambda(\xi_{n+1} - \xi_n)^2}{2(t - t_n)} \right\} d\xi_{n+1} \\ &= H_1 \left(y, 0, \frac{1}{\sqrt{2}}(v_1 + v_2), \frac{1}{\sqrt{2}}(\vec{z}_1 + \vec{z}_2) \right) H_4 \left(\vec{\xi}_n, \frac{1}{\sqrt{2}}(v_1 - v_2), \right. \\ & \quad \left. \frac{1}{\sqrt{2}}(\vec{z}_1 - \vec{z}_2) \right) H_5 \left(\lambda, \frac{1}{\sqrt{2}}(v_1 - v_2), \frac{1}{\sqrt{2}}(\vec{z}_1 - \vec{z}_2) \right) \end{aligned}$$

by Lemma 3.1. Now, by Theorem 2.3, we obtain that

$$\begin{aligned}
& [(\Psi_1 * \Psi_2)_\lambda | X_n](y, \vec{\xi}_n) \\
&= \int_{\mathbb{R}^r} \int_{\mathbb{R}^r} \int_{L_2[0,t]} \int_{L_2[0,t]} H_1\left(y, 0, \frac{1}{\sqrt{2}}(v_1 + v_2), \frac{1}{\sqrt{2}}(\vec{z}_1 + \vec{z}_2)\right) H_2\left(\lambda, \right. \\
&\quad \left. \frac{1}{\sqrt{2}}(v_1 - v_2), \frac{1}{\sqrt{2}}(\vec{z}_1 - \vec{z}_2)\right) H_4\left(\vec{\xi}_n, \frac{1}{\sqrt{2}}(v_1 - v_2), \frac{1}{\sqrt{2}}(\vec{z}_1 - \vec{z}_2)\right) \\
&\quad \times H_5\left(\lambda, \frac{1}{\sqrt{2}}(v_1 - v_2), \frac{1}{\sqrt{2}}(\vec{z}_1 - \vec{z}_2)\right) d\sigma_1(v_1) d\sigma_2(v_2) d\rho_1(\vec{z}_1) d\rho_2(\vec{z}_2).
\end{aligned}$$

By (2.5), the Morera's theorem and the dominated convergence theorem, we have the result. $\square$

4. Relationships between conditional Fourier-Feynman transforms and convolutions

In this section, we investigate the inverse conditional transform of the conditional Fourier-Feynman transforms of the functions as given in the previous sections.

THEOREM 4.1. *Let $1 \leq p \leq \infty$. Then, under the assumptions as given in Theorem 2.2, we have for $P_{X_{n+1}}$ -a.e. $\vec{\xi}_{n+1}, \vec{\zeta}_{n+1} \in \mathbb{R}^{n+2}$*

$$\|T_{\vec{\lambda}}[T_\lambda[\Psi|X_{n+1}](\cdot, \vec{\xi}_{n+1})|X_{n+1}](\cdot, \vec{\zeta}_{n+1}) - \Psi(\cdot + [\vec{\zeta}_{n+1} + \vec{\xi}_{n+1}])\|_p \rightarrow 0$$

as λ approaches to $-iq$ through $\mathbb{C}_+$.

Proof. For $\lambda_1 > 0$, $\lambda \in \mathbb{C}_+$, w_φ -a.e. $y \in C[0, t]$ and $P_{X_{n+1}}$ -a.e. $\vec{\xi}_{n+1}, \vec{\zeta}_{n+1} \in \mathbb{R}^{n+2}$,

$$\begin{aligned}
& I_{T_\lambda[\Psi|X_{n+1}](\cdot, \vec{\xi}_{n+1})}^{\lambda_1}(y, \vec{\zeta}_{n+1}) \\
&= \int_{\mathbb{R}^r} \int_{L_2[0,t]} \int_C H_1(\lambda_1^{-\frac{1}{2}}(x - [x]) + y + [\vec{\zeta}_{n+1}], \vec{\xi}_{n+1}, v, \vec{z}) H_2(\lambda, v, \vec{z}) \\
&\quad dw_\varphi(x) d\sigma(v) d\rho(\vec{z}) \\
&= \int_{\mathbb{R}^r} \int_{L_2[0,t]} \int_C H_1(y, \vec{\zeta}_{n+1} + \vec{\xi}_{n+1}, v, \vec{z}) H_2(\lambda.v.\vec{z}) H_3(\lambda_1, v, \vec{z}, x) \\
&\quad dw_\varphi(x) d\sigma(v) d\rho(\vec{z}) \\
&= \int_{\mathbb{R}^r} \int_{L_2[0,t]} H_1(y, \vec{\zeta}_{n+1} + \vec{\xi}_{n+1}, v, \vec{z}) H_2(\lambda_1, v, \vec{z}) H_2(\lambda, v, \vec{z}) d\sigma(v) d\rho(\vec{z})
\end{aligned}$$

by Lemma 2.1 and Theorem 2.2, where H_1 , H_2 and H_3 are given by (2.3), (2.4) and (2.6), respectively. By (2.5) and the Morera's Theorem, we have for $\lambda_1 \in \mathbb{C}_+$

$$\begin{aligned} & T_{\lambda_1}[T_\lambda[\Psi|X_{n+1}](\cdot, \vec{\xi}_{n+1})|X_{n+1}](y, \vec{\zeta}_{n+1}) \\ &= \int_{\mathbb{R}^r} \int_{L_2[0,t]} H_1(y, \vec{\zeta}_{n+1} + \vec{\xi}_{n+1}, v, \vec{z}) H_2(\lambda_1, v, \vec{z}) H_2(\lambda, v, \vec{z}) d\sigma(v) d\rho(\vec{z}). \end{aligned}$$

Now, we have for $\lambda \in \mathbb{C}_+$

$$\begin{aligned} & T_{\vec{\lambda}}[T_\lambda[\Psi|X_{n+1}](\cdot, \vec{\xi}_{n+1})|X_{n+1}](y, \vec{\zeta}_{n+1}) \\ &= \int_{\mathbb{R}^r} \int_{L_2[0,t]} H_1(y, \vec{\zeta}_{n+1} + \vec{\xi}_{n+1}, v, \vec{z}) H_2\left(\frac{|\lambda|^2}{2\operatorname{Re}\lambda}, v, \vec{z}\right) d\sigma(v) d\rho(\vec{z}) \end{aligned}$$

so that

$$\begin{aligned} & \|T_{\vec{\lambda}}[T_\lambda[\Psi|X_{n+1}](\cdot, \vec{\xi}_{n+1})|X_{n+1}](\cdot, \vec{\zeta}_{n+1}) - \Psi(\cdot + [\vec{\zeta}_{n+1} + \vec{\xi}_{n+1}])\|_p \\ & \leq \int_{\mathbb{R}^r} \int_{L_2[0,t]} \left[1 - H_2\left(\frac{|\lambda|^2}{2\operatorname{Re}\lambda}, v, \vec{z}\right)\right] d|\sigma|(v) d|\rho|(\vec{z}) \end{aligned}$$

which converges to 0 as λ approaches to $-iq$ through $\mathbb{C}_+$ by the dominated convergence theorem. $\square$

THEOREM 4.2. *Let $1 \leq p \leq \infty$ and X_n be given by (1.4). For w_φ -a.e. $x \in C[0, t]$, let $\Psi(x) = F(x)\Phi(x)$, where Φ and F are given by (2.8) and (2.9), respectively. Furthermore, for $y \in C[0, t]$ and $\vec{u}_n \in \mathbb{R}^{n+1}$ let*

$$\begin{aligned} & \Psi_1(y, \vec{u}_n) \\ &= \int_{\mathbb{R}^r} \int_{L_2[0,t]} \exp\{i(v, y)\} \exp\{i\langle \vec{v}, y \rangle, \vec{z} \rangle_{\mathbb{R}^r}\} H_4(\vec{u}_n, v, \vec{z}) d\sigma(v) d\rho(\vec{z}), \end{aligned}$$

where H_4 is given by (3.1). Then, for P_{X_n} -a.e. $\vec{\xi}_n, \vec{\zeta}_n \in \mathbb{R}^{n+1}$

$$\|T_{\vec{\lambda}}[T_\lambda[\Psi|X_n](\cdot, \vec{\xi}_n)|X_n](\cdot, \vec{\zeta}_n) - \Psi_1(\cdot, \vec{\zeta}_n + \vec{\xi}_n)\|_p \rightarrow 0$$

as λ approaches to $-iq$ through $\mathbb{C}_+$.

Proof. For $\vec{\zeta}_n = (\zeta_0, \zeta_1, \dots, \zeta_n) \in \mathbb{R}$, let $\vec{\zeta}_{n+1} = (\zeta_0, \zeta_1, \dots, \zeta_n, \zeta_{n+1})$, where $\zeta_{n+1} \in \mathbb{R}$. Then, for $\lambda_1 > 0$ and $\lambda \in \mathbb{C}_+$

$$\begin{aligned} I_{T_\lambda[\Psi|X_n](\cdot, \vec{\xi}_n)}^{\lambda_1}(y, \vec{\zeta}_{n+1}) &= \int_{\mathbb{R}^r} \int_{L_2[0,t]} H_1(y, \vec{\zeta}_{n+1}, v, \vec{z}) H_2(\lambda_1, v, \vec{z}) \\ &\quad \times H_2(\lambda, v, \vec{z}) H_4(\vec{\xi}_n, v, \vec{z}) H_5(\lambda, v, \vec{z}) d\sigma(v) d\rho(\vec{z}) \end{aligned}$$

by Theorems 2.2 and 3.2, so that by Lemma 3.1

$$\begin{aligned}
& K_{T_\lambda[\Psi|X_n](\cdot, \vec{\xi}_n)}^{\lambda_1}(y, \vec{\zeta}_n) \\
&= \left[\frac{\lambda}{2\pi(t-t_n)} \right]^{\frac{1}{2}} \int_{\mathbb{R}} I_{T_\lambda[\Psi|X_n](\cdot, \vec{\xi}_n)}^{\lambda_1}(y, \vec{\zeta}_{n+1}) \exp \left\{ -\frac{\lambda(\zeta_{n+1} - \zeta_n)^2}{2(t-t_n)} \right\} d\zeta_{n+1} \\
&= \int_{\mathbb{R}^r} \int_{L_2[0,t]} H_1(y, 0, v, \vec{z}) H_2(\lambda_1, v, \vec{z}) H_2(\lambda, v, \vec{z}) H_4(\vec{\zeta}_n + \vec{\xi}_n, v, \vec{z}) \\
&\quad \times H_5(\lambda_1, v, \vec{z}) H_5(\lambda, v, \vec{z}) d\sigma(v) d\rho(\vec{z}),
\end{aligned}$$

where H_1 , H_2 and H_5 are given by (2.3), (2.4) and (3.2), respectively. By (2.5), the Moreras theorem and the dominated convergence theorem, we have the existence of $T_{\lambda_1}[T_\lambda[\Psi|X_n](\cdot, \vec{\xi}_n)|X_n](y, \vec{\zeta}_n)$ with respect to $\lambda_1 \in \mathbb{C}_+$ as the analytic extension of $K_{T_\lambda[\Psi|X_n](\cdot, \vec{\xi}_n)}^{\lambda_1}(y, \vec{\zeta}_n)$. Now, for $\lambda \in \mathbb{C}_+$ and $y \in C[0, t]$

$$\begin{aligned}
& T_{\vec{\lambda}}[T_\lambda[\Psi|X_n](\cdot, \vec{\xi}_n)|X_n](y, \vec{\zeta}_n) \\
&= \int_{\mathbb{R}^r} \int_{L_2[0,t]} H_1(y, 0, v, \vec{z}) H_4(\vec{\zeta}_n + \vec{\xi}_n, v, \vec{z}) H_2\left(\frac{|\lambda|^2}{2\operatorname{Re}\lambda}, v, \vec{z}\right) \\
&\quad \times H_5\left(\frac{|\lambda|^2}{2\operatorname{Re}\lambda}, v, \vec{z}\right) d\sigma(v) d\rho(\vec{z})
\end{aligned}$$

so that we have

$$\begin{aligned}
& \|T_{\vec{\lambda}}[T_\lambda[\Psi|X_n](\cdot, \vec{\xi}_n)|X_n](\cdot, \vec{\zeta}_n) - \Psi_1(\cdot, \vec{\zeta}_n + \vec{\xi}_n)\|_p \\
&\leq \int_{\mathbb{R}^r} \int_{L_2[0,t]} \left[1 - H_2\left(\frac{|\lambda|^2}{2\operatorname{Re}\lambda}, v, \vec{z}\right) H_5\left(\frac{|\lambda|^2}{2\operatorname{Re}\lambda}, v, \vec{z}\right) \right] d|\sigma|(v) d|\rho|(\vec{z})
\end{aligned}$$

which converges to 0 as λ approaches to $-iq$ through $\mathbb{C}_+$ by the dominated convergence theorem. $\square$

THEOREM 4.3. *Let X_{n+1} be given by (1.3) and let $1 \leq p \leq \infty$. Furthermore, let Ψ_1 and Ψ_2 be as given in Theorem 2.3. Then for a nonzero real q , w_φ -a.e. $y \in C[0, t]$ and $P_{X_{n+1}}$ -a.e. $\vec{\xi}_{n+1}, \vec{\zeta}_{n+1} \in \mathbb{R}^{n+2}$,*

$$\begin{aligned}
(4.1) \quad & T_q^{(p)}[(\Psi_1 * \Psi_2)_q|X_{n+1}](\cdot, \vec{\xi}_{n+1})|X_{n+1}](y, \vec{\zeta}_{n+1}) \\
&= \left[T_q^{(p)}[\Psi_1|X_{n+1}]\left(\frac{1}{\sqrt{2}}y, \frac{1}{\sqrt{2}}(\vec{\zeta}_{n+1} + \vec{\xi}_{n+1})\right) \right] \\
&\quad \times \left[T_q^{(p)}[\Psi_2|X_{n+1}]\left(\frac{1}{\sqrt{2}}y, \frac{1}{\sqrt{2}}(\vec{\zeta}_{n+1} - \vec{\xi}_{n+1})\right) \right].
\end{aligned}$$

Proof. Let $\lambda \in \mathbb{C}_+^\sim$. For $\lambda_1 > 0$, w_φ -a.e. $y \in C[0, t]$ and $P_{X_{n+1}}$ -a.e. $\vec{\xi}_{n+1}, \vec{\zeta}_{n+1} \in \mathbb{R}^{n+2}$,

$$\begin{aligned} & I_{[(\Psi_1 * \Psi_2)_\lambda | X_{n+1}](\cdot, \vec{\xi}_{n+1})}^{\lambda_1}(y, \vec{\zeta}_{n+1}) \\ &= \int_{\mathbb{R}^r} \int_{\mathbb{R}^r} \int_{L_2[0, t]} \int_{L_2[0, t]} \int_C H_1 \left(\lambda_1^{-\frac{1}{2}}(x - [x]) + y + [\vec{\zeta}_{n+1}], \vec{\xi}_{n+1}, \right. \\ & \quad \left. \frac{1}{\sqrt{2}}v_1, \frac{1}{\sqrt{2}}\vec{z}_1 \right) H_1 \left(\lambda_1^{-\frac{1}{2}}(x - [x]) + y + [\vec{\zeta}_{n+1}], -\vec{\xi}_{n+1}, \frac{1}{\sqrt{2}}v_2, \frac{1}{\sqrt{2}}\vec{z}_2 \right) \\ & \quad \times H_2 \left(\lambda, \frac{1}{\sqrt{2}}(v_1 - v_2), \frac{1}{\sqrt{2}}(\vec{z}_1 - \vec{z}_2) \right) dw_\varphi(x) d\sigma_1(v_1) d\sigma_2(v_2) d\rho_1(\vec{z}_1) d\rho_2(\vec{z}_2) \end{aligned}$$

by Theorem 2.3, where H_1 and H_2 are given by (2.3) and (2.4), respectively. Now, we have

$$\begin{aligned} & I_{[(\Psi_1 * \Psi_2)_\lambda | X_{n+1}](\cdot, \vec{\xi}_{n+1})}^{\lambda_1}(y, \vec{\zeta}_{n+1}) \\ &= \int_{\mathbb{R}^r} \int_{\mathbb{R}^r} \int_{L_2[0, t]} \int_{L_2[0, t]} \int_C H_1 \left(\frac{1}{\sqrt{2}}y, \frac{1}{\sqrt{2}}(\vec{\zeta}_{n+1} + \vec{\xi}_{n+1}), v_1, \vec{z}_1 \right) \\ & \quad \times H_1 \left(\frac{1}{\sqrt{2}}y, \frac{1}{\sqrt{2}}(\vec{\zeta}_{n+1} - \vec{\xi}_{n+1}), v_2, \vec{z}_2 \right) H_2 \left(\lambda, \frac{1}{\sqrt{2}}(v_1 - v_2), \frac{1}{\sqrt{2}}(\vec{z}_1 - \vec{z}_2) \right) \\ & \quad \times H_3 \left(\lambda_1, \frac{1}{\sqrt{2}}(v_1 + v_2), \frac{1}{\sqrt{2}}(\vec{z}_1 + \vec{z}_2), x \right) \\ & \quad dw_\varphi(x) d\sigma_1(v_1) d\sigma_2(v_2) d\rho_1(\vec{z}_1) d\rho_2(\vec{z}_2), \end{aligned}$$

where H_3 is given by (2.6). By Lemma 2.1, we obtain that

$$\begin{aligned} (4.2) \quad & I_{[(\Psi_1 * \Psi_2)_\lambda | X_{n+1}](\cdot, \vec{\xi}_{n+1})}^{\lambda_1}(y, \vec{\zeta}_{n+1}) \\ &= \int_{\mathbb{R}^r} \int_{\mathbb{R}^r} \int_{L_2[0, t]} \int_{L_2[0, t]} H_1 \left(\frac{1}{\sqrt{2}}y, \frac{1}{\sqrt{2}}(\vec{\zeta}_{n+1} + \vec{\xi}_{n+1}), v_1, \vec{z}_1 \right) \\ & \quad \times H_1 \left(\frac{1}{\sqrt{2}}y, \frac{1}{\sqrt{2}}(\vec{\zeta}_{n+1} - \vec{\xi}_{n+1}), v_2, \vec{z}_2 \right) H_2 \left(\lambda, \frac{1}{\sqrt{2}}(v_1 - v_2), \right. \\ & \quad \left. \frac{1}{\sqrt{2}}(\vec{z}_1 - \vec{z}_2) \right) H_2 \left(\lambda_1, \frac{1}{\sqrt{2}}(v_1 + v_2), \frac{1}{\sqrt{2}}(\vec{z}_1 + \vec{z}_2) \right) d\sigma_1(v_1) \\ & \quad d\sigma_2(v_2) d\rho_1(\vec{z}_1) d\rho_2(\vec{z}_2). \end{aligned}$$

By (2.5), the Morera's theorem and the dominated convergence theorem, we have the analytic extension $T_{\lambda_1}[[(\Psi_1 * \Psi_2)_\lambda | X_{n+1}](\cdot, \vec{\xi}_{n+1}) | X_{n+1}](y, \vec{\zeta}_{n+1})$ of (4.2) as function of $\lambda_1 \in \mathbb{C}_+$. Let $T_q^{(p)}[[(\Psi_1 * \Psi_2)_\lambda | X_{n+1}](\cdot, \vec{\xi}_{n+1}) | X_{n+1}](y, \vec{\zeta}_{n+1})$ be given by the right hand side of (4.2), where λ_1

is replaced by $-iq$, and let $\frac{1}{p} + \frac{1}{p'} = 1$. By (2.5)

$$\begin{aligned} & \|T_{\lambda_1} [[(\Psi_1 * \Psi_2)_\lambda | X_{n+1}](\cdot, \vec{\xi}_{n+1}) | X_{n+1}](\cdot, \vec{\zeta}_{n+1}) \\ & \quad - T_q^{(p)} [[(\Psi_1 * \Psi_2)_\lambda | X_{n+1}](\cdot, \vec{\xi}_{n+1}) | X_{n+1}](\cdot, \vec{\zeta}_{n+1})]\|_{p'} \\ & \leq \int_{\mathbb{R}^r} \int_{\mathbb{R}^r} \int_{L_2[0,t]} \int_{L_2[0,t]} \left| H_2 \left(\lambda_1, \frac{1}{\sqrt{2}}(v_1 + v_2), \frac{1}{\sqrt{2}}(\vec{z}_1 + \vec{z}_2) \right) - \right. \\ & \quad \left. H_2 \left(-iq, \frac{1}{\sqrt{2}}(v_1 + v_2), \frac{1}{\sqrt{2}}(\vec{z}_1 + \vec{z}_2) \right) \right| d|\sigma_1|(v_1) d|\sigma_2|(v_2) d|\rho_1|(\vec{z}_1) d|\rho_2|(\vec{z}_2) \end{aligned}$$

which converges to 0 as λ_1 approaches to $-iq$ through $\mathbb{C}_+$ by the dominated convergence theorem. This shows the existence of $T_q^{(p)} [[(\Psi_1 * \Psi_2)_\lambda | X_{n+1}](\cdot, \vec{\xi}_{n+1}) | X_{n+1}](y, \vec{\zeta}_{n+1})$. Now, we have by (4.2)

$$\begin{aligned} & T_q^{(p)} [[(\Psi_1 * \Psi_2)_q | X_{n+1}](\cdot, \vec{\xi}_{n+1}) | X_{n+1}](y, \vec{\zeta}_{n+1}) \\ & = \int_{\mathbb{R}^r} \int_{\mathbb{R}^r} \int_{L_2[0,t]} \int_{L_2[0,t]} H_1 \left(\frac{1}{\sqrt{2}}y, \frac{1}{\sqrt{2}}(\vec{\zeta}_{n+1} + \vec{\xi}_{n+1}), v_1, \vec{z}_1 \right) H_1 \left(\right. \\ & \quad \left. \frac{1}{\sqrt{2}}y, \frac{1}{\sqrt{2}}(\vec{\zeta}_{n+1} - \vec{\xi}_{n+1}), v_2, \vec{z}_2 \right) H_2(-iq, v_1, \vec{z}_1) H_2(-iq, v_2, \vec{z}_2) \\ & \quad d\sigma_1(v_1) d\sigma_2(v_2) d\rho_1(\vec{z}_1) d\rho_2(\vec{z}_2) \end{aligned}$$

which completes the proof by Theorem 2.2. $\square$

Note that by the same method as used in the proof of Theorem 4.3, we can obtain (4.1), where $-iq$ is replaced by $\lambda \in \mathbb{C}_+$.

Now, we have the final result of our work.

THEOREM 4.4. *If X_{n+1} , $\vec{\xi}_{n+1}$ and $\vec{\zeta}_{n+1}$ in Theorem 4.3 are replaced by X_n , $\vec{\xi}_n \in \mathbb{R}^{n+1}$ and $\vec{\zeta}_n \in \mathbb{R}^{n+1}$, respectively, then the conclusion given by (4.1) is still true.*

Proof. Let $\lambda \in \mathbb{C}_+^\sim$. For $\lambda_1 > 0$, w_φ -a.e. $y \in C[0, t]$ and P_{X_n} -a.e. $\vec{\xi}_n, \vec{\zeta}_n \in \mathbb{R}^{n+1}$,

$$\begin{aligned} & K_{[(\Psi_1 * \Psi_2)_\lambda | X_n](\cdot, \vec{\xi}_n)}^{\lambda_1}(y, \vec{\zeta}_n) \\ & = \int_{\mathbb{R}^r} \int_{\mathbb{R}^r} \int_{L_2[0,t]} \int_{L_2[0,t]} H_1 \left(y, 0, \frac{1}{\sqrt{2}}(v_1 + v_2), \frac{1}{\sqrt{2}}(\vec{z}_1 + \vec{z}_2) \right) H_2 \left(\lambda_1, \right. \end{aligned}$$

$$\begin{aligned}
& \frac{1}{\sqrt{2}}(v_1 + v_2), \frac{1}{\sqrt{2}}(\vec{z}_1 + \vec{z}_2) \Big) H_4 \left(\vec{\zeta}_n, \frac{1}{\sqrt{2}}(v_1 + v_2), \frac{1}{\sqrt{2}}(\vec{z}_1 + \vec{z}_2) \right) H_5 \\
& \left(\lambda_1, \frac{1}{\sqrt{2}}(v_1 + v_2), \frac{1}{\sqrt{2}}(\vec{z}_1 + \vec{z}_2) \right) H_2 \left(\lambda, \frac{1}{\sqrt{2}}(v_1 - v_2), \frac{1}{\sqrt{2}}(\vec{z}_1 - \vec{z}_2) \right) \\
& H_4 \left(\vec{\xi}_n, \frac{1}{\sqrt{2}}(v_1 - v_2), \frac{1}{\sqrt{2}}(\vec{z}_1 - \vec{z}_2) \right) H_5 \left(\lambda, \frac{1}{\sqrt{2}}(v_1 - v_2), \frac{1}{\sqrt{2}}(\vec{z}_1 - \vec{z}_2) \right) \\
& d\sigma_1(v_1) d\sigma_2(v_2) d\rho_1(\vec{z}_1) d\rho_2(\vec{z}_2)
\end{aligned}$$

by Lemma 3.1, Theorems 3.2 and 3.3, where H_1 , H_2 , H_4 and H_5 are given by (2.3), (2.4), (3.1) and (3.2), respectively. By (2.5), the Morera's theorem and the dominated convergence theorem, we have the analytic extension $T_{\lambda_1} [[(\Psi_1 * \Psi_2)_\lambda | X_n](\cdot, \vec{\xi}_n) | X_n](y, \vec{\zeta}_n)$ of $K_{[(\Psi_1 * \Psi_2)_\lambda | X_n](\cdot, \vec{\xi}_n)}^{\lambda_1}(y, \vec{\zeta}_n)$ as function of $\lambda_1 \in \mathbb{C}_+$. Let $T_q^{(p)} [[(\Psi_1 * \Psi_2)_\lambda | X_n](\cdot, \vec{\xi}_n) | X_n](y, \vec{\zeta}_n)$ be given by the right-hand side of the above equality, where λ_1 is replaced by $-iq$, and let $\frac{1}{p} + \frac{1}{p'} = 1$. Then we have

$$\begin{aligned}
& \|T_{\lambda_1} [[(\Psi_1 * \Psi_2)_\lambda | X_n](\cdot, \vec{\xi}_n) | X_n](\cdot, \vec{\zeta}_n) \\
& \quad - T_q^{(p)} [[(\Psi_1 * \Psi_2)_\lambda | X_n](\cdot, \vec{\xi}_n) | X_n](\cdot, \vec{\zeta}_n)\|_{p'} \\
& \leq \int_{\mathbb{R}^r} \int_{\mathbb{R}^r} \int_{L_2[0,t]} \int_{L_2[0,t]} \left| H_2 \left(\lambda_1, \frac{1}{\sqrt{2}}(v_1 + v_2), \frac{1}{\sqrt{2}}(\vec{z}_1 + \vec{z}_2) \right) \right. \\
& \quad \times H_5 \left(\lambda_1, \frac{1}{\sqrt{2}}(v_1 + v_2), \frac{1}{\sqrt{2}}(\vec{z}_1 + \vec{z}_2) \right) - H_2 \left(-iq, \frac{1}{\sqrt{2}}(v_1 + v_2), \right. \\
& \quad \left. \frac{1}{\sqrt{2}}(\vec{z}_1 + \vec{z}_2) \right) H_5 \left(-iq, \frac{1}{\sqrt{2}}(v_1 + v_2), \frac{1}{\sqrt{2}}(\vec{z}_1 + \vec{z}_2) \right) \Big| \\
& \quad d|\sigma_1|(v_1) d|\sigma_1|(v_2) d|\rho_1|(\vec{z}_1) d|\rho_2|(\vec{z}_2)
\end{aligned}$$

which converges to 0 as λ approaches to $-iq$ through $\mathbb{C}_+$ by the dominated convergence theorem. This shows the existence of $T_q^{(p)} [[(\Psi_1 * \Psi_2)_\lambda | X_n](\cdot, \vec{\xi}_n) | X_n](\cdot, \vec{\zeta}_n)$. By (2.3) and (3.1), it is not difficult to show

$$\begin{aligned}
& H_1 \left(y, 0, \frac{1}{\sqrt{2}}(v_1 + v_2), \frac{1}{\sqrt{2}}(\vec{z}_1 + \vec{z}_2) \right) \\
& = H_1 \left(\frac{1}{\sqrt{2}}y, 0, v_1, \vec{z}_1 \right) H_1 \left(\frac{1}{\sqrt{2}}y, 0, v_2, \vec{z}_2 \right)
\end{aligned}$$

and

$$\begin{aligned}
& H_4\left(\vec{\zeta}_n, \frac{1}{\sqrt{2}}(v_1 + v_2), \frac{1}{\sqrt{2}}(\vec{z}_1 + \vec{z}_2)\right) \\
& \times H_4\left(\vec{\xi}_n, \frac{1}{\sqrt{2}}(v_1 - v_2), \frac{1}{\sqrt{2}}(\vec{z}_1 - \vec{z}_2)\right) \\
& = H_4\left(\frac{1}{\sqrt{2}}(\vec{\zeta}_n + \vec{\xi}_n), v_1, \vec{z}_1\right) H_4\left(\frac{1}{\sqrt{2}}(\vec{\zeta}_n - \vec{\xi}_n), v_2, \vec{z}_2\right).
\end{aligned}$$

Furthermore, by (2.4) and (3.2),

$$\begin{aligned}
& H_2\left(\lambda, \frac{1}{\sqrt{2}}(v_1 + v_2), \frac{1}{\sqrt{2}}(\vec{z}_1 + \vec{z}_2)\right) \\
& \times H_2\left(\lambda, \frac{1}{\sqrt{2}}(v_1 - v_2), \frac{1}{\sqrt{2}}(\vec{z}_1 - \vec{z}_2)\right) = H_2(\lambda, v_1, \vec{z}_1) H_2(\lambda, v_2, \vec{z}_2)
\end{aligned}$$

and

$$\begin{aligned}
& H_5\left(\lambda, \frac{1}{\sqrt{2}}(v_1 + v_2), \frac{1}{\sqrt{2}}(\vec{z}_1 + \vec{z}_2)\right) \\
& \times H_5\left(\lambda, \frac{1}{\sqrt{2}}(v_1 - v_2), \frac{1}{\sqrt{2}}(\vec{z}_1 - \vec{z}_2)\right) = H_5(\lambda, v_1, \vec{z}_1) H_5(\lambda, v_2, \vec{z}_2).
\end{aligned}$$

Now, we have the result by Theorem 3.2. $\square$

Note that by the same method as used in the proof of Theorem 4.4, we can show the same equality in the above theorem, where $-iq$ is replaced by $\lambda \in \mathbb{C}_+$.

References

- [1] R. H. Cameron and D. A. Storvick, *Some Banach algebras of analytic Feynman integrable functionals*, Lecture Notes in Mathematics **798**, Springer, Berlin-New York, 1980.
- [2] K. S. Chang, D. H. Cho, B. S. Kim, T. S. Song and I. Yoo, *Conditional Fourier-Feynman transform and convolution product over Wiener paths in abstract Wiener space*, Integral Transforms Spec. Funct. **14** (2003), no. 3, 217-235.
- [3] S. J. Chang and D. Skoug, *The effect of drift on conditional Fourier-Feynman transforms and conditional convolution products*, Int. J. Appl. Math. **2** (2000), no. 4, 505-527.
- [4] D. H. Cho, *A time-independent conditional Fourier-Feynman transform and convolution product on an analogue of Wiener space*, Honam J. Math. (2013), to appear.
- [5] D. H. Cho, *A time-dependent conditional Fourier-Feynman transform and convolution product on an analogue of Wiener space*, Houston J. Math. (2012), submitted.

- [6] D. H. Cho, *A conditional integral transform and conditional convolution product on a function space*, Integral Transforms Spec. Funct. **23** (2012), no 6, 405-420.
- [7] D. H. Cho, *A simple formula for an analogue of conditional Wiener integrals and its applications II*, Czechoslovak Math. J. **59** (2009), no. 2, 431-452.
- [8] D. H. Cho, *A simple formula for an analogue of conditional Wiener integrals and its applications*, Trans. Amer. Math. Soc. **360** (2008), no. 7, 3795-3811.
- [9] D. H. Cho, *Conditional Fourier-Feynman transform and convolution product over Wiener paths in abstract Wiener space: an L_p theory*, J. Korean Math. Soc. **41** (2004), no. 2, 265-294.
- [10] D. H. Cho, B. J. Kim and I. Yoo, *Analogues of conditional Wiener integrals and their change of scale transformations on a function space*, J. Math. Anal. Appl. **359** (2009), no. 2, 421-438.
- [11] M. K. Im and K. S. Ryu, *An analogue of Wiener measure and its applications*, J. Korean Math. Soc. **39** (2002), no. 5, 801-819.
- [12] M. J. Kim, *Conditional Fourier-Feynman transform and convolution product on a function space*, Int. J. Math. Anal. **3** (2009), no. 10, 457-471.
- [13] K. S. Ryu and M. K. Im, *A measure-valued analogue of Wiener measure and the measure-valued Feynman-Kac formula*, Trans. Amer. Math. Soc. **354** (2002), no. 12, 4921-4951.
- [14] K. S. Ryu, M. K. Im, and K. S. Choi, *Survey of the theories for analogue of Wiener measure space*, Interdiscip. Inform. Sci. **15** (2009), no. 3, 319-337.

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Department of Mathematics
 Kyonggi University
 Suwon 443-760, Republic of Korea
E-mail: j94385@kyonggi.ac.kr