POISSON BRACKETS DETERMINED BY JACOBIANS

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ABSTRACT. Fix $n - 2$ elements $h_1, \dots, h_{n-2}$ of the quotient field B of the polynomial algebra $\mathbb{C}[x_1, x_2, \dots, x_n]$. It is proved that B is a Poisson algebra with Poisson bracket defined by $\{f, g\} = \det(\text{Jac}(f, g, h_1, \dots, h_{n-2}))$ for any $f, g \in B$, where $\det(\text{Jac})$ is the determinant of a Jacobian matrix.

In [1], Jordan and the author studied Poisson brackets on the polynomial algebra $\mathbb{C}[x, y, z]$ in three indeterminates x, y, z determined by Jacobians. In particular, for an arbitrary rational function $s/t \in \mathbb{C}(x, y, z)$, they analyzed the Poisson bracket determined by the formula

$$(1) \quad (\{x, y\}, \{y, z\}, \{z, x\}) = t^2 \nabla(s/t), \quad s/t \in \mathbb{C}(x, y, z),$$

where $\nabla = (\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z})$ is the gradient. The general rule is that, for $f, g \in \mathbb{C}[x, y, z]$,

$$\{f, g\} = t^2 \det(\text{Jac}(f, g, s/t)),$$

where $\det(\text{Jac}(f, g, s/t))$ denotes the determinant of the Jacobian matrix of $(f, g, s/t)$.

The purpose of this paper is to generalize the bracket (1) to the general polynomial algebra $A := \mathbb{C}[x_1, x_2, \dots, x_n], n \geq 3$. For fixed $n - 2$ elements $h_1, \dots, h_{n-2}$ of the quotient field B of A , the fact that, for any $f, g \in B$, the bracket defined by

$$(2) \quad \{f, g\} = \det(\text{Jac}(f, g, h_1, \dots, h_{n-2}))$$

is a Poisson bracket is proved in [4] and [2, Theorem 1.4]. But the proof of [4] is not clear and that of [2] uses the Plücker relation and special derivations induced by the n -Jacobi identity in [3]. Here we prove by using only elementary algebraic theories that (2) is a Poisson bracket on B . Fix $s_1, t_1, \dots, s_{n-2}, t_{n-2} \in A$ such that s_i and $t_i \neq 0$ are coprime

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for each $i = 1, 2, \dots, n-2$. Next, as a corollary, we obtain that, for all $f, g \in A$,

$$(3) \quad \{f, g\} = (t_1 \cdots t_{n-2})^2 \det(\text{Jac}(f, g, s_1/t_1, s_2/t_2, \dots, s_{n-2}/t_{n-2}))$$

is a Poisson bracket in A , which is a generalization of (1). The presence of the factor $(t_1 \cdots t_{n-2})^2$ ensures that this restricts to a Poisson bracket on A .

Throughout the paper, A and B denote the algebra $\mathbb{C}[x_1, x_2, \dots, x_n]$ with $n \geq 3$ and the quotient field of A respectively, as above.

NOTATION 1. Let $\mathcal{F} = (\varphi^{ij})$ be an $(n-2) \times n$ -matrix with entries $\varphi^{ij} \in B$.

(a) For any $i, j = 1, \dots, n$, denote by $\mathcal{F}_{ij}$ the determinant of the $n \times n$ -matrix $\begin{pmatrix} e_i \\ e_j \\ \mathcal{F} \end{pmatrix}$, where $\{e_i\}_{i=1}^n$ is the standard basis of B^n .

(b) For $z \in B$, ∇z denotes the row vector of B^n

$$\nabla z = \left(\frac{\partial z}{\partial x_1}, \frac{\partial z}{\partial x_2}, \dots, \frac{\partial z}{\partial x_n} \right) = \frac{\partial z}{\partial x_1} e_1 + \frac{\partial z}{\partial x_2} e_2 + \cdots + \frac{\partial z}{\partial x_n} e_n.$$

LEMMA 2. For all $i, j = 1, \dots, n$, $\mathcal{F}_{ij} = -\mathcal{F}_{ji}$. In particular, $\mathcal{F}_{ii} = 0$.

Proof. It is clear from the elementary linear algebra. $\square$

For any $(n-2) \times n$ -matrix $\mathcal{F}$ in Notation 1 and for any $f, g \in B$, set

$$(4) \quad \{f, g\}^{\mathcal{F}} = \begin{vmatrix} \nabla f \\ \nabla g \\ \mathcal{F} \end{vmatrix} \in B.$$

Then the bilinear product $\{\cdot, \cdot\}^{\mathcal{F}}$ in (4) is antisymmetric and satisfies the Leibniz rule. Thus the algebra B with the bilinear product $\{\cdot, \cdot\}^{\mathcal{F}}$ becomes a Poisson algebra if and only if $\{\cdot, \cdot\}^{\mathcal{F}}$ satisfies the Jacobi identity. In such case, $\{\cdot, \cdot\}^{\mathcal{F}}$ is said to be a Poisson bracket in B .

NOTATION 3. For $a, b, c \in B$, let

$$J_{\mathcal{F}}(a, b, c) = \{\{a, b\}^{\mathcal{F}}, c\}^{\mathcal{F}} + \{\{b, c\}^{\mathcal{F}}, a\}^{\mathcal{F}} + \{\{c, a\}^{\mathcal{F}}, b\}^{\mathcal{F}}.$$

Thus a, b and c satisfy the Jacobi identity for $\{\cdot, \cdot\}^{\mathcal{F}}$ if and only if $J_{\mathcal{F}}(a, b, c) = 0$.

Note that any derivation on an algebra is determined by values of generators.

PROPOSITION 4. [1, Proposition 1.14] *The algebra B is a Poisson algebra under $\{\cdot, \cdot\}^{\mathcal{F}}$ if and only if $J_{\mathcal{F}}(x_i, x_j, x_k) = 0$ for any $1 \leq i, j, k \leq n$.*

LEMMA 5. *Let $1 \leq i, j, k \leq n$. Then $J_{\mathcal{F}}(x_i, x_j, x_k) = 0$ if and only if*

$$(5) \quad \sum_{\ell=1}^n \left[\frac{\partial \mathcal{F}_{ij}}{\partial x_{\ell}} \mathcal{F}_{\ell k} + \frac{\partial \mathcal{F}_{jk}}{\partial x_{\ell}} \mathcal{F}_{\ell i} + \frac{\partial \mathcal{F}_{ki}}{\partial x_{\ell}} \mathcal{F}_{\ell j} \right] = 0.$$

Proof. If $i = j$, $i = k$ or $j = k$ then we have $J_{\mathcal{F}}(x_i, x_j, x_k) = 0$ since $\{\cdot, \cdot\}^{\mathcal{F}}$ is antisymmetric, and (5) holds by Lemma 2. Hence we may assume that $i < j < k$. Observe that

$$\begin{aligned} J_{\mathcal{F}}(x_i, x_j, x_k) &= \left| \begin{array}{c} \nabla \begin{vmatrix} e_i \\ e_j \\ \mathcal{F} \end{vmatrix} \\ e_k \\ \mathcal{F} \end{array} \right| + \left| \begin{array}{c} \nabla \begin{vmatrix} e_j \\ e_k \\ \mathcal{F} \end{vmatrix} \\ e_i \\ \mathcal{F} \end{array} \right| + \left| \begin{array}{c} \nabla \begin{vmatrix} e_k \\ e_i \\ \mathcal{F} \end{vmatrix} \\ e_j \\ \mathcal{F} \end{array} \right| \\ &= \left| \begin{array}{c} \sum_{\ell=1}^n \frac{\partial \mathcal{F}_{ij}}{\partial x_{\ell}} e_{\ell} \\ e_k \\ \mathcal{F} \end{array} \right| + \left| \begin{array}{c} \sum_{\ell=1}^n \frac{\partial \mathcal{F}_{jk}}{\partial x_{\ell}} e_{\ell} \\ e_i \\ \mathcal{F} \end{array} \right| + \left| \begin{array}{c} \sum_{\ell=1}^n \frac{\partial \mathcal{F}_{ki}}{\partial x_{\ell}} e_{\ell} \\ e_j \\ \mathcal{F} \end{array} \right| \\ &= \sum_{\ell=1}^n \left[\frac{\partial \mathcal{F}_{ij}}{\partial x_{\ell}} \mathcal{F}_{\ell k} + \frac{\partial \mathcal{F}_{jk}}{\partial x_{\ell}} \mathcal{F}_{\ell i} + \frac{\partial \mathcal{F}_{ki}}{\partial x_{\ell}} \mathcal{F}_{\ell j} \right]. \end{aligned}$$

Hence the result holds. $\square$

LEMMA 6. *For any $1 \leq i, j, k, \ell \leq n$,*

$$(6) \quad \mathcal{F}_{ij} \mathcal{F}_{\ell k} + \mathcal{F}_{jk} \mathcal{F}_{\ell i} + \mathcal{F}_{ki} \mathcal{F}_{\ell j} = 0.$$

Proof. If any two indices among i, j, k, ℓ are equal, say $\ell = i$, then

$$\mathcal{F}_{ij} \mathcal{F}_{\ell k} + \mathcal{F}_{jk} \mathcal{F}_{\ell i} + \mathcal{F}_{ki} \mathcal{F}_{\ell j} = \mathcal{F}_{ij} \mathcal{F}_{ik} + \mathcal{F}_{jk} \mathcal{F}_{ii} + \mathcal{F}_{ki} \mathcal{F}_{ij} = 0$$

since $\mathcal{F}_{ii} = 0$ and $\mathcal{F}_{ki} = -\mathcal{F}_{ik}$, and thus (6) holds in this case. So we may assume that $n \geq 4$ and that all i, j, k, ℓ are distinct. For any p, q such that $1 \leq p, q \leq n$ and $p \neq q$, denote by X_{pq} the set of all bijective maps from $\{1, \dots, n-2\}$ onto $\{1, \dots, n\} \setminus \{p, q\}$ and, for $p = 1, \dots, n-2, q = 1, \dots, n$, denote by z_{pq} the (p, q) -entry of $\mathcal{F}$. The left hand of (6) is as

follows:

$$\begin{aligned}
& \sum_{\sigma \in X_{ij}, \tau \in X_{\ell k}} \left(\prod_{p=1}^{n-2} z_{p\sigma(p)} z_{p\tau(p)} \right) \begin{vmatrix} e_i \\ e_j \\ e_{\sigma(1)} \\ \vdots \\ e_{\sigma(n-2)} \end{vmatrix} \begin{vmatrix} e_\ell \\ e_k \\ e_{\tau(1)} \\ \vdots \\ e_{\tau(n-2)} \end{vmatrix} \\
& + \sum_{\sigma \in X_{jk}, \tau \in X_{\ell i}} \left(\prod_{p=1}^{n-2} z_{p\sigma(p)} z_{p\tau(p)} \right) \begin{vmatrix} e_j \\ e_k \\ e_{\sigma(1)} \\ \vdots \\ e_{\sigma(n-2)} \end{vmatrix} \begin{vmatrix} e_\ell \\ e_i \\ e_{\tau(1)} \\ \vdots \\ e_{\tau(n-2)} \end{vmatrix} \\
& + \sum_{\sigma \in X_{ki}, \tau \in X_{\ell j}} \left(\prod_{p=1}^{n-2} z_{p\sigma(p)} z_{p\tau(p)} \right) \begin{vmatrix} e_k \\ e_i \\ e_{\sigma(1)} \\ \vdots \\ e_{\sigma(n-2)} \end{vmatrix} \begin{vmatrix} e_\ell \\ e_j \\ e_{\tau(1)} \\ \vdots \\ e_{\tau(n-2)} \end{vmatrix}.
\end{aligned}$$

Let $(\sigma, \tau) \in X_{ij} \times X_{\ell k}$. We will find a unique $(\mu, \nu) \in X_{jk} \times X_{\ell i}$ (or $(\mu, \nu) \in X_{ki} \times X_{\ell j}$) such that $z_{p\sigma(p)} z_{p\tau(p)} = z_{p\mu(p)} z_{p\nu(p)}$ for each $p =$

$1, \dots, n-2$ and that $\begin{vmatrix} e_j \\ e_k \\ e_{\mu(1)} \\ \vdots \\ e_{\mu(n-2)} \end{vmatrix} \begin{vmatrix} e_\ell \\ e_i \\ e_{\nu(1)} \\ \vdots \\ e_{\nu(n-2)} \end{vmatrix}$ is nonzero. (If $(\mu, \nu) \in X_{ki} \times X_{\ell j}$

then $\begin{vmatrix} e_k \\ e_i \\ e_{\mu(1)} \\ \vdots \\ e_{\mu(n-2)} \end{vmatrix} \begin{vmatrix} e_\ell \\ e_j \\ e_{\nu(1)} \\ \vdots \\ e_{\nu(n-2)} \end{vmatrix}$ is nonzero.) There exists a unique p_1 such that

$k = \sigma(p_1)$, and then choose $\tau(p_1)$. If $\tau(p_1) \neq i$ and $\tau(p_1) \neq j$ then there exists a unique p_2 such that $\tau(p_1) = \sigma(p_2)$. If $\tau(p_2) \neq i$ and $\tau(p_2) \neq j$ then choose p_3 such that $\tau(p_2) = \sigma(p_3)$. This process stops only when $\tau(p_r) = i$ or $\tau(p_r) = j$, say $\tau(p_r) = i$, since there does not exist p such that $\sigma(p) = i$ or j . Hence we get a unique sequence

$$k = \sigma(p_1), \tau(p_1) = \sigma(p_2), \tau(p_2) = \sigma(p_3), \dots, \tau(p_r) = i.$$

Note that all terms of the sequence are different since all $\sigma(p_1), \dots, \sigma(p_r), i$ are distinct. Set

$$\mu(q) = \begin{cases} \sigma(q) & q \neq p_m \text{ for all } m = 1, \dots, r \\ \tau(p_m) & q = p_m \end{cases},$$

$$\nu(q) = \begin{cases} \tau(q) & q \neq p_m \text{ for all } m = 1, \dots, r \\ \sigma(p_m) & q = p_m. \end{cases}$$

Then $\mu \in X_{jk}$ and $\nu \in X_{li}$ and it is easy to see that $z_{p\sigma(p)}z_{p\tau(p)} = z_{p\mu(p)}z_{p\nu(p)}$ for each $p = 1, \dots, n-2$ and each row of the matrices

$\begin{pmatrix} e_j \\ e_k \\ e_{\mu(1)} \\ \vdots \\ e_{\mu(n-2)} \end{pmatrix}$ and $\begin{pmatrix} e_\ell \\ e_i \\ e_{\nu(1)} \\ \vdots \\ e_{\nu(n-2)} \end{pmatrix}$ is different, as claimed. Changing suitable rows in the matrices

$$\begin{pmatrix} e_j \\ e_k \\ e_{\mu(1)} \\ \vdots \\ e_{\mu(n-2)} \end{pmatrix} = \begin{pmatrix} e_j \\ e_k \\ e_{\sigma(1)} \\ \vdots \\ e_{\tau(p_1)} = e_{\sigma(p_2)} \\ \vdots \\ e_{\tau(p_m)} = e_{\sigma(p_{m+1})} \\ \vdots \\ e_{\sigma(n-2)} \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} e_\ell \\ e_i \\ e_{\nu(1)} \\ \vdots \\ e_{\nu(n-2)} \end{pmatrix} = \begin{pmatrix} e_\ell \\ e_i \\ e_{\tau(1)} \\ \vdots \\ e_{\sigma(p_1)} = e_k \\ \vdots \\ e_{\sigma(p_m)} = e_{\tau(p_{m-1})} \\ \vdots \\ e_{\tau(n-2)} \end{pmatrix},$$

$$\text{we have } \begin{vmatrix} e_j \\ e_k \\ e_{\mu(1)} \\ \vdots \\ e_{\mu(n-2)} \end{vmatrix} = (-1)^{r+1} \begin{vmatrix} e_i \\ e_j \\ e_{\sigma(1)} \\ \vdots \\ e_{\sigma(n-2)} \end{vmatrix} \quad \text{and} \quad \begin{vmatrix} e_\ell \\ e_i \\ e_{\nu(1)} \\ \vdots \\ e_{\nu(n-2)} \end{vmatrix} = (-1)^r \begin{vmatrix} e_\ell \\ e_k \\ e_{\tau(1)} \\ \vdots \\ e_{\tau(n-2)} \end{vmatrix},$$

hence

$$\begin{vmatrix} e_i \\ e_j \\ e_{\sigma(1)} \\ \vdots \\ e_{\sigma(n-2)} \end{vmatrix} \begin{vmatrix} e_\ell \\ e_k \\ e_{\tau(1)} \\ \vdots \\ e_{\tau(n-2)} \end{vmatrix} = - \begin{vmatrix} e_j \\ e_k \\ e_{\mu(1)} \\ \vdots \\ e_{\mu(n-2)} \end{vmatrix} \begin{vmatrix} e_\ell \\ e_i \\ e_{\nu(1)} \\ \vdots \\ e_{\nu(n-2)} \end{vmatrix}.$$

Therefore (6) holds. $\square$

LEMMA 7. For any $1 \leq i, j, k \leq n$, $J_{\mathcal{F}}(x_i, x_j, x_k) = 0$ if and only if

$$(7) \quad \sum_{\ell=1}^n \left[\mathcal{F}_{ij} \frac{\partial \mathcal{F}_{\ell k}}{\partial x_{\ell}} + \mathcal{F}_{jk} \frac{\partial \mathcal{F}_{\ell i}}{\partial x_{\ell}} + \mathcal{F}_{ki} \frac{\partial \mathcal{F}_{\ell j}}{\partial x_{\ell}} \right] = 0.$$

Proof. Since

$$\begin{aligned} \frac{\partial \mathcal{F}_{ij}}{\partial x_{\ell}} \mathcal{F}_{\ell k} + \frac{\partial \mathcal{F}_{jk}}{\partial x_{\ell}} \mathcal{F}_{\ell i} + \frac{\partial \mathcal{F}_{ki}}{\partial x_{\ell}} \mathcal{F}_{\ell j} &= \frac{\partial}{\partial x_{\ell}} (\mathcal{F}_{ij} \mathcal{F}_{\ell k} + \mathcal{F}_{jk} \mathcal{F}_{\ell i} + \mathcal{F}_{ki} \mathcal{F}_{\ell j}) \\ &\quad - \left[\mathcal{F}_{ij} \frac{\partial \mathcal{F}_{\ell k}}{\partial x_{\ell}} + \mathcal{F}_{jk} \frac{\partial \mathcal{F}_{\ell i}}{\partial x_{\ell}} + \mathcal{F}_{ki} \frac{\partial \mathcal{F}_{\ell j}}{\partial x_{\ell}} \right], \end{aligned}$$

(7) follows from (5) and (6). $\square$

LEMMA 8. For any $\varphi^1, \dots, \varphi^{n-2} \in B$, let $\mathcal{F} = \begin{pmatrix} \nabla \varphi^1 \\ \vdots \\ \nabla \varphi^{n-2} \end{pmatrix}$. Then

$$\sum_{\ell=1}^n \frac{\partial \mathcal{F}_{i\ell}}{\partial x_{\ell}} = 0 \text{ for each } i = 1, \dots, n.$$

Proof. We may assume that $i \neq \ell$ since $\mathcal{F}_{ii} = 0$. Denote by $X_{i\ell}$ the set of all bijective maps from $\{1, \dots, n-2\}$ onto $\{1, \dots, n\} \setminus \{i, \ell\}$ as in the proof of Lemma 6 and set

$$< i, \ell > = \begin{cases} i + \ell - 1 & \text{if } i < \ell, \\ i + \ell & \text{if } i > \ell. \end{cases}$$

Then we have

$$\begin{aligned} &\sum_{\ell=1, \ell \neq i}^n \frac{\partial \mathcal{F}_{i\ell}}{\partial x_{\ell}} \\ &= \sum_{\ell=1, \ell \neq i}^n \sum_{\sigma \in X_{i\ell}} (-1)^{< i, \ell >} \text{sgn}(\sigma) \frac{\partial}{\partial x_{\ell}} \left(\frac{\partial \varphi^1}{\partial x_{\sigma(1)}} \frac{\partial \varphi^2}{\partial x_{\sigma(2)}} \cdots \frac{\partial \varphi^{n-2}}{\partial x_{\sigma(n-2)}} \right) \\ &= \sum_{\ell=1, \ell \neq i}^n \sum_{\sigma \in X_{i\ell}} (-1)^{< i, \ell >} \text{sgn}(\sigma) \left[\frac{\partial^2 \varphi^1}{\partial x_{\ell} \partial x_{\sigma(1)}} \left(\frac{\partial \varphi^2}{\partial x_{\sigma(2)}} \cdots \frac{\partial \varphi^{n-2}}{\partial x_{\sigma(n-2)}} \right) + \cdots \right. \\ &\quad \left. + \left(\frac{\partial \varphi^1}{\partial x_{\sigma(1)}} \cdots \frac{\partial \varphi^{n-3}}{\partial x_{\sigma(n-3)}} \right) \frac{\partial^2 \varphi^{n-2}}{\partial x_{\ell} \partial x_{\sigma(n-2)}} \right]. \end{aligned}$$

Fix a term

$$A = \left(\frac{\partial \varphi^1}{\partial x_{\sigma(1)}} \cdots \frac{\partial \varphi^{k-1}}{\partial x_{\sigma(k-1)}} \right) \frac{\partial^2 \varphi^k}{\partial x_{\ell} \partial x_{\sigma(k)}} \left(\frac{\partial \varphi^{k+1}}{\partial x_{\sigma(k+1)}} \cdots \frac{\partial \varphi^{n-2}}{\partial x_{\sigma(n-2)}} \right).$$

Let $\sigma(k) = j$. Define $\tau \in X_{ij}$ by $\tau(k) = \ell, \tau(q) = \sigma(q)$ for all $q \neq k$. Then the term

$$B = \left(\frac{\partial \varphi^1}{\partial x_{\tau(1)}} \cdots \frac{\partial \varphi^{k-1}}{\partial x_{\tau(k-1)}} \right) \frac{\partial^2 \varphi^k}{\partial x_j \partial x_{\tau(k)}} \left(\frac{\partial \varphi^{k+1}}{\partial x_{\tau(k+1)}} \cdots \frac{\partial \varphi^{n-2}}{\partial x_{\tau(n-2)}} \right)$$

is equal to A since $\frac{\partial^2 \varphi^k}{\partial x_\ell \partial x_j} = \frac{\partial^2 \varphi^k}{\partial x_j \partial x_\ell}$ for all ℓ, j, k , and the coefficients of A and B are $(-1)^{\langle i, \ell \rangle} \text{sgn}(\sigma)$ and $(-1)^{\langle i, \ell \rangle} \text{sgn}(\tau)$, respectively. Let $|\ell - j| = m$. We may assume that $j < \ell < i$. Thus $\ell = j + m$. Let $\sigma(k_1) = \sigma(k) = j, \sigma(k_2) = j + 1, \dots, \sigma(k_m) = j + m - 1$. Hence $\tau(k_1) = \ell, \tau(k_2) = j + 1, \dots, \tau(k_m) = j + m - 1$. Arrange elements of $X_{i\ell}$ and X_{ij} by using order relation:

$$\begin{aligned} X_{i\ell} &= \{p_1 < p_2 < \cdots < p_{n-2}\} \\ &= \{\cdots, j-1, \quad j, \cdots, \quad j+m-1 = \ell-1, \quad \ell+1, \cdots\} \\ X_{ij} &= \{q_1 < q_2 < \cdots < q_{n-2}\} \\ &= \{\cdots, j-1, j+1, \cdots, \quad \ell, \quad \ell+1, \cdots\}. \end{aligned}$$

Identifying p_s and q_s to s for all $s = 1, \dots, n-2$, σ and τ are permutations in $\{1, \dots, n-2\}$ and $\tau^{-1}\sigma$ is defined as follows:

$$\begin{aligned} \tau^{-1}\sigma(k_1) &= k_2, \tau^{-1}\sigma(k_2) = k_3, \tau^{-1}\sigma(k_3) = k_4, \dots, \tau^{-1}\sigma(k_m) = k_1 \\ \tau^{-1}\sigma(p) &= p \text{ for all } p \neq k_q, q = 1, \dots, m. \end{aligned}$$

Thus $\tau^{-1}\sigma$ is the cycle $(k_1 k_2 \cdots k_m)$ in the set $\{1, \dots, n-2\}$. Hence $\text{sgn}(\sigma) = \text{sgn}(\tau)$ if and only if $\ell - j = m$ is odd since $\text{sgn}(\tau) = \text{sgn}(\tau^{-1})$. It follows that the coefficient $(-1)^{\langle i, \ell \rangle} \text{sgn}(\sigma)$ of A is $-(-1)^{\langle i, j \rangle} \text{sgn}(\tau)$ and thus $\sum_{\ell=1, \ell \neq i}^n \frac{\partial \mathcal{F}_{i\ell}}{\partial x_\ell} = 0$. $\square$

THEOREM 9. For any $\varphi^1, \dots, \varphi^{n-2} \in B$, let $\mathcal{F} = \begin{pmatrix} \nabla \varphi^1 \\ \vdots \\ \nabla \varphi^{n-2} \end{pmatrix}$. Then

B is a Poisson algebra under $\{\cdot, \cdot\}^{\mathcal{F}}$.

Proof. It is enough to show that $J_{\mathcal{F}}(x_i, x_j, x_k) = 0$ for all $1 \leq i, j, k \leq n$ by Proposition 4. By Lemma 8, (7) holds since $\mathcal{F}_{i\ell} = -\mathcal{F}_{\ell i}$, and thus we have $J_{\mathcal{F}}(x_i, x_j, x_k) = 0$ for all $1 \leq i, j, k \leq n$ by Lemma 7. $\square$

LEMMA 10. Let $\mathcal{F}$ and $\mathcal{G}$ be $(n-2) \times n$ -matrices in B such that $\mathcal{G}_i = a_i \mathcal{F}_i$ for each i , where $a_i \in B$ and $\mathcal{F}_i$ and $\mathcal{G}_i$ are the i -rows of $\mathcal{F}$ and $\mathcal{G}$ respectively. If $\{\cdot, \cdot\}^{\mathcal{F}}$ is a Poisson bracket on B then $\{\cdot, \cdot\}^{\mathcal{G}}$ is also a Poisson bracket and $\{\cdot, \cdot\}^{\mathcal{G}} = (a_1 \cdots a_{n-2}) \{\cdot, \cdot\}^{\mathcal{F}}$.

Proof. Set $a = a_1 \cdots a_{n-2}$. Since $\mathcal{G}_{ij} = a\mathcal{F}_{ij}$ for all i, j , we have

$$\begin{aligned} \frac{\partial \mathcal{G}_{ij}}{\partial x_\ell} \mathcal{G}_{\ell k} + \frac{\partial \mathcal{G}_{jk}}{\partial x_\ell} \mathcal{G}_{\ell i} + \frac{\partial \mathcal{G}_{ki}}{\partial x_\ell} \mathcal{G}_{\ell j} &= a^2 \left[\frac{\partial \mathcal{F}_{ij}}{\partial x_\ell} \mathcal{F}_{\ell k} + \frac{\partial \mathcal{F}_{jk}}{\partial x_\ell} \mathcal{F}_{\ell i} + \frac{\partial \mathcal{F}_{ki}}{\partial x_\ell} \mathcal{F}_{\ell j} \right] \\ &\quad + a \frac{\partial a}{\partial x_\ell} (\mathcal{F}_{ij} \mathcal{F}_{\ell k} + \mathcal{F}_{jk} \mathcal{F}_{\ell i} + \mathcal{F}_{ki} \mathcal{F}_{\ell j}). \end{aligned}$$

Thus $\{\cdot, \cdot\}^{\mathcal{G}}$ is a Poisson bracket by (5) and (6). $\square$

COROLLARY 11. *Fix $s_1, t_1, \dots, s_{n-2}, t_{n-2} \in A$ such that s_i and $t_i \neq 0$ are coprime for each $i = 1, 2, \dots, n-2$. Then (3) is a Poisson bracket on A .*

Proof. Under the notation of Theorem 9 and Lemma 10, set $\varphi^i = s_i/t_i$ and $a_i = t_i^2$ for all $i = 1, \dots, n-2$. Then the result follows immediately by Theorem 9 and Lemma 10 since each component of $t_i^2 \nabla(s_i/t_i)$ is an element of A . $\square$

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