ON POSITIVE IMPLICATIVE FILTERS IN *BE*-ALGEBRAS

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ABSTRACT. In this paper, we introduce the notion of a positive implicative filter in *BE*-algebras. We show that every positive implicative filter is a filter in *BE*-algebras. We give some examples that a filter may not be a positive implicative filter and also a positive implicative filter may be not an implicative filter in *BE*-algebras. We also give some equivalent conditions of a positive implicative filter in *BE*-algebras.

1. Introduction

Y. Imai and K. Iséki introduced two classes of abstract algebras: *BCK*-algebras and *BCI*-algebras ([6, 7]). It is known that the class of *BCK*-algebras is a proper subclass of the class of *BCI*-algebras. In [4, 5] Q. P. Hu and X. Li introduced a wide class of abstract algebras: *BCH*-algebras. They have shown that the class of *BCI*-algebras is a proper subclass of the class of *BCH*-algebras. J. Neggers and H. S. Kim ([13]) introduced the notion of *d*-algebras which is another generalization of *BCK*-algebras, and also they introduced the notion of *B*-algebras ([14, 15]), which is equivalent in some sense to the groups. Moreover, Y. B. Jun, E. H. Roh and H. S. Kim ([11]) introduced a new notion, called an *BH*-algebra, which is a generalization of *BCH/BCI/BCK*-algebras. A. Walendziak obtained the another equivalent axioms for *B*-algebra ([16]). H. S. Kim, Y. H. Kim and J. Neggers ([10]) introduced the notion a (pre-) Coxeter algebra and showed that a Coxeter algebra is equivalent to an abelian group all of whose elements have order 2, i.e., a Boolean group. C. B. Kim and H. S. Kim ([8]) introduced the notion of a *BM*-algebra which is a specialization of *B*-algebras. They proved

Received April 17, 2014; Accepted July 10, 2014.

2010 Mathematics Subject Classification: Primary 03G25, 20N02, 06F35.

Key words and phrases: *BE*-algebra, (positive implicative) filter, self distributive.

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that the class of BM -algebras is a proper subclass of B -algebras and also showed that a BM -algebra is equivalent to a 0-commutative B -algebra. In [9], H.S. Kim and Y. H. Kim introduced the notion of a BE -algebra as a generalization of a BCK -algebra. Using the notion of upper sets they gave an equivalent condition of the filter in BE -algebras. In [3, 4], S. S. Ahn and K. S. So introduced the notion of ideals in BE -algebras, and then stated and proved several characterizations of such ideals. Also they generalized the notion of upper sets in BE -algebras, and discussed properties of the characterizations of generalized upper sets $A_n(u, v)$ while relating them to the structure of ideals in transitive and self distributive BE -algebras.

In this paper, we introduce the notion of a positive implicative filter in BE -algebras. We show that every positive implicative filter is a filter in BE -algebras. We give some examples that a filter may not be a positive implicative filter and also a positive implicative filter may be not an implicative filter in BE -algebras. We also give equivalent conditions of a positive implicative filter in BE -algebras.

2. Preliminaries

We recall some definitions and results discussed in [2, 3, 9].

DEFINITION 2.1. An algebra $(X; *, 1)$ of type $(2, 0)$ is called a BE -algebra if

- (BE1) $x * x = 1$ for all $x \in X$;
- (BE2) $x * 1 = 1$ for all $x \in X$;
- (BE3) $1 * x = x$ for all $x \in X$;
- (BE4) $x * (y * z) = y * (x * z)$ for all $x, y, z \in X$ (*exchange*)

We introduce a relation " $\leq$ " on X by $x \leq y$ if and only if $x * y = 1$.

PROPOSITION 2.2. If $(X; *, 1)$ is a BE -algebra, then $x * (y * x) = 1$ for any $x, y \in X$.

EXAMPLE 2.3. ([9]) Let $X := \{1, a, b, c, d, 0\}$ be a set with the following table:

$*$	1	a	b	c	d	0
1	1	a	b	c	d	0
a	1	1	a	c	c	d
b	1	1	1	c	c	c
c	1	a	b	1	a	b
d	1	1	a	1	1	a
0	1	1	1	1	1	1

Then $(X; *, 1)$ is a BE -algebra.

DEFINITION 2.4. Let $(X; *, 1)$ be a BE -algebra and let F be a non-empty subset of X . Then F is said to be a *filter* of X if

- (F1) $1 \in F$;
- (F2) $x * y \in F$ and $x \in F$ imply $y \in F$.

In Example 2.3, $F_1 := \{1, a, b\}$ is a filter of X , but $F_2 := \{1, a\}$ is not a filter of X , since $a * b \in F_2$ and $a \in F_2$, but $b \notin F_2$.

PROPOSITION 2.5. Let X be a BE -algebra and let F be a filter of X . If $x \leq y$ and $x \in F$ for any $y \in X$, then $y \in F$.

Proof. Since $x \leq y$, we have $x * y = 1 \in F$. Since F is a filter of X and $x \in F$, we obtain $y \in F$. $\square$

DEFINITION 2.6. A BE -algebra $(X, *, 1)$ is said to be *self distributive* if $x * (y * z) = (x * y) * (x * z)$ for all $x, y, z \in X$.

EXAMPLE 2.7. ([9]) Let $X := \{1, a, b, c, d\}$ be a set with the following table:

$*$	1	a	b	c	d
1	1	a	b	c	d
a	1	1	b	c	d
b	1	a	1	c	c
c	1	1	b	1	b
d	1	1	1	1	1

It is easy to see that X is a BE -algebra satisfying the self distributivity law.

Note that the BE -algebra in Example 2.3 is not self distributive, since $d * (a * 0) = d * d = 1$, while $(d * a) * (d * 0) = 1 * a = a$.

DEFINITION 2.8. A BE -algebra is said to be *transitive* if for any $x, y, z \in X$,

$$y * z \leq (x * y) * (x * z).$$

EXAMPLE 2.9. ([2]) Let $X := \{1, a, b, c\}$ be a set with the following table:

$*$	1	a	b	c
1	1	a	b	c
a	1	1	a	a
b	1	1	1	a
c	1	1	a	1

Then it is easy to see that X is a transitive BE -algebra.

PROPOSITION 2.10. If X is a self distributive BE -algebra, then it is transitive.

The converse Proposition 2.10 need not be true in general. In Example 2.9, X is a transitive BE -algebra, but $a * (a * b) = a * a = 1$, while $(a * a) * (a * b) = 1 * a = a$, showing that X is not self distributive.

PROPOSITION 2.11. If X is a self distributive BE -algebra, then for any $x, y, z \in X$,

- (1) If $x \leq y$, then $z * x \leq z * y$ and $y * z \leq x * z$;
- (2) $y * z \leq (z * x) * (y * z)$.

3. Positive implicative filters

In what follows let X be a BE -algebra unless otherwise specified.

DEFINITION 3.1. Let X be a BE -algebra. A non-empty subset F of X is called an *implicative filter* of X if it satisfies (F1) and

- (I) $x * (y * z) \in F$ and $x * y \in F$ imply $x * z \in F$, for all $x, y, z \in X$.

EXAMPLE 3.2. Let X be the BE -algebra in Example 2.3. Then $F := \{1, a, b\}$ is an implicative filter of X .

Note that every implicative filter is a filter in a BE -algebra, but a filter need not be an implicative filter in general. In Example 2.3, $\{1\}$ is a filter of X , but not an implicative filter of X , since $d * (a * 0) = d * d = 1 \in \{1\}$ and $d * a = 1 \in \{1\}$, but $d * 0 = a \notin \{1\}$.

DEFINITION 3.3. Let X be a BE -algebra. A non-empty subset F of X is called a *positive implicative filter* of X if it satisfies (F1) and

- (Q) $x * ((y * z) * y) \in F$ and $x \in F$ implies $y \in F$ for all $x, y, z \in X$.

EXAMPLE 3.4. Let X be the BE -algebra in Example 2.7. It is easy to check that $F := \{1, a, b\}$ is a positive implicative filter of X .

THEOREM 3.5. *Let X be a BE -algebra. Then every positive implicative filter of X is a filter.*

Proof. Let F be a positive implicative filter of X and let $x * y \in F$ and $x \in F$. Then $x * ((y * y) * y) = x * (1 * y) = x * y \in F$. Since F is a positive implicative filter of X , we have $y \in F$. Thus F is a filter of X . $\square$

REMARK 3.6. The converse of Theorem 3.5 may not be true as shown in the following example. Also every implicative filter of X may not be true an positive implicative filter as in the following example.

EXAMPLE 3.7.

- (1) Let X be the BE -algebra as in Example 2.7. We know that $G := \{1, b\}$ is an implicative filter. But it is not a positive implicative filter, since $b * ((a * d) * a) = b * (d * a) = b * 1 = 1 \in G$ and $b \in G$, but $a \notin G$.
- (2) Let X be the BE -algebra as in Example 2.3. Then $\{1\}$ is a filter but it is neither a positive implicative filter nor an implicative filter of X , since $1 * ((a * b) * a) = 1 * (a * a) = 1, 1 \in \{1\}, a \notin \{1\}$ and $d * (a * 0) = 1, d * a = 1 \in \{1\}, d * 0 = a \notin \{1\}$.

EXAMPLE 3.8. Let $X := \{1, a, b, c\}$ be a BE -algebra with the following table:

$*$	1	a	b	c
1	1	a	b	c
a	1	1	b	c
b	1	a	1	c
c	1	1	b	1

It is easy to check that $\{1\}, \{1, a\}, \{1, b\}, \{1, a, b\}$, and $\{1, a, c\}$ are implicative filters of X . But $\{1, a\}, \{1, a, b\}$, and $\{1, a, c\}$ are positive implicative filters of X .

Now we give some equivalent conditions for a filter to be a positive implicative filter.

THEOREM 3.9. *Let X be a BE -algebra and let F be a filter of X . Then F is a positive implicative filter if and only if, for all $x, y \in X$,*

- (F3) $(x * y) * x \in F$ implies $x \in F$.

Proof. Assume that F is a positive implicative filter and let $(x * y) * x \in F$ for all $x, y \in X$. By (BE3), we have $1 * ((x * y) * x) \in F$. Since $1 \in F$, it follows from (Q) that $x \in F$. Hence (F3) holds.

Conversely, suppose that F satisfies (F3). Let $x * ((y * z) * y) \in F$ and $x \in F$. Using (F2), we have $(y * z) * y \in F$. It follows from (F3) that $y \in F$. Hence F is a positive implicative filter and proof is complete. $\square$

THEOREM 3.10. *Let X be a self distributive BE -algebra. If F is a positive implicative filter of X , then it satisfies*

(F4) $(x * y) * y \in F$ implies $(y * x) * x \in F$ for all $x, y \in X$.

Proof. Suppose that F is a positive implicative filter. Let $(x * y) * y \in F$ for all $x, y \in X$. Since $x \leq (y * x) * x$, it follows from Proposition 2.11 that $((y * x) * x) * y \leq x * y$. Then

$$\begin{aligned} (x * y) * y &\leq (y * x) * ((x * y) * x) \\ &= (x * y) * ((y * x) * x) \\ &\leq (((y * x) * x) * y) * ((y * x) * x). \end{aligned}$$

Thus $1 * (((y * x) * x) * y) * ((y * x) * x) \in F$, whence $(y * x) * x \in F$ by (Q). This completes the proof. $\square$

LEMMA 3.11. *Let X be a BE -algebra. If F is an implicative filter of X , then it satisfies for all $x, y \in X$,*

(F5) $x * (x * y) \in F$ implies $x * y \in F$ for all $x, y \in X$.

Proof. Straightforward. $\square$

THEOREM 3.12. *Let X be a self-distributive BE -algebra. If F is an implicative filter of X satisfying (F4), then F is a positive implicative filter of X .*

Proof. Suppose that F is an implicative filter satisfying (F4). Let $(x * y) * x \in F$ for all $x, y \in X$. It is enough to show that $x \in F$ by Theorem 3.9. It follows from Proposition 2.11 that $x \leq (x * y) * y$ imply $(x * y) * x \leq (x * y) * ((x * y) * y)$. Thus we have $(x * y) * ((x * y) * y) \in F$, since $(x * y) * x \in F$ and F is a filter. Using Lemma 3.11, we have $(x * y) * y \in F$. Since F satisfies (F4), we have

$$(R) \quad (y * x) * x \in F.$$

On the other hand, since $y * (x * y) = 1$, by Proposition 2.11, we get $(x * y) * x \leq y * x = 1 * (y * x)$, and hence $1 * (y * x) \in F$. Since $1 \in F$, it follows from (F2) that $y * x \in F$. By (R) and (F2), we obtain $x \in F$. This completes the proof. $\square$

DEFINITION 3.13. ([1]) Let X be a BE -algebra. X is said to be *commutative* if the following identity holds

$$(C) \quad (x * y) * y = (y * x) * x, \text{ i.e., } x \vee y = y \vee x$$

where $x \vee y = (y * x) * x$, for all $x, y \in X$.

THEOREM 3.14. *Let X be a self distributive commutative *BE*-algebra and let $\emptyset \neq F \subset X$. If F is a positive implicative filter of X , then F is an implicative filter of X .*

Proof. Let $x * (y * z) \in F$ and $x * y \in F$ for any $x, y, z \in X$. Using (BE4) and Proposition 2.11, we have $x * (y * z) = y * (x * z) \leq (x * y) * (x * (x * z))$. Since F is a filter and $x * (y * z) \in F$, we obtain $(x * y) * (x * (x * z)) \in F$. It follows from $x * y \in F$ that $x * (x * z) \in F$. On the other hand, we have

$$\begin{aligned} ((x * z) * z) * (x * z) &= x * ((x * z) * z) * z \\ &= x * ((z * (x * z)) * (x * z)) \\ &= x * (1 * (x * z)) \\ &= x * (x * z) \in F. \end{aligned}$$

Hence $((x * z) * z) * (x * z) \in F$. Since F is a positive implicative filter of X , we obtain $x * z \in F$. Thus F is an implicative filter of X . $\square$

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