A CHARACTERIZATION OF SCREEN CONFORMAL HALF LIGHTLIKE SUBMANIFOLDS

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Abstract. We study the geometry of screen conformal half lightlike submanifolds of a semi-Riemannian manifold. The main result is a classification theorem for screen conformal half lightlike submanifolds of a semi-Riemannian space form with a Killing co-screen distribution.

1. Introduction

It is well known that the radical distribution $Rad(TM) = TM \cap TM^\perp$ of a half lightlike submanifold M of a semi-Riemannian manifold $(\bar{M}, \bar{g})$ is a vector subbundle of the tangent bundle TM and the normal bundle $TM^\perp$ of rank 1. There exist two complementary non-degenerate distributions $S(TM)$ and $\mathcal{D}$ of $Rad(TM)$ in TM and $TM^\perp$ respectively, called a *screen* and co-screen distribution on M such that

$$(1.1) \quad TM = Rad(TM) \oplus_{orth} S(TM), \quad TM^\perp = Rad(TM) \oplus_{orth} \mathcal{D},$$

where the symbol $\oplus_{orth}$ denotes the orthogonal direct sum. We denote such a half lightlike submanifold by $(M, g, S(TM))$. Denote by $F(M)$ the algebra of smooth functions on M and by $\Gamma(E)$ the $F(M)$ module of smooth sections of a vector bundle E over M . Then there exist vector fields $\xi \in \Gamma(Rad(TM))$ and $u \in \Gamma(\mathcal{D})$ such that

$$\bar{g}(u, u) = \epsilon = \pm 1, \quad \bar{g}(\xi, v) = 0, \quad \forall v \in \Gamma(TM^\perp).$$

Consider the orthogonal complementary distribution $S(TM)^\perp$ to $S(TM)$ in TM . Certainly ξ and u belong to $\Gamma(S(TM)^\perp)$. Hence we have the following orthogonal decomposition

$$S(TM)^\perp = \mathcal{D} \oplus_{orth} \mathcal{D}^\perp,$$

where $\mathcal{D}^\perp$ is the orthogonal complementary to $\mathcal{D}$ in $S(TM)^\perp$. We known [3] that, for any null section ξ of $Rad(TM)$ on a neighborhood $\mathcal{U} \subset M$,

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there exists a uniquely defined vector field $N \in \Gamma(ltr(TM))$ satisfying

$$(1.2) \quad \bar{g}(\xi, N) = 1, \quad \bar{g}(N, N) = \bar{g}(N, X) = \bar{g}(N, u) = 0,$$

for all $X \in \Gamma(S(TM)|_{\mathcal{U}})$. We call $ltr(TM)$, N and $tr(TM) = \mathcal{D} \oplus_{orth} ltr(TM)$ the *lightlike transversal vector bundle*, *lightlike transversal vector field* and *transversal vector bundle* of M with respect to $S(TM)$ respectively. $T\bar{M}$ is decomposed as follow :

$$(1.3) \quad T\bar{M} = TM \oplus tr(TM).$$

The purpose of this paper is to study the geometry of screen conformal half lightlike submanifolds of a semi-Riemannian space form $\bar{M}(c)$. We prove a characterization theorem for M such that $\dim M > 3$: If the co-screen distribution is Killing, then $c = 0$ (Theorem 2.1). Using this theorem, we prove an annexed theorem: If $c \neq 0$, then M is totally umbilical and locally a product manifold $M = C \times M^*$, where C is a null curve and M^* is a totally umbilical semi-Riemannian 2-surface or a non-null curve (Theorem 2.3). Recall the following structure equations:

Let $\bar{\nabla}$ be the Levi-Civita connection of $\bar{M}$ and P the projection morphism of $\Gamma(TM)$ on $\Gamma(S(TM))$ with respect to the decomposition (1.1). Then the local Gauss and Weingarten formulas are given by

$$(1.4) \quad \bar{\nabla}_X Y = \nabla_X Y + B(X, Y)N + D(X, Y)u,$$

$$(1.5) \quad \bar{\nabla}_X N = -A_N X + \tau(X)N + \rho(X)u,$$

$$(1.6) \quad \bar{\nabla}_X u = -A_u X + \phi(X)N,$$

$$(1.7) \quad \nabla_X PY = \nabla_X^* PY + C(X, PY)\xi,$$

$$(1.8) \quad \nabla_X \xi = -A_\xi^* X - \tau(X)\xi,$$

for any $X, Y \in \Gamma(TM)$, where ∇ and ∇^* are induced linear connections of M and on $S(TM)$ respectively, the bilinear forms B and D on TM are called the *local transversal second fundamental form* and *local screen second fundamental form* of M respectively, C is called the *local radical second fundamental form* on $S(TM)$. A_N , A_ξ^* and A_u are linear operators on $\Gamma(TM)$ and τ , ρ and ϕ are 1-forms on TM .

Since the connection $\bar{\nabla}$ of $\bar{M}$ is torsion-free, the induced connection ∇ of M is also torsion-free and both B and D are symmetric and

$$(1.9) \quad B(X, \xi) = 0, \quad D(X, \xi) = -\epsilon\phi(X), \quad \forall X \in \Gamma(TM).$$

The induced connection ∇ on M is not metric and satisfies

$$(1.10) \quad (\nabla_X g)(Y, Z) = B(X, Y)\eta(Z) + B(X, Z)\eta(Y),$$

for all $X, Y, Z \in \Gamma(TM)$, where η is a 1-form on TM such that

$$(1.11) \quad \eta(X) = \bar{g}(X, N), \quad \forall X \in \Gamma(TM).$$

But the connection ∇^* on M^* is metric. The above three local second fundamental forms of M and M^* are related to their shape operators by

$$(1.12) \quad B(X, Y) = g(A_\xi^* X, Y), \quad \bar{g}(A_\xi^* X, N) = 0,$$

$$(1.13) \quad C(X, PY) = g(A_N X, PY), \quad \bar{g}(A_N X, N) = 0,$$

$$(1.14) \quad \epsilon D(X, PY) = g(A_u X, PY), \quad \bar{g}(A_u X, N) = \epsilon \rho(X),$$

$$(1.15) \quad \epsilon D(X, Y) = g(A_u X, Y) - \phi(X)\eta(Y).$$

We show, by (1.12) and (1.13), that A_ξ^* and A_N are $\Gamma(S(TM))$ -valued and shape operators related to B and C respectively, called the *radical* and *transversal shape operator* on TM and $S(TM)$ respectively. From (1.12), A_ξ^* is self-adjoint on $\Gamma(TM)$ such that $A_\xi^* \xi = 0$.

A vector field X on a semi-Riemannian manifold $(\bar{M}, \bar{g})$ is said to be a *Killing* if $\bar{\mathcal{L}}_X \bar{g} = 0$, where the symbol $\bar{\mathcal{L}}_X$ denotes the Lie derivative with respect to the vector field X . A distribution D on $\bar{M}$ is said to be a *Killing* if each vector field belonging to D is a Killing. If the co-screen distribution $\mathcal{D}$ is a Killing, using (1.6) and (1.15), we have

$$(\bar{\mathcal{L}}_u \bar{g})(X, Y) = -2\epsilon D(X, Y).$$

Theorem 1.1 [5]. *Let $(M, g, S(TM))$ be a half lightlike submanifold of a semi-Riemannian manifold $(\bar{M}, \bar{g})$. Then the co-screen distribution is a Killing distribution if and only if $D = 0$ on $\Gamma(TM)$.*

We denote by $\bar{R}$, R and R^* the curvature tensors of $\bar{\nabla}$, ∇ and ∇^* respectively. Using the Gauss-Weingarten equations for M and $S(TM)$, we obtain the Gauss-Codazzi equations for M and $S(TM)$:

$$(1.16) \quad \begin{aligned} \bar{g}(\bar{R}(X, Y)Z, PW) &= g(R(X, Y)Z, PW) \\ &+ B(X, Z)C(Y, PW) - B(Y, Z)C(X, PW) \\ &+ \epsilon\{D(X, Z)D(Y, PW) - D(Y, Z)D(X, PW)\}, \end{aligned}$$

$$(1.17) \quad \begin{aligned} \bar{g}(\bar{R}(X, Y)Z, \xi) &= (\nabla_X B)(Y, Z) - (\nabla_Y B)(X, Z) \\ &+ B(Y, Z)\tau(X) - B(X, Z)\tau(Y) \\ &+ D(Y, Z)\phi(X) - D(X, Z)\phi(Y), \end{aligned}$$

$$(1.18) \quad \begin{aligned} \bar{g}(\bar{R}(X, Y)Z, N) &= \bar{g}(R(X, Y)Z, N) \\ &+ \epsilon\{D(X, Z)\rho(Y) - D(Y, Z)\rho(X)\}, \end{aligned}$$

$$(1.19) \quad \bar{g}(R(X, Y)PZ, PW) = g(R^*(X, Y)PZ, PW) \\ + C(X, PZ)B(Y, PW) - C(Y, PZ)B(X, PW),$$

$$(1.20) \quad g(R(X, Y)PZ, N) = (\nabla_X C)(Y, PZ) - (\nabla_Y C)(X, PZ) \\ + C(X, PZ)\tau(Y) - C(Y, PZ)\tau(X).$$

2. Screen conformal half lightlike submanifolds

A half lightlike submanifold $(M, g, S(TM))$ of a semi-Riemannian manifold $(\bar{M}, \bar{g})$ is *screen conformal* [1] if the shape operators A_N and A_ξ^* of M and $S(TM)$ respectively are related by $A_N = \varphi A_\xi^*$, or equivalently,

$$(2.1) \quad C(X, PY) = \varphi B(X, Y), \quad \forall X, Y \in \Gamma(TM)$$

where φ is a non-vanishing smooth function on a neighborhood $\mathcal{U}$ in M .

Note. For a screen conformal half lightlike submanifold M , as C is symmetric, $S(TM)$ is integrable. Thus M is a locally product manifold $C \times M^*$ where C is a null curve and M^* is a leaf of $S(TM)$ [3].

Theorem 2.1. *Let $(M, g, S(TM))$ be an $m(> 3)$ -dimensional screen conformal half lightlike submanifold of a semi-Riemannian space form $\bar{M}(c)$ with a Killing co-screen distribution. Then $c = 0$.*

Proof. As $\bar{M}$ is a space of constant curvature with a Killing co-screen distribution, by Theorem 1.1, (1.9), (1.14) and (1.15), we have

$$D(X, Y) = 0, \quad \phi(X) = 0, \quad A_u X = \epsilon \rho(X) \xi,$$

for any $X, Y \in \Gamma(TM)$. From (1.17), we obtain

$$(2.2) \quad (\nabla_X B)(Y, Z) - (\nabla_Y B)(X, Z) = B(X, Z)\tau(Y) - B(Y, Z)\tau(X),$$

for all $X, Y, Z \in \Gamma(TM)$. Using this, (1.18), (1.20) and (2.1), we get

$$(2.3) \quad \{X[\varphi] - 2\varphi\tau(X)\}B(Y, PZ) - \{Y[\varphi] - 2\varphi\tau(Y)\}B(X, PZ) \\ = c\{g(Y, PZ)\eta(X) - g(X, PZ)\eta(Y)\}.$$

Replacing Y by ξ in (2.3) and using (1.9), we obtain

$$(2.4) \quad \{\xi[\varphi] - 2\varphi\tau(\xi)\}B(X, PZ) = c g(X, PZ).$$

If $\xi[\varphi] - 2\varphi\tau(\xi) = 0$ or $B = 0$, i.e., M is a totally geodesic, then $c = 0$.

Assume that $c \neq 0$. Then $\xi[\varphi] - 2\varphi\tau(\xi) \neq 0$ and $B \neq 0$, that is, M is not a totally geodesic. From (2.1) and (2.4), we have

$$(2.5) \quad B(X, Y) = \sigma g(X, Y), \quad C(X, Y) = \varphi\sigma g(X, Y),$$

for all $X, Y \in \Gamma(TM)$, where $\sigma = c(\xi[\varphi] - 2\varphi\tau(\xi))^{-1} \neq 0$. If $\varphi\sigma = 0$, then $C = 0$ by (2.5). From (2.1), we have $B = 0$. Therefore, we get $\varphi\sigma \neq 0$. Thus M and $S(TM)$ are totally umbilical which are not totally geodesic. Since M is screen conformal, by Note, M is a locally product manifold $C \times M^*$ where C is a null curve and M^* is a leaf of $S(TM)$. Since $\bar{M}$ has a constant curvature, from (1.16), (1.19) and (2.5), we have

$$(2.6) \quad R^*(X, Y)Z = (c + 2\varphi\sigma^2)\{g(Y, Z)X - g(X, Z)Y\},$$

for all $X, Y, Z \in \Gamma(S(TM))$. Thus the leaf M^* of $S(TM)$ is a semi-Riemannian manifold of curvature $(c + 2\varphi\sigma^2)$. Let Ric^* be the induced symmetric Ricci tensor of M^* . From (2.6), we have

$$(2.7) \quad Ric^*(X, Y) = (c + 2\varphi\sigma^2)(m - 2)g(X, Y), \quad \forall X, Y \in \Gamma(S(TM)).$$

Thus M^* is an Einstein manifold. Since $\dim M^* > 2$, $(c + 2\varphi\sigma^2)$ is a constant and M^* has a constant curvature $(c + 2\varphi\sigma^2)$.

Using (1.10), (2.2), (2.5) and the degeneracy of $S(TM)$, we have

$$(2.8) \quad \{X[\sigma] + \sigma\tau(X) - \sigma^2\eta(X)\}PY = \{Y[\sigma] + \sigma\tau(Y) - \sigma^2\eta(Y)\}PX.$$

Now suppose there exists a vector field $X_o \in \Gamma(TM)$ such that $X_o[\sigma] + \sigma\tau(X_o) - \sigma^2\eta(X_o) \neq 0$ at each point $x \in M$, then $PY = fPX_o$ for all $Y \in \Gamma(TM)$, where f is a smooth function. It follows that all vectors from the fiber $S(TM)_x$ are co-linear with $(PX_o)_x$. It is a contradiction as $\dim(S(TM)_x) > 2$. Thus $X[\sigma] + \sigma\tau(X) - \sigma^2\eta(X) = 0$ for all $X \in \Gamma(TM)$. This implies $\xi[\sigma] = \sigma^2 - \sigma\tau(\xi)$. Therefore, $0 = \xi[\varphi\sigma^2] = \sigma(c + 2\varphi\sigma^2)$. Since $(c + 2\varphi\sigma^2)$ is a constant and $\sigma \neq 0$, we have $c + 2\varphi\sigma^2 = 0$. Thus M^* is a semi-Euclidean space and the second fundamental form C of M^* satisfies $C = 0$. Thus, from (2.5), we have $\varphi\sigma = 0$. This means $c = 0$. It is contradiction to $c \neq 0$. Consequently, we have $c = 0$. $\square$

Corollary 1. *There exist no screen conformal half lightlike submanifolds of semi-Riemannian space form $\bar{M}(c)$, $c \neq 0$ with a Killing co-screen distribution and $\dim M > 3$.*

Corollary 2. *There exist no screen conformal totally geodesic half lightlike submanifolds of semi-Riemannian space form $\bar{M}(c)$, $c \neq 0$ with a Killing co-screen distribution.*

Theorem 2.2. *Let $(M, g, S(TM))$ be an $m(> 3)$ -dimensional screen conformal half lightlike submanifold of $\bar{M}(c)$ with a Killing co-screen distribution. Then the curvature tensors R and R^* of M and $S(TM)$ respectively, are related by*

$$(2.9) \quad R(X, Y)Z = \frac{1}{2} R^*(PX, PY)PZ, \quad \forall X, Y, Z \in \Gamma(TM).$$

Proof. Form (1.18) with $D = c = 0$, we have $\bar{g}(R(X, Y)Z, N) = 0$. Thus we see that (2.9) is equivalent with

$$(2.10) \quad g(R(X, Y)Z, PW) = \frac{1}{2} g(R^*(PX, PY)PZ, PW),$$

for all $X, Y, Z, W \in \Gamma(TM)$. Due to (1.17) with $D = c = 0$, we have $g(R(X, Y)\xi, Z) = 0$. Thus we see that (2.10) is true for $Z = \xi$. Using (1.9), (1.16) and (1.19) satisfy $D = c = 0$ and (2.1), we derive (2.10). $\square$

Theorem 2.3. *Let $(M, g, S(TM))$ be a screen conformal half lightlike submanifold of a semi-Riemannian space form $\bar{M}(c)$, $c \neq 0$ with a Killing co-screen distribution. Then M is totally umbilical and locally a product manifold $C \times M^*$, where C is a null curve and M^* is a totally umbilical semi-Riemannian 2-surface or a non-null curve.*

Proof. From the fact that M is a screen conformal lightlike hypersurface with $c \neq 0$, we show that $\dim M \leq 3$ and both M and M^* are totally umbilical which are not totally geodesic by (2.5). By Note, we get $M = C \times M^*$ and M^* is a proper totally umbilical semi-Riemannian 2-surface or a non-null curve. $\square$

References

- [1] Atindogbe, C. and Duggal, K. L. *Conformal screen on lightlike hypersurfaces*, International J. of Pure and Applied Math., 11(4), 2004, 421-442.
- [2] Duggal, K. L., *On canonical screen for lightlike submanifolds of codimension two*, Central European Journal of Math., vol. 5(4), 2007, 710-719.
- [3] Duggal, K. L. and Bejancu, A., *Lightlike Submanifolds of Semi-Riemannian Manifolds and Applications*, Kluwer Acad. Publishers, Dordrecht, 1996.
- [4] Jin, D. H., *Einstein half lightlike submanifolds of codimension 2*, to appear in J. Korea Soc. Math. Educ. Ser. B: Pure Appl. Math., 16, no.1, 2009.
- [5] Jin, D. H., *Einstein half lightlike submanifolds with a Killing co-screen distribution*, Honam Math. J., 30, no.3 2008, p487-504.

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