ARRANGEMENTS OF HYPERPLANES IN $\mathbb{R}^3$ AND THEIR FREENESS

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Abstract. In this paper, we classify essential arrangements $\mathcal{A}$ of hyperplanes in $\mathbb{R}^3$ with $|\mathcal{A}| = 5$ and give a full description of each case in terms of freeness.

1. Introduction

Let V be an l -dimensional vector space over a field $\mathbb{K}$. A hyperplane in V is a codimension one subspace of V . A central l -arrangement $\mathcal{A}$ in V is a finite set of hyperplanes in V such that each hyperplane contains the origin.

Let V^* denote the dual space of V and $S = S(V^*)$ be the symmetric algebra of V^* . Choose a basis $\{e_1, \dots, e_l\}$ in V and let $\{x_1, \dots, x_l\}$ be the dual basis in V^* so that $x_i(e_j) = \delta_{ij}$. If we identify $S(V^*)$ with the polynomial algebra $\mathbb{K}[x_1, \dots, x_l]$, each hyperplane H in V containing the origin is the set of zeros of a polynomial $\alpha_H = a_1x_1 + \dots + a_lx_l$, $a_i \in \mathbb{K}$.

Given a central l -arrangement $\mathcal{A}$ in V , the product $Q(\mathcal{A}) = \prod_{H \in \mathcal{A}} \alpha_H$ is called defining polynomial of $\mathcal{A}$. Suppose $\theta : S \rightarrow S$ is derivation. It is called an $\mathcal{A}$ -derivation if $\theta(Q(\mathcal{A})) \in Q(\mathcal{A})S$. The set of $\mathcal{A}$ -derivations is an S -module, denoted by $D(\mathcal{A})$. The arrangement $\mathcal{A}$ is defined to be free if $D(\mathcal{A})$ is a free S -module. It has been an enduring mystery just what makes an arrangement free, but is known that the property is not generic. Terao showed in [4] that each free arrangement is associated to a collection of nonnegative integers called its exponents, $\exp \mathcal{A} = \{b_1, \dots, b_l\}$ and $\pi(\mathcal{A}, t) = \prod_{i=1}^l (1 + b_i t)$ where π is the Poincaré polynomial of $\mathcal{A}$. He also proved [5] that complex reflection arrangements are free. For these early developments, we refer the reader to [2].

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In this paper, we classify central arrangements $\mathcal{A}$ of hyperplanes in $\mathbb{R}^3$ with $|\mathcal{A}| = 5$ such that $\cap_{H \in \mathcal{A}} H = \{0\}$, so called essential arrangements and provide a full description of each case in terms of freeness where $|\mathcal{A}|$ indicates the cardinality of $\mathcal{A}$.

2. Arrangements $\mathcal{A}$ of Hyperplanes in $\mathbb{R}^3$ with $|\mathcal{A}| = 5$

A fundamental technical tool in this paper is the method of deletion and restriction, which allows induction on the number of hyperplanes in the arrangement. It uses the triple arrangement $(\mathcal{A}, \mathcal{A}', \mathcal{A}'')$ where $\mathcal{A}' = \mathcal{A} \setminus \{H_0\}$ and $\mathcal{A}'' = \{H_0 \cap H : H \in \mathcal{A}'\}$ for some $H_0 \in \mathcal{A}$. The following well-known theorems state how those three arrangements in the triple are related to each other.

Theorem 2.1 (Deletion-Restriction, Brylowski [1]). *Let $(\mathcal{A}, \mathcal{A}', \mathcal{A}'')$ be a triple of arrangements. Then $\pi(\mathcal{A}, t) = \pi(\mathcal{A}', t) + t\pi(\mathcal{A}'', t)$ where π is the Poincaré polynomial of the arrangement. Equivalently, $\chi(\mathcal{A}, t) = \chi(\mathcal{A}', t) - \chi(\mathcal{A}'', t)$ where χ is the characteristic polynomial of the arrangement defined by $\chi(\mathcal{A}, t) = t^l \pi(\mathcal{A}, -t^{-1})$.*

Theorem 2.2 (Addition-Deletion, Terao [4]). *Let $(\mathcal{A}, \mathcal{A}', \mathcal{A}'')$ be a triple of arrangements. Any two of the following statements imply the third:*

- $\mathcal{A}$ is free with $\exp \mathcal{A} = \{b_1, \dots, b_{l-1}, b_l\}$
- $\mathcal{A}'$ is free with $\exp \mathcal{A}' = \{b_1, \dots, b_{l-1}, b_l - 1\}$
- $\mathcal{A}''$ is free with $\exp \mathcal{A}'' = \{b_1, \dots, b_{l-1}\}$.

Theorem 2.3 (Factorization Theorem, Terao [6]). *If $\mathcal{A}$ is a free arrangement with $\exp \mathcal{A} = \{b_1, \dots, b_l\}$, then $\pi(\mathcal{A}, t) = \prod_{i=1}^l (1 + b_i t)$. Equivalently, $\chi(\mathcal{A}, t) = \prod_{i=1}^l (t - b_i)$.*

After a linear change of coordinates, we may assume that any arrangement $\mathcal{A}$ in $\mathbb{R}^3$ with $|\mathcal{A}| = 5$ is defined by $Q(\mathcal{A}) = xyz(a_1x + a_2y + a_3z)(b_1x + b_2y + b_3z)$. Then there are only five different cases to consider.

- (1) $Q(\mathcal{A}) = xyz(x + y)(x + ay)$ where $a \in \mathbb{R} \setminus \{0, 1\}$,
- (2) $Q(\mathcal{A}) = xyz(x + y)(x + z)$,
- (3) $Q(\mathcal{A}) = xyz(x + y)(x + y + z)$,
- (4) $Q(\mathcal{A}) = xyz(x + y)(x + ay + z)$ where $a \in \mathbb{R} \setminus \{0, 1\}$,
- (5) $Q(\mathcal{A}) = xyz(x + y + z)(x + ay + bz)$ where $a, b \in \mathbb{R} \setminus \{0\}$ and both are not 1.

Case 1. $Q(\mathcal{A}) = xyz(x+y)(x+ay)$:
 $xy(x+y)(x+ay)$ defines a 2-arrangement $\mathcal{A}_1$ and z defines a 1-arrangement $\mathcal{A}_2$ such that $\mathcal{A} = \mathcal{A}_1 \times \mathcal{A}_2$. It is clear that $\mathcal{A}_1$ and $\mathcal{A}_2$ are both free with $\exp \mathcal{A}_1 = \{1, 3\}$ and $\exp \mathcal{A}_2 = \{1\}$, respectively. Therefore $\mathcal{A}$ is free with $\exp \mathcal{A} = \{1, 1, 3\}$.

Moreover, let $\theta_1 = xD_x + yD_y$, $\theta_2 = y(x+y)(x+ay)D_y$, and $\theta_3 = zD_z$ where D_x , D_y , and D_z are the usual partial derivatives. Obviously, $\theta_1, \theta_2, \theta_3 \in D(\mathcal{A})$ and

$$\det M(\theta_1, \theta_2, \theta_3) = \begin{vmatrix} x & 0 & 0 \\ y & y(x+y)(x+ay) & 0 \\ 0 & 0 & z \end{vmatrix} = Q(\mathcal{A}).$$

It follows from Saito's criterion [3] that $\theta_1, \theta_2, \theta_3$ form a basis of $D(\mathcal{A})$.

Case 2. $Q(\mathcal{A}) = xyz(x+y)(x+z)$:
 Consider $\theta_1 = xD_x + yD_y + zD_z$, $\theta_2 = y(x+y)D_y$, and $\theta_3 = z(x+z)D_z$. It is easy to check that $\theta_1, \theta_2, \theta_3 \in D(\mathcal{A})$ and

$$\det M(\theta_1, \theta_2, \theta_3) = \begin{vmatrix} x & 0 & 0 \\ y & y(x+y) & 0 \\ z & 0 & z(x+z) \end{vmatrix} = Q(\mathcal{A}).$$

Therefore, $\theta_1, \theta_2, \theta_3$ form a basis of $D(\mathcal{A})$ and $\mathcal{A}$ is free with $\exp \mathcal{A} = \{1, 2, 2\}$.

Case 3. $Q(\mathcal{A}) = xyz(x+y)(x+y+z)$:
 Consider $\theta_1 = xD_x + yD_y + zD_z$, $\theta_2 = -xyD_x + xyD_y$, and $\theta_3 = (x+y+z)(xD_x + yD_y)$. Then $\theta_1, \theta_2, \theta_3 \in D(\mathcal{A})$ and

$$\det M(\theta_1, \theta_2, \theta_3) = \begin{vmatrix} x & 0 & 0 \\ y & y(x+y) & 0 \\ z & 0 & z(x+z) \end{vmatrix} = Q(\mathcal{A}).$$

Therefore, $\theta_1, \theta_2, \theta_3$ form a basis of $D(\mathcal{A})$ and $\mathcal{A}$ is free with $\exp \mathcal{A} = \{1, 2, 2\}$.

In fact, the first three cases above are all in the class $\mathcal{IF}$ of inductively free arrangements that is the smallest class of arrangements satisfying

- (1) $\emptyset \in \mathcal{IF}$,
- (2) if there exists a triple arrangement $(\mathcal{A}, \mathcal{A}', \mathcal{A}'')$ such that $\mathcal{A}'', \mathcal{A}' \in \mathcal{IF}$ and $\exp \mathcal{A}'' \subset \exp \mathcal{A}'$, then $\mathcal{A} \in \mathcal{IF}$.

In order to show that a given arrangement $\mathcal{A}$ is inductively free, we start with an inductively free arrangement, $\emptyset$, and add hyperplanes one at a time satisfying the second condition. The following table shows this process. Each row is one step in the process. The first column gives $\exp \mathcal{A}'$ of the arrangement which is the $\mathcal{A}'$ of the step. The second column gives α_H , where $H : \alpha_H = 0$ is the hyperplane added to $\mathcal{A}'$. The third column gives $\exp \mathcal{A}''$ and the last row shows $\exp \mathcal{A}$.

case 1			case 2			case 3		
$\exp \mathcal{A}'$	α_H	$\exp \mathcal{A}''$	$\exp \mathcal{A}'$	α_H	$\exp \mathcal{A}''$	$\exp \mathcal{A}'$	α_H	$\exp \mathcal{A}''$
0 0 0	x	0 0	0 0 0	x	0 0	0 0 0	x	0 0
0 0 1	y	0 1	0 0 1	y	0 1	0 0 1	y	0 1
0 1 1	z	1 1	0 1 1	z	1 1	0 1 1	x	1 1
1 1 1	$x + y$	1 1	1 1 1	$x + y$	1 1	1 1 1	$x + y$	1 1
1 1 2	$x + ay$	1 1	1 1 2	$x + z$	1 2	1 1 2	$x + y + z$	1 2
1 1 3			1 2 2			1 2 2		

Case 4. $Q(\mathcal{A}) = xyz(x + y)(x + ay + z)$:

Suppose that $\mathcal{A}$ is free. Then $\exp(\mathcal{A}) = \{1, 2, 2\}$ since $Q(\mathcal{A})$ is irreducible. Consider a triple $(\mathcal{A}, \mathcal{A}', \mathcal{A}'')$ with the distinguished hyperplane $H_1 : x + y = 0$. Clearly, $\mathcal{A}''$ is free with $\exp \mathcal{A}'' = \{1, 2\}$. It follows from Addition-Deletion that $\mathcal{A}'$ is free with $\exp \mathcal{A}' = \{1, 1, 2\}$ and hence $\mathcal{A}'$ is a direct product of 2 nonempty irreducible arrangements, which is a contradiction. Therefore $\mathcal{A}$ is not free.

Case 5. $Q(\mathcal{A}) = xyz(x + y + z)(x + ay + bz)$:

Suppose that $\mathcal{A}$ is free. Then $\exp \mathcal{A} = \{1, 2, 2\}$ since $Q(\mathcal{A})$ is irreducible and by Factorization Theorem $\chi(\mathcal{A}, 0) = -4$. Consider a triple $(\mathcal{A}, \mathcal{A}', \mathcal{A}'')$ with $H_1 : x + ay + bz = 0$. Clearly, $\mathcal{A}''$ is free with $\exp \mathcal{A}'' = \{1, 2\}$ or $\{1, 3\}$ and hence $\chi(\mathcal{A}'', 0) = 2$ or 3 . For $\chi(\mathcal{A}', 0)$, consider another triple $(\mathcal{A}', (\mathcal{A}')', (\mathcal{A}')'')$ with $H_2 : x + y + z = 0$. It is easy to see that $(\mathcal{A}')'$ and $(\mathcal{A}')''$ are both free with $\exp(\mathcal{A}')' = \{1, 1, 1\}$ and $\exp(\mathcal{A}')'' = \{1, 2\}$, respectively. It follows from Deletion-Restriction that $\chi(\mathcal{A}', 0) = -3$. This implies that $\chi(\mathcal{A}, 0) = -5$ or -6 , which is a contradiction. Therefore $\mathcal{A}$ is not free.

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