**CHARACTERIZATIONS OF REAL HYPERSURFACES
OF TYPE A IN A NONFLAT COMPLEX SPACE FORM
WHOSE STRUCTURE JACOBI OPERATOR IS
 ξ -PARALLEL**

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Abstract. Let M be a real hypersurface with almost contact metric structure (ϕ, ξ, η, g) of a nonflat complex space form whose structure Jacobi operator $R_\xi = R(\cdot, \xi)\xi$ is ξ -parallel. In this paper, we prove that the condition $\nabla_\xi R_\xi = 0$ characterize the homogeneous real hypersurfaces of type A in a complex projective space $P_n\mathbb{C}$ or a complex hyperbolic space $H_n\mathbb{C}$ when $g(\nabla_\xi \xi, \nabla_\xi \xi)$ is constant.

1. Introduction

A complex n -dimensional Kähler manifold of constant holomorphic sectional curvature c is called a complex space form, which is denoted by $M_n(c)$. As is well-known, a complete and simply connected complex space form is complex analytically isometric to a complex projective space $P_n\mathbb{C}$, a complex Euclidean space $\mathbb{C}_n$ or a complex hyperbolic space $H_n\mathbb{C}$ according as $c > 0$, $c = 0$ and $c < 0$.

In this paper we consider a real hypersurface M in a complex space form $M_n(c)$, $c \neq 0$. Then M has an almost contact metric structure (ϕ, ξ, η, g) induced from the complex structure J and the Kähler metric of $M_n(c)$. A real hypersurface is said to be a *Hopf hypersurface* if the structure vector field ξ of M is principal.

In the study of real hypersurfaces in $P_n\mathbb{C}$, Takagi [15] classified all homogeneous real hypersurfaces and Cecil and Ryan [3] showed that they can be regarded as the tubes of constant radius over Kähler submanifolds when the structure vector field ξ is principal. Such tubes can be divided into six kinds of type A_1, A_2, B, C, D and E .

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In the case of real hypersurfaces in $H_n\mathbb{C}$, the classification of homogeneous real hypersurfaces in $H_n\mathbb{C}$ is obtained by Berndt [1], Berndt and Tamaru [2]. Also such kinds of tubes are said to be real hypersurfaces of *type* A_0, A_1, A_2 or B . Among the several types of real hypersurfaces appeared in Takagi's list or Berndt's one, a particular type of tubes over totally geodesic $P_k\mathbb{C}$ or $H_k\mathbb{C}$ ($0 \leq k \leq n-1$) adding a horosphere in $H_n\mathbb{C}$, which is called *type* A , has a lot of nice geometric properties. For example, Okumura [12](resp. Montiel and Romero [11]) shows that a real hypersurface of type A if and only if the structure operator ϕ commutes with the shape operator $A\phi = \phi A$.

On the other hand, the Jacobi operator field with respect to a tangent vector field X in a Riemannian manifold M is defined by $R_X = R(\cdot, X)X$, where R denotes the Riemannian curvature tensor of M .

When we study a real hypersurface M in $M_n(c)$, we will call the Jacobi operator on M with respect to the structure vector field ξ the *structure Jacobi operator* on M and will denote it by R_ξ . But, recently it is known that there are no real hypersurfaces in $M_n(c)$ with parallel structure Jacobi operator R_ξ , that is $\nabla_X R_\xi = 0$ for any tangent vector field X on M in $M_n(c)$ [13],[14].

Motivated by results mentioned above we consider the parallelism of the structure Jacobi operator R_ξ in the direction of the structure vector field ξ , that is, $\nabla_\xi R_\xi = 0$. In the previous paper [10] we prove the following:

Theorem K. *Let M be a real hypersurface in a nonflat complex space form $M_n(c)$, $c \neq 0$ which satisfies $\mu^2 = g(\nabla_\xi \xi, \nabla_\xi \xi)$ is constant and $\mu^2 + c/24 \neq 0$. Then M holds $\nabla_\xi R_\xi = 0$ if and only if M is locally congruent to one of the following:*

- (1) In case that $M_n(c) = P_n\mathbb{C}$ with $\eta(A\xi) \neq 0$,
 - (A₁) a geodesic hypersphere of radius r , where $0 < r < \pi/2$ and $r \neq \pi/4$,
 - (A₂) a tube of radius r over a totally geodesic $P_k\mathbb{C}$ ($1 \leq k \leq n-2$), where $0 < r < \pi/2$ and $r \neq \pi/4$;
- (2) In case that $M_n(c) = H_n\mathbb{C}$,
 - (A₀) a horosphere,
 - (A₁) a geodesic hypersphere or a tube over a complex hyperbolic hyperplane $H_{n-1}\mathbb{C}$,
 - (A₂) a tube over a totally geodesic $H_k\mathbb{C}$ ($1 \leq k \leq n-2$).

The main purpose of the present paper is to improve Theorem K without the condition $\mu^2 + c/24 \neq 0$. As an application of this result, we give another simple proof of the following theorem:

Theorem L ([8]). *Let M be a real hypersurface in a complex space form $M_n(c)$, $c \neq 0$ satisfying $\nabla_\xi S = 0$ and $\nabla_\xi R_\xi = 0$. Then M is a Hopf hypersurface in $M_n(c)$, where S denotes the Ricci tensor of M .*

2. Preliminaries

Let M be a real hypersurface immersed in a complex space form $(M_n(c), G)$ with almost complex structure J , and the Kähler metric G of constant holomorphic sectional curvature c , and let N be a unit normal vector field on M . The Riemannian connection $\tilde{\nabla}$ in $M_n(c)$ and that ∇ in M are related by the following formulas for any vector fields X and Y on M :

$$\tilde{\nabla}_Y X = \nabla_Y X + g(AY, X)N, \quad \tilde{\nabla}_X N = -AX,$$

where g denotes the Riemannian metric on M induced from that G of $M_n(c)$ and A is the shape operator in the direction of N in $M_n(c)$. For any vector field X tangent to M , we put

$$JX = \phi X + \eta(X)N, \quad JN = -\xi.$$

Then we may see that the structure (ϕ, ξ, η, g) is an almost contact metric structure on M , namely, we have

$$\begin{aligned} \phi^2 X &= -X + \eta(X)\xi, & g(\phi X, \phi Y) &= g(X, Y) - \eta(X)\eta(Y), \\ \eta(\xi) &= 1, & \phi\xi &= 0, & \eta(X) &= g(X, \xi) \end{aligned}$$

for any vector fields X and Y on M .

From the fact $\tilde{\nabla}J = 0$, and above equations, we verify that

$$(2.1) \quad (\nabla_X \phi)Y = \eta(Y)AX - g(AX, Y)\xi, \quad \nabla_X \xi = \phi AX.$$

Since we consider that the ambient manifold is of constant holomorphic sectional curvature c , we have the following Gauss and Codazzi equations respectively.

$$(2.2) \quad \begin{aligned} R(X, Y)Z &= \frac{c}{4} \{g(Y, Z)X - g(X, Z)Y + g(\phi Y, Z)\phi X \\ &\quad - g(\phi X, Z)\phi Y + 2g(\phi X, Y)\phi Z\} \\ &\quad + g(AY, Z)AX - g(AX, Z)AY, \end{aligned}$$

$$(2.3) \quad (\nabla_X A)Y - (\nabla_Y A)X = \frac{c}{4} \{\eta(X)\phi Y - \eta(Y)\phi X - 2g(\phi X, Y)\xi\}$$

for any tangent vector fields X, Y and Z on M , where R denote the Riemann-Christoffel curvature tensor of M . We shall denote the Ricci tensor of type (1,1) by S . Then it follows from (2.2) that

$$(2.4) \quad SX = \frac{c}{4} \{ (2n+1)X - 3\eta(X)\xi \} + hAX - A^2X,$$

where $h = \text{trace } A$. Further, using (2.1), we obtain

$$(2.5) \quad (\nabla_X S)Y = -\frac{3}{4}c \{ g(\phi AX, Y)\xi + \eta(Y)\phi AX \} + (Xh)AY \\ + (hI - A)(\nabla_X A)Y - (\nabla_X A)AY,$$

where I is the identity map.

In what follows, to write our formulas in convention forms, we denote $\alpha = \eta(A\xi)$, $\beta = \eta(A^2\xi)$, and for a function f we denote by ∇f the gradient vector field of f .

We put $U = \nabla_\xi \xi$, then U is orthogonal to the structure vector field ξ . Thus, it is, using (2.1), seen that

$$(2.6) \quad \phi U = -A\xi + \alpha\xi,$$

which enables us to obtain $g(U, U) = \beta - \alpha^2$. Thus we easily see that ξ is a principal curvature vector, that is $A\xi = \alpha\xi$ if and only $\beta - \alpha^2 = 0$.

Now, differentiating (2.6) covariantly along M and using (2.1), we find

$$(2.7) \quad \eta(X)g(AU + \nabla\alpha, Y) + g(\phi X, \nabla_Y U) \\ = g((\nabla_Y A)X, \xi) - g(A\phi AX, Y) + \alpha g(A\phi X, Y),$$

which shows that

$$(2.8) \quad (\nabla_\xi A)\xi = 2AU + \nabla\alpha$$

because of (2.3). From (2.7) we also have

$$(2.9) \quad \nabla_\xi U = 3\phi AU + \alpha A\xi - \beta\xi + \phi\nabla\alpha,$$

where we have used (2.1).

If $\beta - \alpha^2 \neq 0$, then we can put

$$(2.10) \quad A\xi = \alpha\xi + \mu W,$$

where W is a unit vector field orthogonal to ξ . Then by (2.1) we see that $U = \mu\phi W$ and hence $g(U, U) = \mu^2$, and W is also orthogonal to U . Thus, we verify, making use of (2.6) and (2.10), that

$$(2.11) \quad \mu g(\nabla_X W, \xi) = g(AU, X),$$

$$(2.12) \quad g(\nabla_X \xi, U) = \mu g(AW, X).$$

The Gauss equation (2.2) gives the structure Jacobi operator R_ξ :

$$(2.13) \quad R_\xi X = R(X, \xi) \xi = \frac{c}{4} \{X - \eta(X) \xi\} + \alpha AX - \eta(AX) A\xi$$

for any vector field X on M .

3. Real hypersurfaces satisfying $\nabla_\xi R_\xi = 0$

In this section the elemental factors of real hypersurfaces satisfying $\nabla_\xi R_\xi = 0$ in $M_n(c)$ are recalled ([10]). We set $\Omega = \{p \in M | \mu(p) \neq 0\}$ and suppose that $\Omega \neq \emptyset$, that is ξ is not principal curvature vector on M . Hereafter, unless otherwise stated, we discuss our discussions on the open set Ω of M .

Differentiating (2.13) covariantly, and using (2.8), we find

$$(3.1) \quad g((\nabla_\xi R_\xi)Y, Z) = -\frac{c}{4} \{u(Y)\eta(Z) + u(Z)\eta(Y)\} + (\xi\alpha)g(AY, Z) \\ + \alpha g((\nabla_\xi A)Y, Z) - \eta(AZ) \{3g(AU, Y) + Y\alpha\} \\ - \eta(AY) \{3g(AU, Z) + Z\alpha\},$$

where u is a 1-form dual to U with respect to g , that is $u(X) = g(U, X)$.

We assume that $\nabla_\xi R_\xi = 0$. Then from (3.1) we have

$$(3.2) \quad \alpha(\nabla_\xi A)X + (\xi\alpha)AX = \frac{c}{4} (u(X)\xi + \eta(X)U) + \eta(AX)(3AU + \nabla\alpha) \\ + \{3g(AU, X) + X\alpha\}A\xi.$$

Putting $X = \xi$ in this and using (2.8), we find

$$(3.3) \quad \alpha AU + \frac{c}{4}U = 0,$$

which shows that $\alpha \neq 0$ on Ω .

Differentiating (3.3) covariantly and using itself, we obtain

$$(3.4) \quad -\frac{c}{4}(X\alpha)U + \alpha^2(\nabla_X A)U + \alpha^2 A\nabla_X U + \frac{c}{4}\alpha\nabla_X U = 0,$$

or, using (2.3) and (2.6)

$$(3.5) \quad \frac{c}{4} \{(Y\alpha)u(X) - (X\alpha)u(Y)\} + \frac{c}{4}\alpha^2\mu\{\eta(X)w(Y) - \eta(Y)w(X)\} \\ + \alpha^2\{g(A\nabla_X U, Y) - g(A\nabla_Y U, X)\} + \frac{c}{4}\alpha du(X, Y) = 0,$$

where w is a dual 1-form of W with respect to g , that is $w(X) = g(W, X)$. Here, du is the exterior derivative of a 1-form u given by

$$du(X, Y) = Xu(Y) - Yu(X) - u([X, Y]).$$

If we replace X by U in (3.5), then we have

$$(3.6) \quad \frac{c}{4} \{\mu^2\nabla\alpha - (U\alpha)U\} + \alpha^2 A\nabla_U U + \frac{c}{4}\alpha\nabla_U U = 0,$$

because U and W are mutually orthogonal.

Combining (2.7) to (3.2) and using (2.3), we obtain

$$(3.7) \quad \begin{aligned} \alpha^2 \phi \nabla_X U = & \alpha^2 (X\alpha) \xi - \frac{c}{4} \alpha u(X) \xi + \alpha (\xi \alpha) AX + \frac{c}{4} \alpha^2 \phi X \\ & - \eta(AX) \left(\alpha \nabla \alpha - \frac{3}{4} c U \right) - \left\{ \alpha (X\alpha) - \frac{3}{4} c u(X) \right\} A\xi \\ & - \frac{c}{4} \alpha \{u(X) \xi + \eta(X) U\} - \alpha^2 A\phi AX + \alpha^3 \phi AX. \end{aligned}$$

Applying ϕ to this and using (2.12), we have

$$(3.8) \quad \begin{aligned} & \alpha^2 \nabla_X U + \alpha^2 \mu g(AW, X) \xi - \alpha \eta(AX) \phi \nabla \alpha \\ = & -\alpha (\xi \alpha) \phi AX + \frac{c}{4} \alpha^2 \{X - \eta(X) \xi\} + \frac{3}{4} c \mu \eta(AX) W + \alpha (X\alpha) U \\ & - \frac{3}{4} c u(X) U + \alpha^3 AX - \frac{c}{4} \alpha \mu \eta(X) W - \alpha^3 \eta(AX) \xi + \alpha^2 \phi A\phi AX. \end{aligned}$$

On the other hand, differentiating (2.10) covariantly and using (2.1), we find

$$(\nabla_X A) \xi + A\phi AX = (X\alpha) \xi + \alpha \phi AX + (X\mu) W + \mu \nabla_X W,$$

or, using (2.3) and (3.2),

$$(3.9) \quad \begin{aligned} \alpha \mu \nabla_X W = & \frac{c}{4} (u(X) \xi + \eta(X) U) + \eta(AX) (3AU + \nabla \alpha) \\ & + 3g(AU, X) A\xi + \mu (X\alpha) W - \frac{c}{4} \alpha \phi X \\ & + \alpha A\phi AX - \alpha^2 \phi AX - (\xi \alpha) AX - \alpha (X\mu) W. \end{aligned}$$

Putting $X = \alpha U$ in (3.2) and taking account of (3.3), we find

$$\alpha^2 (\nabla_\xi A) U - \frac{c}{4} (\xi \alpha) U = \frac{c}{4} \alpha \mu^2 \xi + \left\{ \alpha (U\alpha) - \frac{3}{4} c \mu^2 \right\} A\xi.$$

Because of (3.3), the equation (2.9) turns out to be

$$\alpha \nabla_\xi U = \frac{3}{4} c \mu W + \alpha^2 A\xi - \alpha \beta \xi + \alpha \phi \nabla \alpha.$$

If we replace $X = \alpha \xi$ in (3.4) and make use of (3.3) and the last two equations, we obtain

$$(3.10) \quad \begin{aligned} & \alpha A\phi \nabla \alpha + \frac{c}{4} \phi \nabla \alpha + (U\alpha) A\xi \\ & + \mu \left(\alpha^2 + \frac{3}{4} c \right) \left\{ AW - \mu \xi - \frac{1}{\alpha} \left(\mu^2 - \frac{c}{4} \right) W \right\} = 0. \end{aligned}$$

4. Real hypersurfaces satisfying $g(\nabla_\xi \xi, \nabla_\xi \xi)$ is constant

In this section we assume that $\nabla_\xi R_\xi = 0$ and at the same time $g(\nabla_\xi \xi, \nabla_\xi \xi)$ is constant, that is, $g(U, U) = \mu^2$ is constant. Then we obtain $\nabla \mu = 0$ on Ω , which implies that $g(\nabla_X U, U) = 0$. Thus, taking an inner product U to (3.8), we get

$$(W\alpha) A\xi = (\xi \alpha) AW + \frac{3}{4\alpha} c \mu U - \mu \nabla \alpha + \alpha A\phi AW,$$

which shows that $W\alpha = 0$ and $(\xi\alpha)g(AW, W) = 0$ because of (3.3). Thus, it follows that

$$(4.1) \quad \mu \left(\nabla\alpha - \frac{3}{4\alpha}cU \right) = (\xi\alpha)AW + \alpha A\phi AW,$$

which implies that

$$(4.2) \quad \alpha(U\alpha) = \frac{c}{4}(3\mu^2 - \alpha\lambda),$$

where we have put $\lambda = g(AW, W)$.

Putting $X = U$ in (3.8) and using (3.3) and (4.2), we find

$$\alpha\nabla_U U = -\frac{c}{4\alpha}\mu(\xi\alpha)W - \frac{c}{4}\lambda U + \frac{c}{4}\mu\phi AW.$$

If we take account of (3.3) and this, then (3.6) turns out to be

$$\mu^2\nabla\alpha - (U\alpha)U = \frac{\mu}{\alpha}(\xi\alpha)\left(\alpha AW + \frac{c}{4}W\right) - \mu\left(\alpha A\phi AW + \frac{c}{4}\phi AW\right),$$

which together with $\lambda(\xi\alpha) = 0$ gives $\xi\alpha = 0$. So we see, using (4.2), that

$$(4.3) \quad \mu\nabla\alpha = \left(\frac{3}{4\alpha}c\mu - \frac{c\lambda}{4\mu}\right)U - \alpha A\phi AW - \frac{c}{4}\phi AW.$$

Combining this to (4.1), we obtain

$$\alpha A\phi AW = -\frac{c}{8}\left(\frac{\lambda}{\mu}U + \phi AW\right)$$

because of $\xi\alpha = 0$. Therefore, (4.3) is reduced to

$$(4.4) \quad \mu\left(\alpha\nabla\alpha - \frac{3}{4}cU\right) = -\frac{c}{8}\left(\alpha\phi AW + \frac{\alpha\lambda}{\mu}U\right).$$

From this, we can verify that

Remark 1. $\alpha = \text{constant}$ on Ω (see, Lemma 3 of [10]).

Using $\alpha = \text{constant}$ on Ω , we see from (4.2) that

$$(4.5) \quad \alpha\lambda = 3\mu^2.$$

Hence, (4.4) shows that $\lambda U = \mu\phi AW$. So we have

$$(4.6) \quad AW = \mu\xi + \lambda W.$$

Differentiating this covariantly, we find

$$(\nabla_X A)W + A\nabla_X W = \mu\phi AX + \lambda\nabla_X W$$

because λ is constant by virtue of Remark 1 and (4.5). Thus we have a Codazzi-type formula for w :

$$(4.7) \quad \begin{aligned} & \lambda (\nabla_X w)(Y) - (\nabla_Y w)(X) \\ &= \frac{c}{4\mu} (u(X)\eta(Y) - u(Y)\eta(X)) + g(A\nabla_X W, Y) \\ & \quad - g(A\nabla_Y W, X) - \mu g((\phi A + A\phi)X, Y), \end{aligned}$$

where we have used (2.3).

On the other hand, replacing X by U in (3.9) and using (3.3), (4.6) and Remark 1, we find

$$(4.8) \quad \alpha \nabla_U W = -\frac{c}{4}\mu\xi.$$

In the same way, we also have

$$(4.9) \quad \alpha g(\nabla_X W, U) = -\mu \left(\alpha^2 + \frac{3}{4}c \right) \eta(X) - \left(\frac{c}{4}\alpha + \frac{c}{2}\lambda + \alpha^2\lambda \right) w(X).$$

If we replace X by U in (4.7) and make use of (3.3) and (4.8), then we obtain

$$\left(\lambda + \frac{c}{4\alpha} \right) g(\nabla_X W, U) = -\mu^2 g(AW, X) + \frac{c}{2\alpha} \mu^2 w(X) - \frac{c}{4\alpha} \lambda \mu \eta(X),$$

which together with (4.9) yields

$$\begin{aligned} & \left(\lambda + \frac{c}{4\alpha} \right) \left\{ \mu \left(\alpha^2 + \frac{3}{4}c \right) \xi + \left(\frac{c}{2}\alpha + \frac{c}{2}\lambda + \alpha^2\lambda \right) W \right\} \\ &= \mu^2 \alpha AW - \frac{c}{2} \mu^2 W + \frac{c}{4} \lambda \mu \xi. \end{aligned}$$

Using (4.6), we see from this

$$(4.10) \quad \left(\lambda + \frac{c}{4\alpha} \right) \left(\alpha^2 + \frac{3}{4}c \right) - \alpha \mu^2 - \frac{c}{4}\lambda = 0$$

and

$$(4.11) \quad \left(\lambda + \frac{c}{4\alpha} \right) \left(\frac{c}{4}\alpha + \frac{c}{2}\lambda + \alpha^2\lambda \right) + \mu^2 \left(\frac{c}{2} - \alpha\lambda \right) = 0.$$

Combining (4.10) to (4.5), we verify that $(\alpha^2 + \frac{3}{4}c)(2\mu^2 + \frac{c}{4}) = 0$, which together with (4.11) implies that

$$(4.12) \quad \alpha^2 + \frac{3}{4}c = 0.$$

From this and (4.5) we have

$$(4.13) \quad \mu^2 \alpha = -\frac{c}{4}\lambda,$$

which connected to (4.11) gives $6\lambda = \alpha$. So we see, using (4.13), that (for detail, see [?])

$$(4.14) \quad 6\mu^2 + \frac{c}{4} = 0.$$

Using (3.3) and (4.12), the equation (3.2) can be written as

$$(4.15) \quad (\nabla_\xi A)X = \frac{c}{4\alpha} \{u(X)\xi + \eta(X)U\} + \eta(AX)U + u(X)A\xi,$$

which shows that

$$(4.16) \quad (\nabla_\xi A)W = \mu U.$$

Because of (3.3) and (4.12), the equation (3.9) turns out to be

$$(4.17) \quad \begin{aligned} \mu \nabla_X W &= \mu(u(X)W + w(X)U) - \frac{c}{2\alpha}(u(X)\xi + \eta(X)U) - \frac{c}{4}\phi X \\ &\quad + A\phi AX - \alpha\phi AX, \end{aligned}$$

which implies that

$$\begin{aligned} \mu \{(\nabla_X w)Y - (\nabla_Y w)X\} &= -\frac{c}{2}g(\phi X, Y) + 2g(A\phi AX, Y) \\ &\quad - \alpha g((A\phi - \phi A)X, Y). \end{aligned}$$

Putting $X = U$ in this and taking account of (3.3) and (4.12), we find

$$(4.18) \quad (\nabla_U w)Y - (\nabla_Y w)U = \frac{c}{4} \left(w(Y) - \frac{1}{\alpha} g(AW, Y) \right).$$

If we take account of (4.6) and (4.12), then (3.8) becomes

$$(4.19) \quad \begin{aligned} &\nabla_X U + \mu g(A\xi, X)\xi + g(A\xi, X)A\xi \\ &= \alpha AX + \frac{c}{4}(X - \eta(X)\xi) + u(X)U - \frac{c}{4\alpha}\mu\eta(X)W + \phi A\phi AX. \end{aligned}$$

Replacing X by U in this and making use of (3.3), (4.6) and (4.12), we obtain

$$(4.20) \quad \nabla_U U = 0.$$

Thus, (3.4) implies that

$$(4.21) \quad (\nabla_U A)U = 0$$

by virtue of (4.12).

From (2.3), (3.4) and (4.12) we have

$$\frac{c}{4}\mu(\eta(X)w(Y) - \eta(Y)w(X)) + \alpha\{g(A\nabla_X U, Y) - g(A\nabla_Y U, X)\} + \frac{c}{4}du(X, Y) = 0,$$

or, using (4.19)

$$(4.22) \quad du(X, Y) = \mu\lambda(\eta(Y)w(X) - \eta(X)w(Y)).$$

Using above equations, we are now going to prove that $\Omega = \emptyset$.

In the following, let $\{e_1, \dots, e_{2n-1}\}$ be an orthonormal frame field of M . And the coefficients of tensors are defined by $\xi^i = g(\xi, e_i)$, $A_i^j = g(Ae_i, e_j)$, $\phi_i^j = g(\phi e_i, e_j)$, $\nabla_i U^j = g(\nabla_{e_i} U, e_j)$ and so on. Furthermore, we use the Einstein rule, that is, we take the summation for the same indices in the equation. So (4.17) can be written as

(4.23)

$$\begin{aligned} \mu \nabla_j W_i &= \left(\mu W_i - \frac{c}{2\alpha} \xi_i \right) U_j + \left(\mu W_j - \frac{c}{2\alpha} \xi_j \right) U_i + \alpha A_{jr} \phi_i^r - \frac{c}{4} \phi_{ji} \\ &\quad - A_i^r A_{js} \phi_r^s. \end{aligned}$$

Differentiating this covariantly and using (2.1), we find

$$\begin{aligned} \mu \nabla_k \nabla_j W_i &= \left(\mu \nabla_k W_i + \frac{c}{2\alpha} A_{kr} \phi_i^r \right) U_j + \left(\mu W_i - \frac{c}{2\alpha} \xi_i \right) \nabla_k U_j \\ &\quad + \left(\mu \nabla_k W_j + \frac{c}{2\alpha} A_{kr} \phi_j^r \right) U_i + \left(\mu W_j - \frac{c}{2\alpha} \xi_j \right) \nabla_k U_i \\ &\quad + \alpha (\nabla_k A_{jr}) \phi_i^r + \alpha \xi_i A_{jk}^2 - \alpha (A_{jr} \xi^r) A_{ki} + \frac{c}{4} (\xi_i A_{kj} - \xi_j A_{ki}) \\ &\quad - (\nabla_k A_i^r) A_{js} \phi_i^s - A_i^r (\nabla_k A_{js}) \phi_r^s + (A_{jr} \xi^r) A_{ki}^2 - (A_{ir} \xi^r) A_{kj}^2, \end{aligned}$$

where we have used (4.2) and $\mu = \text{const}$. If we take the symmetric part with respect to indices k and j in this, and make use of (2.3), (4.22) and the Ricci-identity, we obtain

(4.24)

$$\begin{aligned} -\mu R_{kjih} W^h &= \left(\mu \nabla_k W_i + \frac{c}{2\alpha} A_{kr} \phi_i^r \right) U_j - \left(\mu \nabla_j W_i + \frac{c}{2\alpha} A_{jr} \phi_i^r \right) U_k \\ &\quad + \left\{ \mu (\nabla_k W_j - \nabla_j W_k) + \frac{c}{2\alpha} (A_{kr} \phi_j^r - A_{jr} \phi_k^r) \right\} U_i \\ &\quad + \mu \lambda \left(\mu W_i - \frac{c}{2\alpha} \xi_i \right) (\xi_k W_j - \xi_j W_k) + \frac{c}{4} \alpha (\xi_k g_{ji} - \xi_j g_{ki}) \\ &\quad + \left(\mu W_i - \frac{c}{2\alpha} \xi_i \right) \nabla_k U_i - \left(\mu W_k - \frac{c}{2\alpha} \xi_k \right) \nabla_j U_i \\ &\quad + A_{kr} \xi^r (\alpha A_{ji} - A_{ji}^2) - A_{jr} \xi^r (\alpha A_{ki} - A_{ki}^2) \\ &\quad + (\nabla_j A_i^r) A_{ks} \phi_r^s - (\nabla_k A_i^r) A_{js} \phi_r^s. \end{aligned}$$

On the other hand from (2.2) we have

$$R_{kjih} U^k W^h = -\frac{c}{4} (3W_j U_i + W_i U_j + \mu \phi_{ji}) + \frac{c}{4\alpha} U_i A_{jr} W^r,$$

where we have used (2.6), (2.10) and (3.3).

Multiplying U^k to (4.24) and summing for k , and using (3.3), (4.8), (4.12), (4.18) and the last equation, we have

$$\begin{aligned}
 (4.25) \quad & \frac{c}{12} \mu (U_j W_i + U_i W_j) + \frac{c}{4} \mu^2 \phi_{ji} + \mu^2 \left(\mu \nabla_j W_i + \frac{c}{2\alpha} A_{jr} \phi_i^r \right) \\
 &= -\frac{c}{4\alpha} \mu^2 \xi_i U_j + \left(\frac{c}{2\alpha} \mu A_{jr} W^r - \frac{c}{12} A_{jr} \xi^r \right) U_i - \frac{c}{4\alpha} \mu (\nabla_j A_{ir}) W^r \\
 &\quad - (\nabla_U A_i^r) A_{js} \phi_r^s.
 \end{aligned}$$

Since we see, using (4.23), that $\nabla_\xi W = 0$, if we multiply ξ^j to this and summing for j , we obtain $\mu^2 + \frac{c}{12} = 0$, where we have used (4.12), (4.16) and (4.21). This contradicts (4.14). Hence we verify that $\Omega = \emptyset$. Thus, we have

Lemma 1. *Let M be a real hypersurface satisfies $\nabla_\xi R_\xi = 0$ and $g(\nabla_\xi \xi, \nabla_\xi \xi) = \text{const.}$ in $M_n(c)$, $c \neq 0$. Then M is a Hopf hypersurface in $M_n(c)$.*

According to Lemma 1, we have $A\xi = \alpha\xi$ on M . So we see in addition that α is constant on M ([9]). Thus, we, using (3.2), verify that $\alpha \nabla_\xi A = 0$, which together with (2.1) and (2.3) implies that $\alpha(A\phi - \phi A) = 0$.

Here, we note the case $\alpha = 0$ corresponds to the case of tube of radius $\frac{\pi}{4}$ in $P_n\mathbb{C}$ (see [3]). But, in the case of $H_n\mathbb{C}$ it is known that α never vanishes for Hopf hypersurfaces (cf. [1]). Owing to Okumura [12] or Montiel and Romero [11] stated in Introduction, we have

Theorem 2. *Let M be a real hypersurface in a complex space form $M_n(c)$, $c \neq 0$ which satisfies $g(\nabla_\xi \xi, \nabla_\xi \xi)$ is constant. Then M holds $\nabla_\xi R_\xi = 0$ if and only if M is locally congruent to one of the following:*

- (I) *In case that $M_n(c) = P_n\mathbb{C}$ with $\eta(A\xi) \neq 0$,*
 - (A₁) *a geodesic hypersphere of radius r , where $0 < r < \pi/2$ and $r \neq \pi/4$,*
 - (A₂) *a tube of radius r over a totally geodesic $P_k\mathbb{C}$ ($1 \leq k \leq n-2$), where $0 < r < \pi/2$ and $r \neq \pi/4$.*
- (II) *In case that $M_n(c) = H_n\mathbb{C}$,*
 - (A₀) *a horosphere;*
 - (A₁) *a geodesic hypersphere or a tube over a complex hyperbolic hyperplane $H_{n-1}\mathbb{C}$;*
 - (A₂) *a tube over a totally geodesic $H_k\mathbb{C}$ ($1 \leq k \leq n-2$).*

5. Proof of Theorem L

Let M be a real hypersurface in $M_n(c)$, $c \neq 0$ satisfying $\nabla_\xi R_\xi = 0$. Then (3.1) $\sim$ (3.10) are valid on Ω .

Further, we assume $\nabla_\xi S = 0$ on M . Then replacing X by ξ in (2.5) and using the Codazzi equation (2.3), we obtain

$$(5.1) \quad \begin{aligned} & \frac{3}{4}c \{u(X) \eta(Y) + u(Y) \eta(X)\} + \frac{c}{4} \{g(AY, \phi X) + g(AX, \phi Y)\} \\ &= (\xi h) g(AX, Y) + hg((\nabla_Y A)X, \xi) - \frac{c}{4}hg(\phi X, Y) \\ & \quad - g(AY, (\nabla_X A)\xi) - g(AX, (\nabla_Y A)\xi). \end{aligned}$$

Combining (5.1) to (3.2) and making use of (3.3), we find

$$(5.2) \quad \begin{aligned} & \frac{c}{4} \left(3\alpha^2 - h\alpha - \frac{c}{4} \right) (u(X)\xi + \eta(X)U) - 2\alpha(\xi\alpha)A^2X + \alpha \{h(\xi\alpha) - \alpha(\xi h)\}AX \\ &= g(A\xi, X) \left\{ h\alpha\nabla\alpha - \alpha A\nabla\alpha + \frac{3}{4}c(AU - hU) - \frac{c}{4}\alpha U \right\} \\ & \quad + \left\{ h\alpha(X\alpha) - \alpha g(A\nabla\alpha, X) + \frac{3}{4}c(g(AU, X) - hu(X)) - \frac{c}{4}\alpha u(X) \right\} A\xi \\ & \quad + \left\{ \frac{3}{4}cu(X) - \alpha(X\alpha) \right\} A^2\xi + \left\{ \frac{3}{4}cU - \alpha\nabla\alpha \right\} g(A^2\xi, X). \end{aligned}$$

Since U is orthogonal to ξ , we see, using (2.10) and (3.3), that $g(A^2\xi, U) = 0$, by replacing X by U in (5.2) and taking account of (3.3), we get

$$(5.3) \quad \begin{aligned} & \left\{ \frac{3}{4}c\mu^2 - \alpha(U\alpha) \right\} A^2\xi + \left\{ \left(h\alpha + \frac{c}{4} \right) U\alpha - \frac{3}{4}c \left(\frac{c}{4\alpha} + h \right) \mu^2 - \frac{c}{4}\alpha\mu^2 \right\} A\xi \\ &= \frac{c}{4} \left(3\alpha^2 - h\alpha - \frac{c}{4} \right) \mu^2\xi - \frac{c}{4} \left\{ \left(\frac{c}{2\alpha} + h \right) \xi\alpha - \alpha(\xi h) \right\} U, \end{aligned}$$

which shows that

$$(5.4) \quad \left(h\alpha + \frac{c}{2} \right) \xi\alpha - \alpha^2(\xi h) = 0.$$

We notice here that $\alpha(U\alpha) - \frac{3}{4}c\mu^2 \neq 0$ on Ω . In fact, if not, then we have $\alpha A\xi = (h\alpha + \frac{c}{4} - 3\alpha^2)\xi$ because of (5.3) and (5.4). So we have $A\xi = \alpha\xi$, a contradiction. Thus, $\alpha(U\alpha) - \frac{3}{4}c\mu^2 \neq 0$ is satisfied everywhere. Therefore, we have from (5.3) and (5.4) the following:

$$(5.5) \quad A^2\xi = \rho A\xi + (\beta - \rho\alpha)\xi,$$

where the function ρ is defined by $\mu\rho = g(A^2\xi, W)$.

Combining (2.10) to (5.5), we verify that

$$(5.6) \quad AW = \mu\xi + \lambda W$$

because of $\mu \neq 0$, where we have put $\lambda = \rho - \alpha$.

If we take an inner product (3.10) with W and make use of (5.6), then we obtain

$$(5.7) \quad \left(\beta - \rho\alpha - \frac{c}{4}\right) \left\{ \alpha(U\alpha) - \mu^2 \left(\alpha^2 + \frac{3}{4}c\right) \right\} = 0.$$

Remark 2. It is proved in [5] that $\Omega = \emptyset$ if a real hypersurface M satisfies $\nabla_\xi R\xi = 0$ and $AR\xi - R_\xi A = 0$ in $M_n(c)$, $c \neq 0$.

From (5.7) we have

$$(5.8) \quad \alpha(U\alpha) = \mu^2 \left(\alpha^2 + \frac{3}{4}c\right).$$

Indeed, if not, we have $\beta - \rho\alpha = \frac{c}{4}$. Thus, (5.5) becomes $A^2\xi = \rho A\xi + \frac{c}{4}\xi$. From this and (2.13) we see that $AR_\xi - R_\xi A = 0$ on this subset. By Remark 2, it is contradictory. Hence, (5.8) is proved.

Using (5.3) and (5.8), we obtain

$$A^2\xi = hA\xi + \frac{c}{4\alpha^2} \left(h\alpha + \frac{c}{4} - 3\alpha^2\right) \xi,$$

which enables us to obtain

$$(5.9) \quad \alpha^2 \left(\beta - h\alpha + \frac{3}{4}c\right) = \frac{c}{4} \left(h\alpha + \frac{c}{4}\right).$$

Further, from (5.5) we see that $\rho = h$ and hence

$$(5.10) \quad \lambda = h - \alpha.$$

Putting $X = \xi$ in (2.4) and using (5.5), we find

$$S\xi = \left\{ \frac{c}{2}(n-1) + h\alpha - \beta \right\} \xi.$$

Differentiating this covariantly, we get

$$(\nabla_X S)\xi + S\nabla_X \xi = X(h\alpha - \beta)\xi + \left\{ \frac{c}{2}(n-1) + h\alpha - \beta \right\} \nabla_X \xi.$$

If we replace X by ξ in this, and make use of (2.4) and $\nabla_\xi S = 0$, then we have

$$A^2U - hAU - \left(\beta - h\alpha + \frac{3}{4}c\right)U + \xi(h\alpha - \beta)\xi = 0,$$

which implies that

$$(5.11) \quad \xi\beta = \xi(h\alpha).$$

Putting $X = U$ in (3.8) and using (3.3), (5.6) and (5.8), we have

$$\alpha^2 \nabla_U U = -\frac{c}{4} \mu(\xi\alpha) W + \alpha \left(\alpha \mu^2 + \frac{c}{4} \lambda \right) U,$$

which, together with (3.4) and (5.8) yields

$$(5.12) \quad \alpha \nabla \alpha = \left(\alpha^2 + \frac{3}{4} c \right) U + \frac{\xi\alpha}{\mu} \left(\alpha A W + \frac{c}{4} W \right).$$

Now, we are going to prove $\xi\alpha = 0$ on Ω . Since we have $\rho = h$, we can write (5.5) as $A^2\xi = hA\xi + (\beta - h\alpha)\xi$. Differentiating this covariantly and making use of the second equation of (2.1), we find

$$(5.13) \quad \begin{aligned} g((\nabla_X A) A\xi, Y) + g(A(\nabla_X A)\xi, Y) + g(A^2\phi AX, Y) - hg(A\phi AX, Y) \\ = (Xh)g(A\xi, Y) + hg((\nabla_X A)\xi, Y) \\ + X(\beta - h\alpha)\eta(Y) + (\beta - h\alpha)g(\phi AX, Y), \end{aligned}$$

which together with (2.3), (2.9) and (5.6) gives,

$$(\nabla_\xi A) A\xi = hAU - \frac{c}{4}U + \frac{1}{2}\nabla\beta.$$

If we replace X by ξ in (5.13) and use (2.9) and the last equation, then we get

$$3A^2U - 2hAU - \left(\beta - h\alpha + \frac{c}{4} \right) U = (\xi h) A\xi - A\nabla\alpha + h\nabla\alpha - \frac{1}{2}\nabla\beta,$$

which, connected to (3.3) and (5.9) yields

$$\frac{1}{2}\alpha^2\nabla\beta + \frac{c}{4}\left(2\alpha^2 + h\alpha + \frac{c}{2}\right)U = \alpha^2(\xi h)A\xi - \alpha^2A\nabla\alpha + h\alpha^2\nabla\alpha.$$

Substituting (5.12) into this, we obtain

$$(5.14) \quad \begin{aligned} \frac{1}{2}\alpha^2\nabla\beta + \left\{ \frac{c}{4}\left(2\alpha^2 + h\alpha + \frac{c}{2}\right) - \left(h\alpha + \frac{c}{4}\right)\left(\alpha^2 + \frac{3}{4}c\right) \right\} U \\ = \alpha^2(\xi h)A\xi - \frac{\alpha}{\mu}(\xi\alpha)\left(\alpha A^2W + \frac{c}{4}AW\right) + \frac{h\alpha}{\mu}(\xi\alpha)\left(\alpha AW + \frac{c}{4}W\right). \end{aligned}$$

Differentiating (5.9) and using itself, we find

$$(5.15) \quad \alpha^3(Y\beta) - \alpha^2\left(\alpha^2 + \frac{c}{4}\right)Yh = \left(h\alpha^3 - \frac{c}{4}h\alpha - \frac{c^2}{8}\right)Y\alpha,$$

which implies that

$$3\alpha \{(X\alpha)(Y\beta) - (Y\alpha)(X\beta)\} - \left(3\alpha^2 + \frac{c}{4}\right) \{(X\alpha)(Yh) - (Y\alpha)(Xh)\} = 0.$$

Combining (5.15) to this, we obtain $(X\alpha)(Yh) - (Y\alpha)(Xh) = 0$. This, together with (5.14) yields

$$\xi\alpha \left\{ \alpha^2(Xh) - \left(h\alpha + \frac{c}{2}\right) X\alpha \right\} = 0.$$

Thus, we have $\xi\alpha = 0$. In fact, if not, then we have

$$\alpha^2(Xh) = \left(h\alpha + \frac{c}{2}\right) X\alpha$$

on this subset. From this and (5.15) we see that $\alpha(X\beta) = \left(2h\alpha + \frac{c}{2}\right) X\alpha$, which together with (5.8) gives $\alpha^2(U\beta) = \left(2h\alpha + \frac{c}{2}\right) \left(\alpha^2 + \frac{3}{4}c\right) \mu^2$. If we take the inner product U to (5.14) and take account of the last equation, we obtain $2\alpha^2 + h\alpha + \frac{c}{2} = 0$ on the subset, a contradiction by virtue of (5.4). Therefore, $\xi\alpha = 0$ is established everywhere on Ω .

From this fact, (5.4), (5.10) and (5.11) imply respectively to

$$(5.16) \quad \xi h = 0, \quad \xi\lambda = 0, \quad \xi\beta = 0.$$

Further, (5.12) becomes

$$(5.17) \quad \alpha\nabla\alpha = \left(\alpha^2 + \frac{3}{4}c\right) U.$$

Since we have $g(\nabla_X U, U) = \mu(X\mu)$, taking the inner product U to (3.8) and making use of (3.3), (5.13) and the fact that $\xi\alpha = 0$, we obtain $\alpha\nabla\mu = \mu\alpha U + \frac{c}{4}\phi AW$. This, together with (5.6) gives

$$(5.18) \quad \frac{1}{2}\alpha\nabla\mu^2 = \left(\alpha\mu^2 + \frac{c}{4}\lambda\right) U.$$

Differentiating (5.18) covariantly and taking the symmetric part obtained, we find

$$\frac{c}{4} \{(X\lambda)u(Y) - (Y\lambda)u(X)\} + \left(\alpha\mu^2 + \frac{c}{4}\mu\right) du(X, Y) = 0,$$

where we have used (5.17) and (5.18). If we replace Y by ξ to this, and make use of (5.16), we obtain

$$\left(\alpha\mu^2 + \frac{c}{4}\lambda\right) du(X, \xi) = 0.$$

Finally, we suppose that $\alpha\mu^2 + \frac{c}{4}\lambda \neq 0$ on Ω , and that we restrict the arguments on such a place. Then we have from the last equation

$$g(\nabla_\xi U, X) + g(\nabla_X \xi, U) = 0.$$

This, together with (2.9), (2.10), (5.6) and (5.10) imply that

$$\phi(3AU + \nabla\alpha) + \mu hW = 0,$$

which enables to obtain

$$\nabla\alpha = hU - 3AU.$$

Thus, it follows that

$$\alpha\nabla\alpha = \left(h\alpha + \frac{3}{4}c\right)U$$

by virtue of (3.3). Comparing this with (5.17), we see that $h = \alpha$ and hence $\lambda = 0$ because of (5.10). Therefore (5.9) is reduced to

$$(5.19) \quad \alpha^2\mu^2 + \frac{c}{2}\alpha^2 = \left(\frac{c}{4}\right)^2.$$

Further, (5.18) turns out to be $\nabla\mu = \mu U$.

Differentiating (5.19) along Ω , and using (5.17) and the last equation, we obtain

$$\left(2\alpha^2 + \frac{3}{4}c\right)\mu^2 + \frac{c}{2}\left(\alpha^2 + \frac{3}{4}c\right) = 0,$$

which, connected to (5.19) implies that α is constant. Hence, μ is constant, a contradiction because of the fact that $\nabla\mu = \mu U$. Consequently we have $\alpha\mu^2 + \frac{c}{4}\lambda = 0$ on Ω , which together with (5.18) yield that $\nabla\mu = 0$. Owing to Lemma 1, M is a Hopf hypersurface in $M_n(c)$. This completes the proof.

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