ON SYMMETRIC GENERALIZED 3-DERIVATIONS AND COMMUTATIVITY IN PRIME NEAR-RINGS

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Abstract. In this note, we introduce a symmetric generalized 3-derivation in near-rings and investigate some conditions for a near-ring to be a commutative ring.

1. Introduction and preliminaries

A non-empty set R with two binary operations $+$ (addition) and $\cdot$ (multiplication) is called a *near-ring* if it satisfies the following axioms:

- i) $(R, +)$ is a group (not necessarily abelian),
- ii) $(R, \cdot)$ is a semigroup,
- iii) $x \cdot (y + z) = x \cdot y + x \cdot z$ for all $x, y, z \in R$.

Precisely speaking, it is a *left near-ring* because it satisfies the left distributive law. We will use the word *near-ring* to mean *left near-ring* and denote xy instead of $x \cdot y$.

For a near-ring R , the set $R_0 = \{x \in R : 0x = 0\}$ is called the *zero-symmetric part* of R . A near-ring R is said to be *zero-symmetric* if $R = R_0$. Throughout this note, R will be a zero-symmetric near-ring and R is called *prime* if $xRy = \{0\}$ implies $x = 0$ or $y = 0$. Recall that R is called *n -torsion-free*, where n is a positive integer, if $nx = 0$ implies $x = 0$ for all $x \in R$. The symbol C will represent the multiplicative center of R , that is, $C = \{x \in R : xy = yx \text{ for all } y \in R\}$. For $x \in R$, the symbol $C(x)$ will denote the centralizer of x in R . As usual, for $x, y \in R$, $[x, y]$ will denote the commutator $xy - yx$, while $\langle x, y \rangle$ will indicate the additive-group commutator $x + y - x - y$. As for terminologies concerning near-rings used here without special mention, we refer to G. Pilz [10].

An additive map $d : R \rightarrow R$ is called a *derivation* if the Leibniz rule $d(xy) = d(x)y + xd(y)$ holds for all $x, y \in R$. By a *bi-derivation* we mean

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a bi-additive map $D : R \times R \rightarrow R$ (i.e., D is additive in both arguments) which satisfies the relations

$$D(xy, z) = D(x, z)y + xD(y, z)$$

and

$$D(x, yz) = D(x, y)z + yD(x, z)$$

for all $x, y, z \in R$. Let D be symmetric, that is, $D(x, y) = D(y, x)$ for all $x, y \in R$. The map $\tau : R \rightarrow R$ defined by $\tau(x) = D(x, x)$ for all $x \in R$ is called the trace of D .

Derivations and bi-derivations in rings and near-rings have been studied by many mathematicians in several ways [1, 2, 3, 4, 5, 6, 7, 8, 11, 12, 15, 16]. Furthermore, M. Uçkun and M.A. Öztürk [14] investigated symmetric bi- Γ -derivations and commutativity in Γ -near-rings.

A 3-additive map $\Delta : R \times R \times R \rightarrow R$ is called a *3-derivation* if the relations

$$\Delta(x_1x_2, y, z) = \Delta(x_1, y, z)x_2 + x_1\Delta(x_2, y, z),$$

$$\Delta(x, y_1y_2, z) = \Delta(x, y_1, z)y_2 + y_1\Delta(x, y_2, z)$$

and

$$\Delta(x, y, z_1z_2) = \Delta(x, y, z_1)z_2 + z_1\Delta(x, y, z_2)$$

are fulfilled for all $x, y, z, x_i, y_i, z_i \in R$, $i = 1, 2$.

Let a map $F : R \times R \times R \rightarrow R$ be *symmetric*, namely, the equation $F(x_1, x_2, x_3) = F(x_{\pi(1)}, x_{\pi(2)}, x_{\pi(3)})$ holds for all $x_1, x_2, x_3 \in R$ and for every permutation $\{\pi(1), \pi(2), \pi(3)\}$. A map $f : R \rightarrow R$ defined by $f(x) = F(x, x, x)$ for all $x \in R$, is the *trace* of F . It is obvious that, in the case when $F : R \times R \times R \rightarrow R$ is a symmetric map which is also 3-additive (i.e., additive in each argument), the trace f of F satisfies the relation

$$f(x+y) = f(x) + 2F(x, x, y) + F(x, y, y) + F(x, x, y) + 2F(x, y, y) + f(y)$$

for all $x, y \in R$.

Since we have

$$F(0, y, z) = F(0+0, y, z) = F(0, y, z) + F(0, y, z)$$

for all $y, z \in R$, we obtain $F(0, y, z) = 0$ for all $y, z \in R$. Hence we get

$$0 = F(0, y, z) = F(x-x, y, z) = F(x, y, z) + F(-x, y, z)$$

and so we see that $F(-x, y, z) = -F(x, y, z)$ for all $x, y, z \in R$. This tells us that f is an odd function. If Δ is symmetric, then the above three relations are equivalent to each other.

An additive map $h : R \rightarrow R$ is called a *generalized derivation* if there exists a derivation $d : R \rightarrow R$ such that $h(xy) = d(x)y + xh(y)$ for all $x, y \in R$. M. Brešar [4] defined this concept in rings. This notion is found in [13] and other properties of generalized derivations in rings and near-rings were given by B. Hvala [9], H.E. Bell [2] and Ö. Gölbaşı [8], respectively.

A symmetric bi-additive map $H : R \times R \rightarrow R$ is said to be a *symmetric generalized bi-left derivation* (resp. *symmetric generalized bi-right derivation*) associated with D if there exists a symmetric bi-derivation $D : R \times R \rightarrow R$ such that $H(xy, z) = D(x, z)y + xH(y, z)$ (resp. $H(xy, z) = H(x, z)y + xD(y, z)$) for all $x, y, z \in R$. H is called a *symmetric generalized bi-derivation* associated with D if it is both a symmetric generalized bi-left derivation and a symmetric generalized bi-right derivation associated with D (see [14]).

Here we introduce the following map:

By a *symmetric generalized 3-left derivation* (resp. *symmetric generalized 3-right derivation*) associated with Δ , we mean a symmetric 3-additive map $\mathcal{G} : R \times R \times R \rightarrow R$ such that there exists a symmetric 3-derivation $\Delta : R \times R \times R \rightarrow R$ satisfying $\mathcal{G}(xy, z, w) = \Delta(x, z, w)y + x\mathcal{G}(y, z, w)$ (resp. $\mathcal{G}(xy, z, w) = \mathcal{G}(x, z, w)y + x\Delta(y, z, w)$) for all $w, x, y, z \in R$. The map $\mathcal{G}$ is called a *symmetric generalized 3-derivation* associated with Δ if it is both a symmetric generalized 3-left derivation and a symmetric generalized 3-right derivation associated with Δ . We here will denote the generalized 3-(left or right) derivation $\mathcal{G}$ associated with the symmetric 3-derivation Δ by the tuple $(\mathcal{G}, \Delta)$. Note that the symmetric generalized 3-derivation (Δ, Δ) is the symmetric 3-derivation Δ .

As a simple example, let N be a commutative near-ring and let

$$R = \left\{ \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} \mid a, b \in N \right\}.$$

Then it is easy to see that R is a near-ring under matrix addition and matrix multiplication and that the map $\Delta : R \times R \times R \rightarrow R$ defined by

$$\left(\begin{pmatrix} a_1 & b_1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} a_2 & b_2 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} a_3 & b_3 \\ 0 & 0 \end{pmatrix} \right) \mapsto \begin{pmatrix} 0 & a_1 a_2 a_3 \\ 0 & 0 \end{pmatrix}$$

is a symmetric 3-derivation.

Furthermore, we define a map $\mathcal{G} : R \times R \times R \rightarrow R$ by

$$\left(\begin{pmatrix} a_1 & b_1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} a_2 & b_2 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} a_3 & b_3 \\ 0 & 0 \end{pmatrix} \right) \mapsto \begin{pmatrix} 0 & a_1 a_2 a_3 + b_1 b_2 b_3 \\ 0 & 0 \end{pmatrix}.$$

Then $(\mathcal{G}, \Delta)$ is a symmetric generalized 3-left derivation.

In this note, we investigate the conditions for a near-ring with symmetric generalized 3-derivations to be a commutative ring.

2. Lemmata

We need the following lemmas to obtain our main results.

Lemma 2.1 ([3, Lemma 3]). *Let R be a prime near-ring. If $C \setminus \{0\}$ contains an element z for which $z + z \in C$, then $(R, +)$ is abelian.*

Lemma 2.2. *Let R be a 2 and 3-torsion free near-ring. Suppose that there exists a symmetric 3-additive map $F : R \times R \times R \rightarrow R$ such that $f(x) = 0$ for all $x \in R$, where f is the trace of F . Then we have $F = 0$.*

Proof. For any $x, y \in R$,

$$f(x+y) = f(x) + 2F(x, x, y) + F(x, y, y) + F(x, x, y) + 2F(x, y, y) + f(y)$$

and so, by the hypothesis, we get

$$(2.1) \quad 2F(x, x, y) + F(x, y, y) + F(x, x, y) + 2F(x, y, y) = 0$$

for all $x, y \in R$. Putting $-x$ instead of x in (2.1), we obtain

$$(2.2) \quad 2F(x, x, y) - F(x, y, y) + F(x, x, y) - 2F(x, y, y) = 0$$

for all $x, y \in R$.

On the other hand, for any $x, y \in R$,

$$f(y+x) = f(y) + 2F(y, y, x) + F(y, x, x) + F(y, y, x) + 2F(y, x, x) + f(x)$$

and thus, by the hypothesis, we have

$$(2.3) \quad 2F(x, y, y) + F(x, x, y) + F(x, y, y) + 2F(x, x, y) = 0$$

for all $x, y \in R$ since F is symmetric. Comparing (2.1) with (2.2), we get

$$2F(x, y, y) + F(x, x, y) + F(x, y, y) = F(x, x, y) - 3F(x, y, y)$$

which implies that

$$\begin{aligned} & 2F(x, y, y) + F(x, x, y) + F(x, y, y) + 2F(x, x, y) \\ &= F(x, x, y) - 3F(x, y, y) + 2F(x, x, y) \end{aligned}$$

for all $x, y \in R$. Hence it follows from (2.3) that

$$(2.4) \quad F(x, x, y) - 3F(x, y, y) + 2F(x, x, y) = 0$$

for all $x, y \in R$. The substitution $x = -x$ in (2.4) leads to

$$(2.5) \quad F(x, x, y) + 3F(x, y, y) + 2F(x, x, y) = 0$$

for all $x, y \in R$. Combining (2.4) and (2.5), we obtain

$$(2.6) \quad F(x, y, y) = 0$$

for all $x, y \in R$ since R is 6-torsion free. The replacement $y = y + z$ to linearize (2.6) yields

$$F(x, y, z) = 0$$

for all $x, y, z \in R$, i.e., $F = 0$ which completes the proof. $\square$

Lemma 2.3. *Let R be a 2 and 3-torsion free prime near-ring and let $x \in R$. Suppose that there exists a nonzero symmetric generalized 3-derivation $(\mathcal{G}, \Delta) : R \times R \times R \rightarrow R$ such that $xg(y) = 0$ for all $y \in R$, where g is the trace of $\mathcal{G}$. Then we have $x = 0$.*

Proof. Since we have

$$g(y + z) = g(y) + 2\mathcal{G}(y, y, z) + \mathcal{G}(y, z, z) + \mathcal{G}(y, y, z) + 2\mathcal{G}(y, z, z) + g(z)$$

for all $y, z \in R$, the hypothesis gives

$$(2.7) \quad 2x\mathcal{G}(y, y, z) + x\mathcal{G}(y, z, z) + x\mathcal{G}(y, y, z) + 2x\mathcal{G}(y, z, z) = 0$$

for all $y, z \in R$. Setting $y = -y$ in (2.7), it follows that

$$(2.8) \quad 2x\mathcal{G}(y, y, z) - x\mathcal{G}(y, z, z) + x\mathcal{G}(y, y, z) - 2x\mathcal{G}(y, z, z) = 0$$

for all $y, z \in R$.

On the other hand, for any $y, z \in R$,

$$g(z + y) = g(z) + 2\mathcal{G}(z, z, y) + \mathcal{G}(z, y, y) + \mathcal{G}(z, z, y) + 2\mathcal{G}(z, y, y) + g(y)$$

and so, by the hypothesis, we have

$$(2.9) \quad 2x\mathcal{G}(y, z, z) + x\mathcal{G}(y, y, z) + x\mathcal{G}(y, z, z) + 2x\mathcal{G}(y, y, z) = 0$$

for all $y, z \in R$ since $\mathcal{G}$ is symmetric. Comparing (2.8) with (2.9), we get

$$2x\mathcal{G}(y, z, z) + x\mathcal{G}(y, y, z) + x\mathcal{G}(y, z, z) = x\mathcal{G}(y, y, z) - 3x\mathcal{G}(y, z, z)$$

which means that

$$\begin{aligned} & 2x\mathcal{G}(y, z, z) + x\mathcal{G}(y, y, z) + x\mathcal{G}(y, z, z) + 2x\mathcal{G}(y, y, z) \\ &= x\mathcal{G}(y, y, z) - 3x\mathcal{G}(y, z, z) + 2x\mathcal{G}(y, y, z) \end{aligned}$$

for all $y, z \in R$. Now, from (2.9), we obtain

$$(2.10) \quad x\mathcal{G}(y, y, z) - 3x\mathcal{G}(y, z, z) + 2x\mathcal{G}(y, y, z) = 0$$

for all $y, z \in R$. Taking $y = -y$ in (2.10) leads to

$$(2.11) \quad x\mathcal{G}(y, y, z) + 3x\mathcal{G}(y, z, z) + 2x\mathcal{G}(y, y, z) = 0$$

for all $y, z \in R$. Combining (2.10) and (2.11), we obtain

$$(2.12) \quad x\mathcal{G}(y, z, z) = 0$$

for all $y \in R$ since R is 6-torsion free. Replacing $z = z + w$ to linearize (2.12) and using the conditions show that

$$(2.13) \quad x\mathcal{G}(w, y, z) = 0$$

for all $w, y, z \in R$. Assume first that $\Delta = 0$. Since $(\mathcal{G}, \Delta)$ is a nonzero symmetric generalized 3-left derivation, substituting wv for w in (2.13) implies that

$$xw\mathcal{G}(v, y, z) = 0$$

for all $v, w, y, z \in R$. By the primeness of R , we get $x = 0$. On the other hand, assume that $\Delta \neq 0$. Since $(\mathcal{G}, \Delta)$ is also a symmetric generalized 3-right derivation, again substituting wv for w in (2.13) implies that

$$xw\Delta(v, y, z) = 0$$

for all $v, w, y, z \in R$. From the primeness of R , we have $x = 0$ which gives the conclusion. $\square$

Lemma 2.4. *Let R be a near-ring and let $(\mathcal{G}, \Delta) : R \times R \times R \rightarrow R$ be a symmetric generalized 3-derivation. Then for all $v, w, x, y, z \in R$, we have, respectively,*

- (i) $[\mathcal{G}(x, z, w)y + x\Delta(y, z, w)]v = \mathcal{G}(x, z, w)yv + x\Delta(y, z, w)v;$
- (ii) $[\Delta(x, z, w)y + x\mathcal{G}(y, z, w)]v = \Delta(x, z, w)yv + x\mathcal{G}(y, z, w)v.$

Proof. (i) Since we have

$$\mathcal{G}(xy, z, w) = \mathcal{G}(x, z, w)y + x\Delta(y, z, w)$$

for all $w, x, y, z \in R$, the associative law gives

$$(2.14) \quad \begin{aligned} \mathcal{G}((xy)v, z, w) &= \mathcal{G}(xy, z, w)v + xy\Delta(v, z, w) \\ &= [\mathcal{G}(x, z, w)y + x\Delta(y, z, w)]v + xy\Delta(v, z, w) \end{aligned}$$

for all $v, w, x, y, z \in R$ and

$$(2.15) \quad \begin{aligned} \mathcal{G}(x(yv), z, w) &= \mathcal{G}(x, z, w)yv + x\Delta(yv, z, w) \\ &= \mathcal{G}(x, z, w)yv + x[\Delta(y, z, w)v + y\Delta(v, z, w)] \\ &= \mathcal{G}(x, z, w)yv + x\Delta(y, z, w)v + xy\Delta(v, z, w) \end{aligned}$$

for all $v, w, x, y, z \in R$. Comparing (2.14) and (2.15), we see that

$$[\mathcal{G}(x, z, w)y + x\Delta(y, z, w)]v = \mathcal{G}(x, z, w)yv + x\Delta(y, z, w)v$$

for all $v, w, x, y, z \in R$.

- (ii) It is proved by the similar way to (i). $\square$

3. Main results

Now we are ready to prove our main results.

Theorem 3.1. *Let R be a 2 and 3-torsion free prime near-ring. Suppose that there exists a nonzero symmetric generalized 3-derivation $(\mathcal{G}, \Delta) : R \times R \times R \rightarrow R$ such that*

$$\mathcal{G}(x, y, z) \in C$$

for all $x, y, z \in R$. Then R is a commutative ring.

Proof. Assume that $\mathcal{G}(x, y, z) \in C$ for all $x, y, z \in R$. Since $(\mathcal{G}, \Delta)$ is nonzero, there exist $x_0, y_0, z_0 \in R$ such that $\mathcal{G}(x_0, y_0, z_0) \in C \setminus \{0\}$ and

$$\mathcal{G}(x_0, y_0, z_0) + \mathcal{G}(x_0, y_0, z_0) = \mathcal{G}(x_0, y_0, z_0 + z_0) \in C.$$

Hence $(R, +)$ is abelian by Lemma 2.1.

Let $\Delta = 0$. Then $\mathcal{G}(xy, z, w) = \mathcal{G}(x, z, w)y \in C$ holds for all $w, x, y \in R$ and hence we have $\mathcal{G}(x, z, w)yv = v\mathcal{G}(x, z, w)y$ for all $v, w, x, y, z \in R$ which implies that $[y, v]g(x) = 0$ for all $v, x, y \in R$. Reminding of Lemma 2.3, we get $[y, v] = 0$ for all $v, y \in R$, i.e., R is a commutative ring.

Let $\Delta \neq 0$. Since the hypothesis implies that

$$(3.1) \quad w\mathcal{G}(x, y, z) = \mathcal{G}(x, y, z)w$$

for all $w, x, y, z \in R$, we replace x by xv in (3.1) to get

$$w[\mathcal{G}(x, y, z)v + x\Delta(v, y, z)] = [\mathcal{G}(x, y, z)v + x\Delta(v, y, z)]w$$

and thus, from Lemma 2.4(i) and the hypothesis, it follows that

$$\mathcal{G}(x, y, z)wv + wx\Delta(v, y, z) = \mathcal{G}(x, y, z)vw + x\Delta(v, y, z)w$$

which means that

$$(3.2) \quad \mathcal{G}(x, y, z)[w, v] = [x\Delta(v, y, z), w]$$

for all $v, w, x, y, z \in R$. Since $(\mathcal{G}, \Delta)$ is a symmetric generalized 3-left derivation and $g(x) \in C$, setting $g(x)u$ in place of x in (3.2) and using Lemma 2.4(ii) yield

$$\Delta(g(x), y, z)u[w, v] + g(x)\mathcal{G}(u, y, z)[w, v] = g(x)[u\Delta(v, y, z), w]$$

for all $u, w, x, y, z \in R$. So the relation (3.2) gives

$$(3.3) \quad \Delta(g(x), y, z)u[w, v] = 0$$

for all $u, w, x, y, z \in R$.

Suppose that R is not commutative. Since R is prime, we get, from (3.3),

$$(3.4) \quad \Delta(g(x), y, z) = 0$$

for all $x, y, z \in R$. Let us $x = x + w$ in (3.4). Then we have

$$\begin{aligned} 0 &= \Delta(g(x + w), y, z) \\ &= \Delta(g(x) + g(w) + 3\mathcal{G}(x, x, w) + 3\mathcal{G}(w, w, x), y, z) \\ &= 3\Delta(\mathcal{G}(x, x, w), y, z) + 3\Delta(\mathcal{G}(x, w, w), y, z), \end{aligned}$$

that is,

$$(3.5) \quad \Delta(\mathcal{G}(x, x, w), y, z) + \Delta(\mathcal{G}(x, w, w), y, z) = 0$$

for all $w, x, y, z \in R$. Letting $x = -x$ in (3.5) and then comparing the result with (3.5), we see that

$$(3.6) \quad \Delta(\mathcal{G}(x, x, w), y, z) = 0$$

for all $w, x, y, z \in R$. We replace x by $x + v$ in (3.6) to obtain

$$(3.7) \quad \Delta(\mathcal{G}(x, v, w), y, z) = 0$$

for all $v, w, x, y, z \in R$. Substituting xu for x in (3.7), we get

$$\begin{aligned} 0 &= \Delta(\mathcal{G}((xu, v, w), y, z) \\ &= \Delta(\mathcal{G}(x, v, w)u + x\Delta(u, v, w), y, z) \\ &= \Delta(\mathcal{G}(x, v, w)u, y, z) + \Delta(x\Delta(u, v, w), y, z) \\ &= \Delta(\mathcal{G}(x, v, w), y, z)u + \mathcal{G}(x, v, w)\Delta(u, y, z) \\ &\quad + \Delta(x, y, z)\Delta(u, v, w) + x\Delta(\Delta(u, v, w), y, z) \end{aligned}$$

and so the equality (3.7) gives

$$(3.8) \quad \mathcal{G}(x, v, w)\Delta(u, y, z) + \Delta(x, y, z)\Delta(u, v, w) + x\Delta(\Delta(u, v, w), y, z) = 0$$

for all $u, v, w, x, y, z \in R$. Let us write $g(w)$ in place of w in (3.8). Then it follows from (3.4) that

$$(3.9) \quad \mathcal{G}(g(w), x, v)\Delta(u, y, z) = 0$$

for all $u, v, w, x, y, z \in R$. Putting tu in substitute for u in (3.9) yields

$$\mathcal{G}(g(w), x, v)t\Delta(u, y, z) = 0$$

for all $t, u, v, w, x, y, z \in R$ which implies that

$$(3.10) \quad \mathcal{G}(g(w), x, v) = 0$$

for all $v, x, w \in R$ by the primeness of R and $\Delta \neq 0$. Taking $y + w$ instead of w in (3.10), we get

$$\begin{aligned} 0 &= \mathcal{G}(g(y + w), x, v) \\ &= \mathcal{G}(g(y) + g(w) + 3\mathcal{G}(y, y, w) + 3\mathcal{G}(y, w, w), x, v) \\ &= 3\mathcal{G}(\mathcal{G}(y, y, w), x, v) + 3\mathcal{G}(\mathcal{G}(y, w, w), x, v), \end{aligned}$$

that is,

$$(3.11) \quad \mathcal{G}(\mathcal{G}(y, y, w), x, v) + \mathcal{G}(\mathcal{G}(y, w, w), x, v) = 0$$

for all $v, w, x, y \in R$. Setting $y = -y$ in (3.11) and then combining the result with (3.11), we see that

$$(3.12) \quad \mathcal{G}(\mathcal{G}(y, y, w), x, v) = 0$$

for all $v, w, x, y \in R$. We replace y by $y + z$ in (3.12) to obtain

$$(3.13) \quad \mathcal{G}(\mathcal{G}(y, z, w), x, v) = 0$$

for all $v, w, x, y, z \in R$. Substituting yu for y in (3.13) gives the relation

$$\begin{aligned} 0 &= \mathcal{G}(\mathcal{G}(yu, z, w), x, v) \\ &= \mathcal{G}(\Delta(y, z, w)u + y\mathcal{G}(u, z, w), x, v) \\ &= \mathcal{G}(\Delta(y, z, w)u, x, v) + \mathcal{G}(y\mathcal{G}(u, z, w), x, v) \\ &= \mathcal{G}(\Delta(y, z, w), x, v)u + \Delta(y, z, w)\Delta(u, x, v) \\ &\quad + \Delta(y, x, v)\mathcal{G}(u, z, w) + y\mathcal{G}(\mathcal{G}(u, z, w), x, v) \end{aligned}$$

from which the equality (3.13) leads to the relation

$$(3.14) \quad \mathcal{G}(\Delta(y, z, w), x, v)u + \Delta(y, z, w)\Delta(u, x, v) + \Delta(y, x, v)\mathcal{G}(u, z, w) = 0$$

for all $u, v, w, x, y, z \in R$. Letting $u = g(u)$ in (3.14), we have, by (3.4) and (3.10),

$$(3.15) \quad \mathcal{G}(\Delta(y, z, w), x, v)g(u) = 0$$

for all $u, v, w, x, y, z \in R$. Now, from Lemma 2.3 and $\mathcal{G} \neq 0$, the equality (3.15) implies that

$$(3.16) \quad \mathcal{G}(\Delta(y, z, w), x, v) = 0$$

for all $v, w, x, y, z \in R$ and putting $y = z = w$ in (3.16) yields

$$(3.17) \quad \mathcal{G}(\delta(y), x, v) = 0$$

for all $v, x, y \in R$. Hence, by (3.16), the equality (3.14) reduces to the relation

$$(3.18) \quad \Delta(y, z, w)\Delta(u, x, v) + \Delta(y, x, v)\mathcal{G}(u, z, w) = 0$$

for all $u, v, w, x, y, z \in R$. Replacing u by $\delta(u)$ in (3.18), the equality (3.17) tells us that

$$(3.19) \quad \Delta(y, z, w)\Delta(\delta(u), x, v) = 0$$

for all $u, v, w, x, y, z \in R$ and so taking zt instead of z in (3.19) gives

$$\Delta(y, z, w)t\Delta(\delta(u), x, v) = 0$$

for all $t, u, v, w, x, y, z \in R$. The primeness of R now leads to

$$(3.20) \quad \Delta(\delta(u), x, v) = 0$$

for all $u, v, x \in R$ in view of $\Delta \neq 0$. By (3.20), we see that

$$\Delta(\delta(u+x), u+x, u+x) + \Delta(\delta(-u+x), -u+x, -u+x) = 0$$

for all $u, x \in R$ from which we use the relation (3.20) to get

$$(3.21) \quad \Delta(\Delta(u, u, x), u, u) + \Delta(\Delta(u, u, x), x, x) + 2\Delta(\Delta(u, x, x), u, x) = 0$$

for all $u, x \in R$. Writing $u+x$ for x in (3.21) and utilizing (3.20), we obtain

$$(3.22) \quad \begin{aligned} 0 = & 6\Delta(\Delta(u, u, x), u, u) + 6\Delta(\Delta(u, u, x), u, x) + \Delta(\Delta(u, u, x), x, x) \\ & + 2\Delta(\Delta(u, x, x), u, u) + 2\Delta(\Delta(u, x, x), u, x) \end{aligned}$$

for all $u, x \in R$ and so substituting $-u$ for u in (3.22) and combining the result with (3.23) yield

$$(3.23) \quad \Delta(\Delta(u, x, x), u, u) + 3\Delta(\Delta(u, u, x), u, x) = 0$$

for all $u, x \in R$. Taking $u = u+x$ in (3.23) and using (3.20), we have

$$(3.24) \quad \begin{aligned} 0 = & \Delta(\Delta(u, x, x), u, u) + 3\Delta(\Delta(u, u, x), u, x) + 3\Delta(\Delta(u, u, x), x, x) \\ & + 3\Delta(\Delta(u, x, x), u, x) + 3\Delta(\Delta(u, x, x), x, x) \end{aligned}$$

for all $u, x \in R$. Let us replace x by $-x$ in (3.24) and compare the result with (3.24). Then we obtain

$$(3.25) \quad \Delta(\Delta(u, u, x), x, x) + \Delta(\Delta(u, x, x), u, x) = 0$$

for all $u, x \in R$. Putting $u = u+x$ in (3.25) and using (3.20), the relation (3.25) reduces to

$$(3.26) \quad \Delta(\Delta(u, u, x), x, x) + 3\Delta(\Delta(u, x, x), x, x) + \Delta(\Delta(u, x, x), u, x) = 0$$

for all $u, x \in R$. Replacing x by $-x$ in (3.26) and combining the result with (3.26), we have

$$(3.27) \quad \Delta(\Delta(u, x, x), x, x) = 0$$

for all $u, x \in R$. We substitute ux for u in (3.27) and then employ (3.20) and (3.27) to arrive at

$$(3.28) \quad \Delta(u, x, x)\delta(x) = 0$$

for all $u, x \in R$. Finally, taking xy instead of u in (3.28), it follows that

$$\delta(x)y\delta(x) = 0$$

for all $x, y \in R$. Since R is prime, we see that $\delta = 0$ which implies $\Delta = 0$ by Lemma 2.2. This is a contradiction because of $\Delta \neq 0$. Hence R is commutative. This completes the proof. $\square$

Theorem 3.2. *Let R be a 2 and 3-torsion free prime near-ring. Suppose that there exists a nonzero symmetric generalized 3-derivation $(\mathcal{G}, \Delta) : R \times R \times R \rightarrow R$ such that*

$$g(x), g(x) + g(x) \in C(\mathcal{G}(u, v, w))$$

for all $u, v, w, x, y \in R$, where g is the trace of $\mathcal{G}$. Then R is a commutative ring.

Proof. Assume that

$$(3.29) \quad g(x), g(x) + g(x) \in C(\mathcal{G}(u, v, w))$$

for all $u, v, w, x \in R$. Thus we get, from (3.29),

$$\begin{aligned} & \mathcal{G}(u + p, v, w)(g(x) + g(x)) \\ &= (g(x) + g(x))\mathcal{G}(u + p, v, w) \\ &= (g(x) + g(x))[\mathcal{G}(u, v, w) + \mathcal{G}(p, v, w)] \\ &= (g(x) + g(x))\mathcal{G}(u, v, w) + (g(x) + g(x))\mathcal{G}(p, v, w) \\ &= g(x)\mathcal{G}(u, v, w) + g(x)\mathcal{G}(u, v, w) + g(x)\mathcal{G}(p, v, w) + g(x)\mathcal{G}(p, v, w) \\ &= g(x)[\mathcal{G}(u, v, w) + \mathcal{G}(u, v, w) + \mathcal{G}(p, v, w) + \mathcal{G}(p, v, w)] \\ (3.30) \quad &= [\mathcal{G}(u, v, w) + \mathcal{G}(u, v, w) + \mathcal{G}(p, v, w) + \mathcal{G}(p, v, w)]g(x) \end{aligned}$$

for all $p, u, v, w, x \in R$ and

$$\begin{aligned} & \mathcal{G}(u + p, v, w)(g(x) + g(x)) \\ &= \mathcal{G}(u + p, v, w)g(x) + \mathcal{G}(u + p, v, w)g(x) \\ &= g(x)\mathcal{G}(u + p, v, w) + g(x)\mathcal{G}(u + p, v, w) \\ &= g(x)[\mathcal{G}(u, v, w) + \mathcal{G}(p, v, w)] + g(x)[\mathcal{G}(u, v, w) + \mathcal{G}(p, v, w)] \\ &= g(x)[\mathcal{G}(u, v, w) + \mathcal{G}(p, v, w) + \mathcal{G}(u, v, w) + \mathcal{G}(p, v, w)] \\ (3.31) \quad &= [\mathcal{G}(u, v, w) + \mathcal{G}(p, v, w) + \mathcal{G}(u, v, w) + \mathcal{G}(p, v, w)]g(x) \end{aligned}$$

for all $p, u, v, w, x \in R$. Comparing (3.30) and (3.31), we obtain

$$\mathcal{G}(\langle u, p \rangle, v, w)g(x) = 0$$

for all $p, u, v, w, x \in R$. From Lemma 2.3, it follows that

$$(3.32) \quad \mathcal{G}(\langle u, p \rangle, v, w) = 0$$

for all $p, u, v, w \in R$.

First, let us consider the case $\Delta = 0$. Since $(\mathcal{G}, \Delta)$ is a symmetric generalized 3-right derivation, we substitute uz for u and up for p in (3.32) to get

$$\begin{aligned} 0 &= \mathcal{G}(u\langle z, p \rangle, v, w) \\ &= \mathcal{G}(u, v, w)\langle z, p \rangle + u\Delta(\langle z, p \rangle, v, w) \\ &= \mathcal{G}(u, v, w)\langle z, p \rangle \end{aligned}$$

and so we see that

$$(3.33) \quad g(u)\langle z, p \rangle = 0$$

for all $p, u, z \in R$. Letting $z = wz$ and $p = wp$ in (3.33) yields

$$(3.34) \quad g(u)w\langle z, p \rangle = 0$$

for all $p, u, z, w \in R$. Since Lemma 2.2 tells us that $\mathcal{G} \neq 0$ implies $g \neq 0$, we conclude, from (3.34) and the primeness of R , that $\langle z, p \rangle = 0$ is fulfilled for all $p, z \in R$. Therefore $(R, +)$ is abelian.

Since we have $\Delta = 0$ and $(\mathcal{G}, \Delta)$ is a symmetric generalized 3-right derivation, we obtain

$$\mathcal{G}(xy, z, t) = \mathcal{G}(x, z, t)y \in C(\mathcal{G}(u, v, w))$$

for all $t, u, v, w, x, y, z \in R$. Thus we have

$$\mathcal{G}(x, z, t)y\mathcal{G}(u, v, w) = \mathcal{G}(u, v, w)\mathcal{G}(x, z, t)y$$

for all $t, u, v, w, x, y, z \in R$ which implies that

$$\mathcal{G}(x, z, t)[y, \mathcal{G}(u, v, w)] = 0$$

for all $t, u, v, w, x, y, z \in R$. Taking xs in place of x in this equation, we get

$$\mathcal{G}(x, z, t)s[y, \mathcal{G}(u, v, w)] = 0$$

for all $s, t, u, v, w, x, y \in R$ since $(\mathcal{G}, \Delta)$ is a symmetric generalized 3-right derivation. Since R is prime and $\mathcal{G} \neq 0$, we obtain $[y, \mathcal{G}(u, v, w)] = 0$ for all $u, v, w, y \in R$, i.e., $\mathcal{G}(u, v, w) \in C$ for all $u, v, w \in R$. Invoking Theorem 3.1, R is a commutative ring.

Let $\Delta \neq 0$ and let δ be the trace of Δ . Since $(\mathcal{G}, \Delta)$ is a symmetric generalized 3-left derivation, replacing u by uz and p by up in (3.32) yields that

$$\begin{aligned} 0 &= \mathcal{G}(u\langle z, p \rangle, v, w) \\ &= \Delta(u, v, w)\langle z, p \rangle + u\mathcal{G}(\langle z, p \rangle, v, w) \\ &= \Delta(u, v, w)\langle z, p \rangle \end{aligned}$$

and so we obtain

$$(3.35) \quad \Delta(u, v, w)\langle z, p \rangle = 0$$

for all $p, u, z \in R$. Letting $z = sz$ and $p = sp$ in (3.35) yields

$$(3.36) \quad \Delta(u, v, w)s\langle z, p \rangle = 0$$

for all $p, s, u, v, w, z \in R$. Since $\Delta \neq 0$, we conclude, from (3.36) and the primeness of R , that $\langle z, p \rangle = 0$ is fulfilled for all $p, z \in R$. Hence $(R, +)$ is abelian.

We claim that R is a commutative ring. Indeed, from the hypothesis, we see that

$$(3.37) \quad [g(x), \mathcal{G}(u, v, w)] = 0$$

for all $u, v, w, x \in R$. Hence if we let $x = x + y$ in (3.37), then we deduce from (3.37) that

$$(3.38) \quad [\mathcal{G}(x, x, y), \mathcal{G}(u, v, w)] + [\mathcal{G}(x, y, y), \mathcal{G}(u, v, w)] = 0$$

for all $u, v, w, x, y \in R$. Setting $y = -y$ into (3.38) and comparing the result with (3.38), we obtain

$$(3.39) \quad [\mathcal{G}(x, y, y), \mathcal{G}(u, v, w)] = 0$$

for all $u, v, w, x, y \in R$. Replacing y by $y + z$ in (3.39) and using (3.39), we have

$$[\mathcal{G}(x, y, z), \mathcal{G}(u, v, w)] = 0$$

since $\mathcal{G}$ is permuting, i.e.,

$$(3.40) \quad \mathcal{G}(x, y, z)\mathcal{G}(u, v, w) = \mathcal{G}(u, v, w)\mathcal{G}(x, y, z)$$

for all $u, v, w, x, y, z \in R$. Taking ut instead of u in (3.40) and applying Lemma 2.4(ii), we obtain

$$\begin{aligned} (3.41) \quad &\Delta(u, v, w)t\mathcal{G}(x, y, z) - \mathcal{G}(x, y, z)\Delta(u, v, w)t \\ &+ u\mathcal{G}(t, v, w)\mathcal{G}(x, y, z) - \mathcal{G}(x, y, z)u\mathcal{G}(t, v, w) = 0 \end{aligned}$$

for all $t, u, v, w, x, y, z \in R$. Substituting $g(u)$ for u in (3.41) and then utilizing the hypothesis and (3.40), we get

$$(3.42) \quad \Delta(g(u), v, w)[t, \mathcal{G}(x, y, z)] = 0$$

for all $t, u, v, w, x, y, z \in R$. Let us write in (3.42) ws instead of w . Then we have

$$\Delta(g(u), v, ws)[t, \mathcal{G}(x, y, z)] = 0$$

for all $s, t, u, v, w, x, y, z \in R$. Since R is prime, we arrive at either $\Delta(g(u), v, w) = 0$ or $[t, \mathcal{G}(x, y, z)] = 0$ for all $t, u, v, w, x, y, z \in R$.

Following the same process among (3.4)~(3.28) in the proof of Theorem 3.1, the case when $\Delta(g(u), v, w) = 0$ holds for all $u, v, w \in R$ leads to the contradiction.

Consequently, we obtain

$$[t, \mathcal{G}(x, y, z)] = 0$$

for all $t, x, y, z \in R$, i.e. $\mathcal{G}(x, y, z) \in C$ for all $x, y, z \in R$. Therefore, Theorem 3.1 tells us that R is a commutative ring which is complete the proof. $\square$

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