ON FUNCTIONAL INEQUALITIES CONNECTED WITH INTERTWINING MAPPINGS

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Abstract. For a mapping satisfying the inequality

$$\|\lambda f(x) + 2\lambda f(y) + 2f(\lambda z)\| \leq \left\| 2f\left(\lambda\left(\frac{x}{2} + y + z\right)\right) \right\| + \phi(x, y, z),$$

we will study the stability problem of this mapping.

1. Introduction

In 1940, Ulam [19] proposed the following question concerning the stability of group homomorphisms :

Let G_1 be a group and let G_2 be a metric group with the metric d . Given $\epsilon > 0$, does there exist a $\delta > 0$ such that if a function $h : G_1 \rightarrow G_2$ satisfies the inequality $d(h(xy), h(x)h(y)) < \delta$ for all $x, y \in G_1$, then there exists a homomorphism $H : G_1 \rightarrow G_2$ with $d(h(x), H(x)) < \epsilon$ for all $x \in G_1$?

In the following year 1941, Hyers [7] answered the problem of Ulam under the assumption that the groups are Banach spaces. A generalized version of the theorem of Hyers for approximately additive mappings was given by Aoki [1] and for the theorem of Hyers for approximately linear mappings it was presented by Rassias [16]. Since then, the stability problems of various functional equations have been extensively investigated by a number of mathematicians (for instances, [5, 6, 8, 9, 10, 11, 14, 15, 17, 18]). Moreover, the problems concerned with the generalizations and the applications of the stability to a number of functional equations and mappings have been developed as well.

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By regarding a large influence of Ulam, Hyers and Rassias on the investigation of stability problems of functional equations the stability phenomenon that was introduced and proved by Rassias [16] in the year 1978 is called the *generalized Hyers-Ulam stability* or the *Hyers-Ulam-Rassias stability*.

Let A be a Banach algebra and let X be a weak Banach A -bimodule. By a weak Banach A -module we mean a Banach space X which is an A -bimodule such that for each $a \in A$, the mappings $z \mapsto az$ and $z \mapsto za$ defined on X are continuous. A linear mapping $L : A \rightarrow X$ is called *intertwining* if for each $a \in A$, the mappings $x \mapsto L(ax) - aL(x)$ and $x \mapsto L(xa) - L(x)a$ from A into X are continuous. These mappings were first introduced by Bade and Curtis [2] and were generalized by Laursen [13]. Evidently, every A -bimodule homomorphism of A into X and any derivation of A into X are intertwining (cf. [3, 4]).

In this paper, we will take account of the stability problem of a mapping satisfying the inequality

$$\|\lambda f(x) + 2\lambda f(y) + 2f(\lambda z)\| \leq \left\| 2f\left(\lambda\left(\frac{x}{2} + y + z\right)\right) \right\| + \phi(x, y, z).$$

This functional inequality is based on the inequality introduced in [18].

2. Stability on a Banach algebra

In this section, let A be a Banach algebra and X be a weak Banach A -module, respectively. We also denote by $\mathbb{T} := \{z \in \mathbb{C} : |z| = 1\}$. Let $\phi : A^3 \rightarrow [0, \infty)$ be a function such that

$$\sum_{j=0}^{\infty} \frac{1}{2^j} \phi(-2^{j+1}(x+y), 2^j x, 2^j y) < K$$

for all $x, y \in A$ and some finite constant K .

Theorem 2.1. *Let a mapping $f : A \rightarrow X$ satisfy the inequality*

$$(2.1) \quad \|\lambda f(x) + 2\lambda f(y) + 2f(\lambda z)\| \leq \left\| 2f\left(\lambda\left(\frac{x}{2} + y + z\right)\right) \right\| + \phi(x, y, z)$$

for all $x, y, z \in A$ and all $\lambda \in \mathbb{T}$. Suppose that for each $a \in A$, mappings

$$(2.2) \quad f_a(x) = f(ax) - af(x), \quad {}_a f(x) = f(ax) - f(x)a$$

are continuous. Then there exists exactly one intertwining mapping $L : A \rightarrow X$ such that

$$(2.3) \quad \|L(x) - f(x)\| \leq \rho_1(x),$$

where

$$\begin{aligned} \rho_1(x) = & \sum_{j=0}^{\infty} \frac{1}{2^{j+1}} \left[\phi(-2^{j+1}x, 2^jx, 0) + \frac{1}{2} \phi(0, 2^{j+1}x, -2^{j+1}x) \right. \\ & \left. + \frac{11}{6} \phi(0, 0, 0) \right]. \end{aligned}$$

Proof. Fix $a \in A$. We demonstrate from (2.1) that

$$\begin{aligned} (2.4) \quad \|\lambda f_a(x) + 2\lambda f_a(y) + 2f_a(\lambda z)\| & \leq \left\| 2f\left(\lambda a\left(\frac{x}{2} + y + z\right)\right) \right\| \\ & + \|a\| \left\| 2f\left(\lambda\left(\frac{x}{2} + y + z\right)\right) \right\| + \phi(ax, ay, az) + \|a\| \phi(x, y, z). \end{aligned}$$

Set $\lambda = 1$ in (2.4). Then we get

$$\begin{aligned} (2.5) \quad \|f_a(x) + 2f_a(y) + 2f_a(z)\| & \leq \left\| 2f\left(a\left(\frac{x}{2} + y + z\right)\right) \right\| \\ & + \|a\| \left\| 2f\left(\frac{x}{2} + y + z\right) \right\| + \phi(ax, ay, az) + \|a\| \phi(x, y, z). \end{aligned}$$

Putting $\lambda = 1$ and $x = y = z = 0$ in (2.1) gives the inequality

$$\|f(0)\| \leq \frac{1}{3} \phi(0, 0, 0).$$

By letting $x = y = z = 0$ in (2.5), then we arrive at

$$\|f_a(0)\| \leq \left(\frac{1}{3} + \frac{1}{3}\|a\|\right) \phi(0, 0, 0).$$

And by letting $x = -2x, y = x$ and $z = 0$ in (2.5), we lead to

$$\begin{aligned} \|f_a(-2x) + 2f_a(x)\| & \leq \phi(-2ax, ax, 0) + \|a\| \phi(-2x, x, 0) \\ & + \left(\frac{4}{3} + \frac{4}{3}\|a\|\right) \phi(0, 0, 0). \end{aligned}$$

This implies that

$$\begin{aligned} (2.6) \quad \left\| \frac{1}{2} f_a(-2x) + f_a(x) \right\| & \leq \frac{1}{2} \phi(-2ax, ax, 0) + \frac{1}{2} \|a\| \phi(-2x, x, 0) \\ & + \left(\frac{2}{3} + \frac{2}{3}\|a\|\right) \phi(0, 0, 0). \end{aligned}$$

Also, by taking $x = 0, y = x$ and $z = -x$ in (2.5), we obtain

$$\begin{aligned} (2.7) \quad \|f_a(x) + f_a(-x)\| & \leq \frac{1}{2} \phi(0, ax, -ax) + \frac{1}{2} \|a\| \phi(0, x, -x) \\ & + \left(\frac{1}{2} + \frac{1}{2}\|a\|\right) \phi(0, 0, 0). \end{aligned}$$

Combine (2.6) and (2.7) to find that

$$\begin{aligned}
\left\| \frac{f_a(2^l x)}{2^l} - \frac{f_a(2^m x)}{2^m} \right\| &\leq \sum_{j=l}^{m-1} \left\| \frac{f_a(2^j x)}{2^j} - \frac{f_a(2^{j+1} x)}{2^{j+1}} \right\| \\
&\leq \sum_{j=l}^{m-1} \left[\left\| \frac{f_a(2^j x)}{2^j} + \frac{f_a(-2^{j+1} x)}{2^{j+1}} \right\| + \left\| \frac{f_a(2^{j+1} x)}{2^{j+1}} + \frac{f_a(-2^{j+1} x)}{2^{j+1}} \right\| \right] \\
&\leq \sum_{j=l}^{m-1} \frac{1}{2^{j+1}} \left[\phi(-2^{j+1} ax, 2^j ax, 0) + \frac{1}{2} \phi(0, 2^{j+1} ax, -2^{j+1} ax) \right. \\
&\quad \left. + \|a\| \phi(-2^{j+1} x, 2^j x, 0) + \frac{1}{2} \|a\| \phi(0, 2^{j+1} x, -2^{j+1} x) \right. \\
&\quad \left. + \left(\frac{11}{6} + \frac{11}{6} \|a\| \right) \phi(0, 0, 0) \right]
\end{aligned}$$

for all nonnegative integers m and l with $m > l$ and all $x \in A$. This means that the sequence $\{f_a(2^n x)/2^n\}$ is a Cauchy sequence. Since X is complete, the sequence $\{f_a(2^n x)/2^n\}$ converges. Thus we may define the mapping $L_a : A \rightarrow X$ by $L_a(x) := \lim_{n \rightarrow \infty} f_a(2^n x)/2^n$ for all $x \in A$. In particular, by setting $l = 0$ and passing the limit as $m \rightarrow \infty$, we have

$$(2.8) \quad \|f_a(x) - L_a(x)\| \leq \sigma_1(x),$$

where

$$\begin{aligned}
\sigma_1(x) &= \sum_{j=0}^{\infty} \frac{1}{2^{j+1}} \left[\phi(-2^{j+1} ax, 2^j ax, 0) + \frac{1}{2} \phi(0, 2^{j+1} ax, -2^{j+1} ax) \right. \\
&\quad \left. + \|a\| \phi(-2^{j+1} x, 2^j x, 0) + \frac{1}{2} \|a\| \phi(0, 2^{j+1} x, -2^{j+1} x) \right. \\
&\quad \left. + \left(\frac{11}{6} + \frac{11}{6} \|a\| \right) \phi(0, 0, 0) \right].
\end{aligned}$$

First of all, we note due to (2.7) that

$$\begin{aligned}
\|L_a(x) + L_a(-x)\| &= \lim_{n \rightarrow \infty} \frac{1}{2^n} \|f_a(2^n x) + f_a(-2^n x)\| \\
&\leq \lim_{n \rightarrow \infty} \left[\frac{1}{2^{n+1}} \phi(0, 2^n ax, -2^n ax) + \frac{1}{2^{n+1}} \|a\| \phi(0, 2^n x, -2^n x) \right. \\
&\quad \left. + \frac{1}{2^n} \left(\frac{1}{2} + \frac{1}{2} \|a\| \right) \phi(0, 0, 0) \right] = 0,
\end{aligned}$$

from the assumption of ϕ . So we have $L_a(-x) = -L_a(x)$.

Now, we claim that the mapping $L_a : A \rightarrow X$ is an additive mapping :
Use (2.5), (2.6) and (2.7) to obtain that

$$\begin{aligned}
& \|L_a(x) + L_a(y) - L_a(x+y)\| \\
&= \lim_{n \rightarrow \infty} \frac{1}{2^n} \|f_a(2^n x) + f_a(2^n y) + f_a(2^n(-x-y))\| \\
&\leq \lim_{n \rightarrow \infty} \left\{ \frac{1}{2^n} \left\| \frac{1}{2} f_a(2^{n+1}(-x-y)) + f_a(2^n x) + f_a(2^n y) \right\| \right. \\
&\quad + \frac{1}{2^n} \left\| f_a(2^n(-x-y)) + \frac{1}{2} f_a(2^{n+1}(x+y)) \right\| \\
&\quad \left. + \frac{1}{2^{n+1}} \|f_a(2^{n+1}(-x-y)) + f_a(2^{n+1}(x+y))\| \right\} \\
&\leq \lim_{n \rightarrow \infty} \left\{ \frac{1}{2^{n+1}} \left[\phi(2^{n+1}a(-x-y), 2^n ax, 2^n ay) \right. \right. \\
&\quad + \|a\| \phi(2^{n+1}(-x-y), 2^n x, 2^n y) \Big] \\
&\quad + \frac{1}{2^{n+1}} \left[\phi(2^{n+1}a(x+y), 2^n a(-x-y), 0) \right. \\
&\quad + \frac{1}{2} \phi(0, 2^{n+1}a(-x-y), 2^{n+1}a(x+y)) \Big] \\
&\quad + \frac{1}{2^{n+1}} \|a\| \left[\phi(2^{n+1}(x+y), 2^n(-x-y), 0) \right. \\
&\quad + \frac{1}{2} \phi(0, 2^{n+1}(-x-y), 2^{n+1}(x+y)) \Big] \\
&\quad \left. + \frac{1}{2^n} \left(\frac{5}{4} + \frac{5}{4} \|a\| \right) \phi(0, 0, 0) \right\} = 0,
\end{aligned}$$

from the assumption of ϕ .

In order to prove that the mapping $L_a(x)$ is unique, let us assume that $S_a : A \rightarrow X$ be another additive mapping satisfying (2.8). Then it follows that

$$\begin{aligned}
\|L_a(x) - S_a(x)\| &= \frac{1}{2^n} \|L_a(2^n x) - S_a(2^n x)\| \\
&\leq \frac{1}{2^n} [\|L_a(2^n x) - f_a(2^n x)\| + \|S_a(2^n x) - f_a(2^n x)\|] \\
&\leq \frac{1}{2^{n-1}} \sigma_1(2^n x) \rightarrow 0
\end{aligned}$$

as $n \rightarrow \infty$, for all $x \in A$, because

$$\begin{aligned} & \lim_{n \rightarrow \infty} \frac{1}{2^{n-1}} \sigma_1(2^n x) \\ &= \lim_{n \rightarrow \infty} \sum_{j=n}^{\infty} \frac{1}{2^{j+1}} \left[\phi(-2^{j+1}ax, 2^j ax, 0) + \frac{1}{2} \phi(0, 2^{j+1}ax, -2^{j+1}ax) \right. \\ & \quad + \|a\| \phi(-2^{j+1}x, 2^j x, 0) + \frac{1}{2} \|a\| \phi(0, 2^{j+1}x, -2^{j+1}x) \\ & \quad \left. + \left(\frac{11}{6} + \frac{11}{6} \|a\| \right) \phi(0, 0, 0) \right] = 0. \end{aligned}$$

Consequently, we conclude that there exists exactly one additive mapping $L_a : A \rightarrow X$ satisfying (2.8).

On the other hand, we have by letting $x = 0, y = x$ and $z = -x$ in (2.5) that

$$\begin{aligned} (2.9) \quad & \|\lambda f_a(x) + f_a(-\lambda x)\| \leq \frac{1}{2} \phi(0, ax, -ax) + \frac{1}{2} \|a\| \phi(0, x, -x) \\ & + \left(\frac{1}{2} + \frac{1}{2} \|a\| \right) \phi(0, 0, 0). \end{aligned}$$

Replace x by $2^n x$ in (2.9) and divide by 2^n to get

$$\begin{aligned} & \left\| \lambda \frac{f_a(2^n x)}{2^n} + \frac{f_a(-2^n \lambda x)}{2^n} \right\| \leq \frac{1}{2^{n+1}} \phi(0, 2^n ax, -2^n ax) \\ & + \frac{1}{2^{n+1}} \|a\| \phi(0, 2^n x, -2^n x) + \frac{1}{2^n} \left(\frac{1}{2} + \frac{1}{2} \|a\| \right) \phi(0, 0, 0). \end{aligned}$$

By taking $n \rightarrow \infty$, we find that $L_a(\lambda x) = \lambda L_a(x)$ for all $x \in A$ and all $\lambda \in \mathbb{T}$, by assumption of ϕ . Here and now, let $\mu \in \mathbb{C}$ be a nonzero and M an integer greater than $4|\mu|$. Then $|\frac{\mu}{M}| < \frac{1}{4} < 1 - \frac{2}{3}$. According to the theorem in [12], we see that there exist elements $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{T}$ such that $3\frac{\mu}{M} = \lambda_1 + \lambda_2 + \lambda_3$. Observe that $L_a(\frac{1}{3}x) = \frac{1}{3}L_a(x)$ for all $x \in A$. Thus it follows that

$$\begin{aligned} L_a(\mu x) &= L_a\left(\frac{M}{3} \cdot 3\frac{\mu}{M}x\right) = M L_a\left(\frac{1}{3} \cdot 3\frac{\mu}{M}x\right) = \frac{M}{3} L_a\left(3\frac{\mu}{M}x\right) \\ &= \frac{M}{3} L_a(\lambda_1 x + \lambda_2 x + \lambda_3 x) = \frac{M}{3} (L_a(\lambda_1 x) + L_a(\lambda_2 x) + L_a(\lambda_3 x)) \\ &= \frac{M}{3} (\lambda_1 + \lambda_2 + \lambda_3) L_a(x) = \frac{M}{3} \cdot 3\frac{\mu}{M} \cdot L_a(x) = \mu L_a(x) \end{aligned}$$

for all $x \in A$. Hence, by the additivity of L_a , we obtain that

$$L_a(\zeta x + \eta y) = \zeta L_a(x) + \eta L_a(y)$$

for all $x \in A$ and for all $\zeta, \eta \in \mathbb{C}$ with $\zeta \neq 0, \eta \neq 0$. In particular, $L_a(0x) = 0 = 0L_a(x)$. Therefore, L_a is a linear mapping.

Now, we need to show that the mapping L_a is continuous: Suppose that f_a is continuous at a point $x_0 \in A$ and there exists a point such that L_a is not continuous at this point. Then there exist a positive number η and a sequence $\{x_n\}$ in A such that $\lim_{n \rightarrow \infty} x_n = 0$ and $\|L_a(x_n)\| > \eta$. Set $\epsilon > \frac{3\alpha}{\eta}$, where $\sigma_1(x) < \alpha$ for all $x \in A$. Note that

$$\|L_a(\epsilon x_n + x_0) - L_a(x_0)\| = \|L_a(\epsilon x_n)\| > 3\alpha$$

for all n . Since $\lim_{n \rightarrow \infty} f_a(\epsilon x_n + x_0) = f_a(x_0)$, there is an integer N such that for all $n > N$

$$\begin{aligned} \|L_a(\epsilon x_n + x_0) - L_a(x_0)\| &\leq \|L_a(\epsilon x_n + x_0) - f_a(\epsilon x_n + x_0)\| \\ &\quad + \|f_a(\epsilon x_n + x_0) - f_a(x_0)\| + \|f_a(x_0) - L_a(x_0)\| < 3\alpha, \end{aligned}$$

which is a contradiction.

Let $\lambda = 1$ in (2.1). Following the similar method as the previous proof, there is exactly one additive mapping $L : A \rightarrow X$ satisfying the inequality (2.3), where $L(x) := \lim_{n \rightarrow \infty} f(2^n x)/2^n$ for all $x \in A$.

We also imply by putting $x = 0, y = x$ and $z = -x$ in (2.1) that

$$(2.10) \quad \|\lambda f(x) + f(-\lambda x)\| \leq \frac{1}{2}\phi(0, x, -x) + \frac{1}{2}\phi(0, 0, 0).$$

Replace x by $2^n x$ in (2.10) and divide by 2^n to get

$$\left\| \lambda \frac{f(2^n x)}{2^n} + \frac{f(-2^n \lambda x)}{2^n} \right\| \leq \frac{1}{2^{n+1}}\phi(0, 2^n x, -2^n x) + \frac{1}{2^{n+1}}\phi(0, 0, 0).$$

Taking the limit as $n \rightarrow \infty$, we have $L(\lambda x) = \lambda L(x)$ for all $x \in A$ and all $\lambda \in \mathbb{T}$, from assumption of ϕ . By the same reasoning as the above proof, L is a linear mapping.

Let us replace x by $2^n x$ in (2.2) and then divide by 2^n . Then, by utilizing the definition of L_a and L , we yield that

$$L_a(x) = \lim_{n \rightarrow \infty} \frac{f_a(2^n x)}{2^n} = \lim_{n \rightarrow \infty} \left[\frac{f(2^n a x)}{2^n} - a \frac{f(2^n x)}{2^n} \right] = L(ax) - aL(x).$$

In similar manner, one can show that for each $a \in A$, a linear mapping ${}_a L : A \rightarrow X$ defined by ${}_a L(x) := \lim_{n \rightarrow \infty} {}_a f(2^n x)/2^n$ is continuous on A and ${}_a L(x) = L(ax) - L(x)a$.

Thus L is an intertwining mapping. The proof of Theorem is complete. $\square$

Corollary 2.2. *Let $p, q < 1$ and $r < 1$ be real number and let a mapping $f : A \rightarrow X$ satisfy the inequality*

$$\begin{aligned} \|\lambda f(x) + 2\lambda f(y) + 2f(\lambda z)\| &\leq \left\| 2f\left(\lambda\left(\frac{x}{2} + y + z\right)\right) \right\| \\ &+ \theta(\|x\|^p + \|y\|^q + \|z\|^r) \end{aligned}$$

for all $x, y, z \in A$ and all $\lambda \in \mathbb{T}$. Suppose that for each $a \in A$, the mappings (2.2) are continuous. Then there exists exactly one intertwining mapping $L : A \rightarrow X$ such that

$$\|L(x) - f(x)\| \leq \theta \left[\frac{2^p}{2 - 2^p} \|x\|^p + \frac{1 + 2^{q-1}}{2 - 2^q} \|x\|^q + \frac{2^r}{2 - 2^r} \|x\|^r \right].$$

Corollary 2.3. *Let a mapping $f : A \rightarrow X$ satisfy the inequality*

$$\|\lambda f(x) + 2\lambda f(y) + 2f(\lambda z)\| \leq \left\| 2f\left(\lambda\left(\frac{x}{2} + y + z\right)\right) \right\| + \theta$$

for all $x, y, z \in A$ and all $\lambda \in \mathbb{T}$. Suppose that for each $a \in A$, the mappings (2.2) are continuous. Then there exists exactly one intertwining mapping $L : A \rightarrow X$ such that

$$\|L(x) - f(x)\| \leq \frac{10}{3}\theta.$$

Theorem 2.4. *Let a mapping $f : A \rightarrow X$ satisfy the inequality (2.1). Suppose that for each $a \in A$, the mappings (2.2) are continuous. Then there exists exactly one intertwining mapping $L : A \rightarrow X$ such that*

$$(2.11) \quad \left\| L(x) - \frac{f(x) - f(-x)}{2} \right\| \leq \rho_2(x),$$

where

$$\rho_2(x) = \sum_{j=0}^{\infty} \frac{1}{2^{j+2}} \left[\phi(-2^{j+1}x, 2^jx, 0) + \phi(2^{j+1}x, -2^jx, 0) + \frac{8}{3}\phi(0, 0, 0) \right].$$

Proof. For each $a \in A$, we define a mapping $g_a : A \rightarrow X$ by

$$g_a(x) = \frac{f_a(x) - f_a(-x)}{2}$$

for all $x \in A$. Due to (2.6), we obtain

$$(2.12) \quad \begin{aligned} \left\| \frac{1}{2}f_a(x) + f_a\left(-\frac{x}{2}\right) \right\| &\leq \frac{1}{2}\phi\left(ax, -\frac{ax}{2}, 0\right) + \frac{1}{2}\|a\|\phi\left(x, -\frac{x}{2}, 0\right) \\ &+ \left(\frac{2}{3} + \frac{2}{3}\|a\|\right)\phi(0, 0, 0). \end{aligned}$$

So we have by (2.12)

$$\begin{aligned} \left\| \frac{g_a(2^l x)}{2^l} - \frac{g_a(2^m x)}{2^m} \right\| &\leq \sum_{j=l}^{m-1} \left\| \frac{g_a(2^j x)}{2^j} - \frac{g_a(2^{j+1} x)}{2^{j+1}} \right\| \\ &\leq \sum_{j=l}^{m-1} \frac{1}{2^{j+2}} \left[\phi(-2^{j+1} ax, 2^j ax, 0) + \phi(2^{j+1} ax, -2^j ax, 0) \right. \\ &\quad \left. + \|a\| \phi(-2^{j+1} x, 2^j x, 0) + \|a\| \phi(2^{j+1} x, -2^j x, 0) \right. \\ &\quad \left. + \left(\frac{8}{3} + \frac{8}{3} \|a\| \right) \phi(0, 0, 0) \right] \end{aligned}$$

for all nonnegative integers m and l with $m > l$ and all $x \in A$. It means that the sequence $\{g_a(2^n x)/2^n\}$ is a Cauchy sequence. Since X is complete, the sequence $\{g_a(2^n x)/2^n\}$ converges. So we can define the mapping $L_a : A \rightarrow X$ by $L_a(x) := \lim_{n \rightarrow \infty} g_a(2^n x)/2^n$ for all $x \in A$. Furthermore, by letting $l = 0$ and passing the limit as $m \rightarrow \infty$, we get

$$(2.13) \quad \|g_a(x) - L_a(x)\| \leq \sigma_2(x),$$

where

$$\begin{aligned} \sigma_2(x) &= \sum_{j=0}^{\infty} \frac{1}{2^{j+2}} \left[\phi(-2^{j+1} ax, 2^j ax, 0) + \phi(2^{j+1} ax, -2^j ax, 0) \right. \\ &\quad \left. + \|a\| \phi(-2^{j+1} x, 2^j x, 0) + \|a\| \phi(2^{j+1} x, -2^j x, 0) + \left(\frac{8}{3} + \frac{8}{3} \|a\| \right) \phi(0, 0, 0) \right]. \end{aligned}$$

Next, we claim that the mapping $L_a : A \rightarrow X$ is an additive mapping: We first note that $L_a(-x) = -L_a(x)$, because g_a is an odd function. If use the same argument as in the proof of Theorem 2.1, then we deduce

$$\lim_{n \rightarrow \infty} \frac{1}{2^n} \|f_a(2^n x) + f_a(2^n y) + f_a(2^n(-x - y))\| = 0.$$

From this, we can show that

$$\begin{aligned} &\|L_a(x) + L_a(y) - L_a(x + y)\| \\ &= \lim_{n \rightarrow \infty} \frac{1}{2^n} \|g_a(2^n x) + g_a(2^n y) + g_a(2^n(-x - y))\| \\ &\leq \lim_{n \rightarrow \infty} \left[\frac{1}{2^{n+1}} \|f_a(2^n x) + f_a(2^n y) + f_a(2^n(-x - y))\| \right. \\ &\quad \left. + \frac{1}{2^{n+1}} \|f_a(-2^n x) + f_a(-2^n y) + f_a(2^n(x + y))\| \right] = 0. \end{aligned}$$

Now, to prove the uniqueness of the mapping $L_a(x)$, let us assume that $T_a : A \rightarrow X$ be another additive mapping satisfying (2.13). Then

we find that

$$\begin{aligned}
& \lim_{n \rightarrow \infty} \frac{1}{2^{n-1}} \sigma_2(2^n x) \\
&= \lim_{n \rightarrow \infty} 2 \sum_{j=n}^{\infty} \frac{1}{2^{j+2}} \left[\phi(-2^{j+1}ax, 2^j ax, 0) + \phi(2^{j+1}ax, -2^j ax, 0) \right. \\
&\quad + \|a\| \phi(-2^{j+1}x, 2^j x, 0) + \|a\| \phi(2^{j+1}x, -2^j x, 0) \\
&\quad \left. + \left(\frac{8}{3} + \frac{8}{3} \|a\| \right) \phi(0, 0, 0) \right] = 0,
\end{aligned}$$

which implies that

$$\begin{aligned}
\|L_a(x) - T_a(x)\| &= \frac{1}{2^n} \|L_a(2^n x) - T_a(2^n x)\| \\
&\leq \frac{1}{2^n} [\|L_a(2^n x) - f_a(2^n x)\| + \|T_a(2^n x) - f_a(2^n x)\|] \\
&\leq \frac{1}{2^{n-1}} \sigma_2(2^n x) \rightarrow 0
\end{aligned}$$

as $n \rightarrow \infty$, for all $x \in A$.

So that there exists exactly one additive mapping $L_a : A \rightarrow X$ satisfying (2.13).

By (2.9), we have

$$\begin{aligned}
& \|\lambda g_a(x) + g_a(-\lambda x)\| \\
&\leq \frac{1}{2} \|\lambda f_a(x) + f_a(-\lambda x)\| + \frac{1}{2} \|\lambda f_a(-x) + f_a(\lambda x)\| \\
&\leq \frac{1}{4} \phi(0, ax, -ax) + \frac{1}{4} \|a\| \phi(0, x, -x) + \frac{1}{4} \phi(0, -ax, ax) \\
&\quad + \frac{1}{4} \|a\| \phi(0, -x, x) + \left(\frac{1}{2} + \frac{1}{2} \|a\| \right) \phi(0, 0, 0).
\end{aligned}$$

Substitute $x := 2^n x$ in the above inequality and multiply $\frac{1}{2^n}$ to obtain

$$\begin{aligned}
& \left\| \lambda \frac{g_a(2^n x)}{2^n} + \frac{g_a(-2^n \lambda x)}{2^n} \right\| \\
&\leq \frac{1}{2^{n+2}} \phi(0, 2^n ax, -2^n ax) + \frac{1}{2^{n+2}} \|a\| \phi(0, 2^n x, -2^n x) \\
&\quad + \frac{1}{2^{n+2}} \phi(0, -2^n ax, 2^n ax) + \frac{1}{2^{n+2}} \|a\| \phi(0, -2^n x, 2^n x) \\
&\quad + \frac{1}{2^n} \left(\frac{1}{2} + \frac{1}{2} \|a\| \right) \phi(0, 0, 0).
\end{aligned}$$

In this inequality, let $n \rightarrow \infty$. Then $L_a(\lambda x) = \lambda L_a(x)$, where $x \in A$ and $\lambda \in \mathbb{T}$. As the proof of Theorem 2.1, we can show that L_a is a linear mapping.

On the other hand, continuity of f_a at $x_0 \in A$ guarantees continuity of f_a at $-x_0 \in A$, since f_a is continuous on A . Thus, by the definition of g_a , we see that g_a is continuous at $x_0 \in A$.

It remains to show that L_a is continuous at $x_0 \in A$. If we use the contrapositive, then, as in the proof of Theorem 2.1, we can prove that L_a is continuous at $x_0 \in A$.

Put $\lambda = 1$ in (2.1). Similarly, as we did in the precedent proof, there exists exactly one additive mapping $L : A \rightarrow X$ satisfying the inequality (2.11), where

$$L(x) = \lim_{n \rightarrow \infty} \frac{g(2^n x)}{2^n}, \quad g(x) = \frac{f(x) - f(-x)}{2}$$

for all $x \in A$.

Now it follows from (2.10) that

(2.14)

$$\begin{aligned} \|\lambda g(x) + g(-\lambda x)\| &\leq \frac{1}{2} \|\lambda f(x) + f(-\lambda x)\| + \frac{1}{2} \|\lambda f(-x) + f(\lambda x)\| \\ &\leq \frac{1}{4} \phi(0, x, -x) + \frac{1}{4} \phi(0, -x, x) + \frac{1}{2} \phi(0, 0, 0). \end{aligned}$$

Replacing x by $2^n x$ in (2.14) and dividing by 2^n , we yield that

$$\begin{aligned} \left\| \lambda \frac{g(2^n x)}{2^n} + \frac{g(-2^n \lambda x)}{2^n} \right\| &\leq \frac{1}{2^{n+2}} \phi(0, 2^n x, -2^n x) + \frac{1}{2^{n+2}} \phi(0, -2^n x, 2^n x) \\ &\quad + \frac{1}{2^{n+1}} \phi(0, 0, 0). \end{aligned}$$

Passing the limit, we see that $L(\lambda x) = \lambda L(x)$ for all $x \in A$ and all $\lambda \in \mathbb{T}$, by assumption of ϕ . By the same way as the proof of Theorem 2.1, we have

$$L(\zeta x + \eta y) = \zeta L(x) + \eta L(y)$$

for all $x \in A$ and all $\zeta, \eta \in \mathbb{C}$, that is, L is a linear mapping.

Replacing x by $2^n x$ in (2.2) and then dividing by 2^n , it follows from the definition of L_a and L that

$$L_a(x) = \lim_{n \rightarrow \infty} \frac{g_a(2^n x)}{2^n} = \lim_{n \rightarrow \infty} \left[\frac{g(2^n a x)}{2^n} - a \frac{g(2^n x)}{2^n} \right] = L(ax) - aL(x).$$

In similar fashion, we can show that for each $a \in A$, the linear mapping ${}_a L : A \rightarrow X$ defined by ${}_a L(x) := \lim_{n \rightarrow \infty} \frac{1}{2^n} g_a(2^n x)$ is continuous on A and ${}_a L = L(ax) - L(x)a$.

So L is an intertwining mapping. This completes the proof of the theorem. $\square$

Corollary 2.5. *Let $p, q < 1$ and $r < 1$ be real number and let a mapping $f : A \rightarrow X$ satisfy the inequality*

$$\begin{aligned} \|\lambda f(x) + 2\lambda f(y) + 2f(\lambda z)\| &\leq \left\| 2f\left(\lambda\left(\frac{x}{2} + y + z\right)\right) \right\| \\ &\quad + \theta(\|x\|^p + \|y\|^q + \|z\|^r) \end{aligned}$$

for all $x, y, z \in A$ and all $\lambda \in \mathbb{T}$. Suppose that for each $a \in A$, the mappings (2.2) are continuous. Then there exists exactly one intertwining mapping $L : A \rightarrow X$ such that

$$\left\| L(x) - \frac{f(x) - f(-x)}{2} \right\| \leq \theta \left[\frac{2^p}{2 - 2^p} \|x\|^p + \frac{2^q}{2 - 2^q} \|x\|^q \right].$$

Corollary 2.6. *Let a mapping $f : A \rightarrow X$ satisfy the inequality*

$$\|\lambda f(x) + 2\lambda f(y) + 2f(\lambda z)\| \leq \left\| 2f\left(\lambda\left(\frac{x}{2} + y + z\right)\right) \right\| + \theta$$

for all $x, y, z \in A$ and all $\lambda \in \mathbb{T}$. Suppose that for each $a \in A$, the mappings (2.2) are continuous. Then there exists exactly one intertwining mapping $L : A \rightarrow X$ such that

$$\left\| L(x) - \frac{f(x) - f(-x)}{2} \right\| \leq \frac{7}{3}\theta.$$

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