POSITIVE INTERPOLATION ON

$Ax = y$ AND $AX = Y$ IN $\text{Alg}\mathcal{L}$

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ABSTRACT. Let $\mathcal{L}$ be a subspace lattice on a Hilbert space $\mathcal{H}$.

Let x and y be vectors in $\mathcal{H}$ and let P_x be the projection onto $sp(x)$. If $P_x E = E P_x$ for each $E \in \mathcal{L}$, then the following are equivalent.

(1) There exists an operator A in $\text{Alg}\mathcal{L}$ such that $Ax = y$, $Af = 0$ for all f in $sp(x)^\perp$ and $A \geq 0$.

(2) $\sup \left\{ \frac{\|E^\perp y\|}{\|E^\perp x\|} : E \in \mathcal{L} \right\} < \infty$ and $\langle x, y \rangle \geq 0$.

Let X and Y be operators in $\mathcal{B}(\mathcal{H})$. Let P be the projection onto $\overline{\text{range} X}$. If $PE = EP$ for each $E \in \mathcal{L}$, then the following are equivalent:

(1) $\sup \left\{ \frac{\|E^\perp Yf\|}{\|E^\perp Xf\|} : f \in \mathcal{H}, E \in \mathcal{L} \right\} < \infty$ and $\langle Xf, Yf \rangle \geq 0$ for all f in $\mathcal{H}$.

(2) There exists a positive operator A in $\text{Alg}\mathcal{L}$ such that $AX = Y$.

1. Introduction

Let $\mathcal{C}$ be a collection of operators acting on a Hilbert space $\mathcal{H}$ and let x and y be vectors on $\mathcal{H}$. An *interpolation question* for $\mathcal{C}$ asks for which x and y is there a bounded operator $A \in \mathcal{C}$ such that $Ax = y$. Given operators X and Y acting on a Hilbert space $\mathcal{H}$, an interpolating operator is a bounded operator A such that $AX = Y$. A variation, the ‘ n -vector interpolation problem’, asks for an operator A such that $Ax_i = y_i$ for fixed finite collections $\{x_1, x_2, \dots, x_n\}$ and $\{y_1, y_2, \dots, y_n\}$. The n -vector interpolation problem was considered for a C^* -algebra $\mathcal{U}$ by Kadison[8]. In case $\mathcal{U}$ is a nest algebra, the

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(one-vector) interpolation problem was solved by Lance[9]: his result was extended by Hopenwasser[4] to the case that $\mathcal{U}$ is a CSL-algebra. Munch[10] obtained conditions for interpolation in case A is required to lie in the ideal of Hilbert-Schmidt operators in a nest algebra. Hopenwasser[5] once again extended the interpolation condition to the ideal of Hilbert-Schmidt operators in a CSL-algebra. Hopenwasser's paper also contains a sufficient condition for interpolation n -vectors. We[6] obtained conditions for interpolation in the case A is in $\text{Alg}\mathcal{L}$ when $\mathcal{L}$ is a CSL. Again we [7] showed an interpolation condition to the case that $\mathcal{L}$ is a subspace lattice.

We establish some notations and conventions. Let $\mathcal{H}$ be a Hilbert space. A *subspace lattice* $\mathcal{L}$ is a strongly closed lattice of orthogonal projections on $\mathcal{H}$. A *commutative subspace lattice* $\mathcal{L}$, or CSL $\mathcal{L}$ is a subspace lattice whose elements commute each other. We assume that the projections 0 and I lie in $\mathcal{L}$. We usually identify projections and their ranges, so that it makes sense to speak of an operator as leaving a projection invariant. Let $\mathcal{L}$ be a subspace lattice on a Hilbert space $\mathcal{H}$. Then $\text{Alg}\mathcal{L}$ denotes the algebra of bounded operators on $\mathcal{H}$ that leave invariant every projection in $\mathcal{L}$; $\text{Alg}\mathcal{L}$ is a weakly closed subalgebra of the algebra $\mathcal{B}(\mathcal{H})$ of all bounded operators acting on $\mathcal{H}$. Let x and y be vectors in $\mathcal{H}$. Then $\langle x, y \rangle$ means the inner product of vectors x and y . Let $\mathbb{N}$ be the set of all natural numbers and let $\mathbb{C}$ be the set of all complex numbers. In this paper, we use the convention $\frac{0}{0} = 0$, when necessary.

2. $Ax = y$ in $\text{Alg}\mathcal{L}$

Let $\mathcal{H}$ be a Hilbert space and $\mathcal{L}$ be a subspace lattice of orthogonal projections acting on $\mathcal{H}$ containing 0 and I . Let $\mathcal{M}$ be a subset of a Hilbert space $\mathcal{H}$. Then $\overline{\mathcal{M}}$ means the closure of $\mathcal{M}$, $\mathcal{M}^\perp$ the orthogonal complement of $\mathcal{M}$ and $[\mathcal{M}]$ the closed subspace of $\mathcal{H}$ generated by $\mathcal{M}$. Let x be a vector of $\mathcal{H}$. Then $sp(x) = \{\alpha x \mid \alpha \in \mathbb{C}\}$. An operator A in $\mathcal{B}(\mathcal{H})$ is *positive*, $A \geq 0$, if $\langle Ax, x \rangle \geq 0$ for all x in $\mathcal{H}$.

THEOREM 2.1. *Let $\mathcal{L}$ be a subspace lattice on a Hilbert space $\mathcal{H}$. Let x and y be vectors in $\mathcal{H}$ and let P_x be the projection onto $sp(x)$. If $P_x E = E P_x$ for each $E \in \mathcal{L}$, then the following are equivalent.*

- (1) *There exists an operator A in $\text{Alg}\mathcal{L}$ such that $Ax = y$, $Af = 0$ for all f*

in $\text{sp}(x)^\perp$ and $A \geq 0$.

$$(2) \sup \left\{ \frac{\|E^\perp y\|}{\|E^\perp x\|} : E \in \mathcal{L} \right\} < \infty \text{ and } \langle x, y \rangle \geq 0.$$

Proof. (1) $\Rightarrow$ (2). If we assume that (1) holds, then $\sup \left\{ \frac{\|E^\perp y\|}{\|E^\perp x\|} : E \in \mathcal{L} \right\} < \infty$

by Theorem 3.4 [7]. Since $A \geq 0$, $\langle x, y \rangle = \langle x, Ax \rangle \geq 0$.

(2) $\Rightarrow$ (1). If $\sup \left\{ \frac{\|E^\perp y\|}{\|E^\perp x\|} : E \in \mathcal{L} \right\} < \infty$, then there exists an operator A in

$\text{Alg}\mathcal{L}$ such that $Ax = y$, $Af = 0$ for all f in $\text{sp}(x)^\perp$ by Theorem 3.4 [7]. Since $\langle x, y \rangle \geq 0$, $\langle x, Ax \rangle \geq 0$. Since $Af = 0$ for all $f \in \text{sp}(x)^\perp$, $\langle g, Ag \rangle \geq 0$ for all $g \in \mathcal{H}$. So $A \geq 0$. $\square$

THEOREM 2.2. Let $\mathcal{L}$ be a subspace lattice on a Hilbert space $\mathcal{H}$ and $x_1, \dots, x_n, y_1, \dots, y_n$ be vectors in $\mathcal{H}$. Let $\mathcal{M} = \left\{ \sum_{i=1}^n \alpha_i x_i : \alpha_i \in \mathbb{C} \right\}$ and $P_{\mathcal{M}}$ be the projection onto $\mathcal{M}$. If $P_{\mathcal{M}}E = EP_{\mathcal{M}}$ for each $E \in \mathcal{L}$, then the following are equivalent.

(1) There is an operator A in $\text{Alg}\mathcal{L}$ such that $y_i = Ax_i (i = 1, 2, \dots, n)$, $Ag = 0$ for all g in $\mathcal{M}^\perp$ and $A \geq 0$.

(2) $\sup \left\{ \frac{\|E^\perp(\sum_{i=1}^n \alpha_i y_i)\|}{\|E^\perp(\sum_{i=1}^n \alpha_i x_i)\|} : \alpha_i \in \mathbb{C} \text{ and } E \in \mathcal{L} \right\} < \infty$ and $\langle x_k, y_k \rangle \geq 0$ for

all $k = 1, 2, \dots, n$.

Proof. (1) $\Rightarrow$ (2). If we assume that (1) holds, then

$$\sup \left\{ \frac{\|E^\perp(\sum_{i=1}^n \alpha_i y_i)\|}{\|E^\perp(\sum_{i=1}^n \alpha_i x_i)\|} : \alpha_i \in \mathbb{C} \text{ and } E \in \mathcal{L} \right\} < \infty \text{ by Theorem 3.5 [7]. Since } A \geq 0,$$

$\langle x_k, y_k \rangle = \langle x_k, Ax_k \rangle \geq 0$ for all $k = 1, 2, \dots, n$.

(2) $\Rightarrow$ (1). If $\sup \left\{ \frac{\|E^\perp(\sum_{i=1}^n \alpha_i y_i)\|}{\|E^\perp(\sum_{i=1}^n \alpha_i x_i)\|} : \alpha_i \in \mathbb{C} \text{ and } E \in \mathcal{L} \right\} < \infty$, then there exists

an operator A in $\text{Alg}\mathcal{L}$ such that $Ax_k = y_k$ for all $k = 1, 2, \dots, n$ and $Af = 0$ for all f in $\mathcal{M}^\perp$ by Theorem 3.5 [7]. Since $\langle x_k, y_k \rangle \geq 0$, $\langle x_k, Ax_k \rangle \geq 0$ for all $k = 1, 2, \dots, n$. Since $Af = 0$ for all $f \in \mathcal{M}^\perp$, $\langle g, Ag \rangle \geq 0$ for all $g \in \mathcal{H}$. So $A \geq 0$. $\square$

With the similar proof as Theorem 2.2, we can get the following theorem.

THEOREM 2.3. *Let $\mathcal{L}$ be a subspace lattice on a Hilbert space $\mathcal{H}$ and $\{x_n\}$ and $\{y_n\}$ be two infinite sequences of vectors in $\mathcal{H}$. Let*

$\mathcal{N} = \left\{ \sum_{i=1}^n \alpha_i x_i : \alpha_i \in \mathbb{C}, n \in \mathbb{N} \right\}$ and $P_{\overline{\mathcal{N}}}$ be the projection onto $\overline{\mathcal{N}}$. If $P_{\overline{\mathcal{N}}}E = EP_{\overline{\mathcal{N}}}$ for each $E \in \mathcal{L}$, then the following are equivalent.

(1) *There is an operator A in $\text{Alg}\mathcal{L}$ such that $y_i = Ax_i (i = 1, 2, \dots)$, $Ag = 0$ for all g in $\mathcal{N}^\perp$ and $A \geq 0$.*

(2) $\sup \left\{ \frac{\|E^\perp(\sum_{i=1}^n \alpha_i y_i)\|}{\|E^\perp(\sum_{i=1}^n \alpha_i x_i)\|} : n \in \mathbb{N}, \alpha_i \in \mathbb{C} \text{ and } E \in \mathcal{L} \right\} < \infty$ and $\langle x_k, y_k \rangle \geq 0$

for all $k = 1, 2, \dots$.

3. $AX = Y$ in $\text{Alg}\mathcal{L}$

Let $\mathcal{L}$ be a subspace lattice (i.e. a complete lattice of orthogonal projections which contains 0 and I) on $\mathcal{H}$. $\text{Alg}\mathcal{L}$ is the algebra of all bounded operators on $\mathcal{H}$ which leave invariant each projection E in $\mathcal{L}$. Assume that X and Y are operators in $\mathcal{B}(\mathcal{H})$ and A is an operator in $\text{Alg}\mathcal{L}$ such that $AX = Y$. Then $\|E^\perp Yf\| = \|E^\perp AXf\| = \|E^\perp AE^\perp Xf\| \leq \|A\| \|E^\perp Xf\|$, for all $E \in \mathcal{L}$. If, for convenience, we adopt the convention that a fraction whose numerator and denominator are both zero is equal to zero, then the inequalities above may be stated in the form

$$\sup_{E \in \mathcal{L}} \frac{\|E^\perp Yf\|}{\|E^\perp Xf\|} \leq \|A\|.$$

In [7], we showed that the above fact is a necessary and sufficient condition for existence of an interpolating operator in $\text{Alg}\mathcal{L}$.

THEOREM A[7]. *Let $\mathcal{L}$ be a subspace lattice on $\mathcal{H}$ and let X and Y be operators in $\mathcal{B}(\mathcal{H})$. Let P be the projection onto $\overline{\text{range}X}$. If $PE = EP$ for every $E \in \mathcal{L}$, then the following are equivalent:*

(1) There exists an operator A in $\text{Alg}\mathcal{L}$ such that $AX = Y$.

(2) $\sup \left\{ \frac{\|E^\perp Yf\|}{\|E^\perp Xf\|} : f \in \mathcal{H}, E \in \mathcal{L} \right\} = K < \infty$.

Moreover, if condition (2) holds, we may choose an operator A such that $\|A\| = K$.

Now, we are going to investigate positive interpolation problems in $\text{Alg}\mathcal{L}$.

THEOREM 3.1. *Let $\mathcal{L}$ be a subspace lattice acting on a Hilbert space $\mathcal{H}$ and let X and Y be operators in $\mathcal{B}(\mathcal{H})$. Let P be the projection onto the $\overline{\text{range} X}$. If $PE = EP$ for each $E \in \mathcal{L}$, then the following are equivalent:*

(1) $\sup \left\{ \frac{\|E^\perp Yf\|}{\|E^\perp Xf\|} : f \in \mathcal{H}, E \in \mathcal{L} \right\} < \infty$ and $\langle Xf, Yf \rangle \geq 0$ for all $f \in \mathcal{H}$.

(2) There exists an operator A in $\text{Alg}\mathcal{L}$ such that $AX = Y$ and $A \geq 0$.

PROOF. Assume that $\sup \left\{ \frac{\|E^\perp Yf\|}{\|E^\perp Xf\|} : f \in \mathcal{H}, E \in \mathcal{L} \right\} < \infty$. Then there exists an

operator A in $\text{Alg}\mathcal{L}$ such that $AX = Y$ by Theorem A. Since $\langle Xf, Yf \rangle \geq 0$ for all f in $\mathcal{H}$, $0 \leq \langle Xf, Yf \rangle = \langle Xf, AXf \rangle$. If f is a vector in $\overline{\text{range} X}^\perp$, then $\langle f, Af \rangle = \langle f, 0 \rangle = 0$. Hence A is positive.

Conversely, if there exists a positive operator A in $\text{Alg}\mathcal{L}$ such that $AX = Y$,

then $\sup \left\{ \frac{\|E^\perp Yf\|}{\|E^\perp Xf\|} : f \in \mathcal{H}, E \in \mathcal{L} \right\} < \infty$ and also for all f in $\mathcal{H}$,

$\langle Xf, Yf \rangle = \langle Xf, AXf \rangle \geq 0$. □

THEOREM 3.2. *Let $\mathcal{L}$ be a subspace lattice acting on a Hilbert space $\mathcal{H}$ and let X_i and Y_i be operators in $\mathcal{B}(\mathcal{H})$ for $i = 1, 2, \dots, n$. Let P_j be the projection onto $\overline{\text{range} X_j}$ for all $j = 1, 2, \dots, n$. If $P_k E = E P_k$ for some k in $\{1, 2, \dots, n\}$, then the following are equivalent:*

(1) $\sup \left\{ \frac{\|E^\perp (\sum_{i=1}^n Y_i f_i)\|}{\|E^\perp (\sum_{i=1}^n X_i f_i)\|} : f \in \mathcal{H}, E \in \mathcal{L} \right\} < \infty$ and $\langle X_k f, Y_k f \rangle \geq 0$ for

all f in $\mathcal{H}$.

(2) There exists a positive operator A in $\text{Alg}\mathcal{L}$ such that $AX_i = Y_i$ for $i = 1, 2, \dots, n$.

PROOF. Assume that $\sup \left\{ \frac{\|E^\perp(\sum_{i=1}^n Y_i f_i)\|}{\|E^\perp(\sum_{i=1}^n X_i f_i)\|} : f \in \mathcal{H}, E \in \mathcal{L} \right\} < \infty$.

Then there

exists an operator A in $\text{Alg}\mathcal{L}$ such that $AX_i = Y_i$ for $i = 1, 2, \dots, n$. Since $\langle X_k f, Y_k f \rangle \geq 0$ for given k and all f in $\mathcal{H}$, $0 \leq \langle X_k f, Y_k f \rangle = \langle X_k f, AX_k f \rangle$. If f is a vector in $\overline{\text{range } X_k}^\perp$, then $\langle f, Af \rangle = \langle f, 0 \rangle = 0$. Hence A is positive.

Conversely, if there exists a positive operator A in $\text{Alg}\mathcal{L}$ such that $AX_i = Y_i$ for

$i = 1, 2, \dots, n$, then $\sup \left\{ \frac{\|E^\perp(\sum_{i=1}^n Y_i f_i)\|}{\|E^\perp(\sum_{i=1}^n X_i f_i)\|} : f \in \mathcal{H}, E \in \mathcal{L} \right\} < \infty$ and also for all f in $\mathcal{H}$ and $i = 1, 2, \dots, n$, $\langle X_i f, Y_i f \rangle = \langle X_i f, AX_i f \rangle \geq 0$. $\square$

THEOREM 3.3. Let $\mathcal{L}$ be a subspace lattice acting on a Hilbert space $\mathcal{H}$ and let X_i and Y_i be operators in $\mathcal{B}(\mathcal{H})$ for $i = 1, 2, \dots$. Let P_j be the projection onto $\overline{\text{range } X_j}$ for all $j = 1, 2, \dots$. If $P_k E = E P_k$ for some k in $\mathbb{N}$, then the following are equivalent:

$$(1) \sup \left\{ \frac{\|E^\perp(\sum_{i=1}^n Y_i f_i)\|}{\|E^\perp(\sum_{i=1}^n X_i f_i)\|} : f \in \mathcal{H}, n \in \mathbb{N}, E \in \mathcal{L} \right\} < \infty \text{ and } \langle X_k f, Y_k f \rangle \geq 0$$

for all f in $\mathcal{H}$.

(2) There exists a positive operator A in $\text{Alg}\mathcal{L}$ such that $AX_i = Y_i$ for $i = 1, 2, \dots$.

PROOF. Assume that $\sup \left\{ \frac{\|E^\perp(\sum_{i=1}^n Y_i f_i)\|}{\|E^\perp(\sum_{i=1}^n X_i f_i)\|} : f \in \mathcal{H}, n \in \mathbb{N}, E \in \mathcal{L} \right\} < \infty$.

Then

there exists an operator A in $\text{Alg}\mathcal{L}$ such that $AX_i = Y_i$ for $i = 1, 2, \dots$ (See Theorem 2.3[6]). Since $\langle X_k f, Y_k f \rangle \geq 0$ for given k and all f in $\mathcal{H}$, $0 \leq \langle X_k f, Y_k f \rangle = \langle X_k f, AX_k f \rangle$. If f is a vector in $\overline{\text{range } X_k}^\perp$, then $\langle f, Af \rangle = \langle f, 0 \rangle = 0$. Hence A is positive.

Conversely, if there exists a positive operator A in $\text{Alg}\mathcal{L}$ such that $AX_i = Y_i$ for

$i = 1, 2, \dots$, then $\sup \left\{ \frac{\|E^\perp(\sum_{i=1}^n Y_i f_i)\|}{\|E^\perp(\sum_{i=1}^n X_i f_i)\|} : f \in \mathcal{H}, n \in \mathbb{N}, E \in \mathcal{L} \right\} < \infty$ and also

for all f in $\mathcal{H}$ and $i = 1, 2, \dots$, $\langle X_if, Yif \rangle = \langle X_if, AX_if \rangle \geq 0$. $\square$

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