REGULAR COVERING SPACE IN DIGITAL COVERING THEORY AND ITS APPLICATIONS

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Abstract. As a survey-type article, the paper reviews some results on a regular covering space in digital covering theory. The recent paper [10](see also [12]) established the notion of regular covering space in digital covering theory and studied its various properties. Besides, the papers [14, 16] developed a discrete Deck's transformation group of a digital covering. In this paper we study further their properties. By using these properties, we can classify digital covering spaces. Finally, the paper proposes an open problem.

1. Introduction

Let $\mathbf{N}$, $\mathbf{Z}$, and $\mathbf{R}$ denote the sets of natural numbers, integers, and real numbers, respectively. Let $\mathbf{Z}^n$ denote the set of points in the Euclidean n -dimensional space with integer coordinates. Useful tools from algebraic topology and geometric topology for studying digital topological properties of a (binary) digital space include a digital covering space, the (digital) k -fundamental group, a digital k -surface, and so forth. These have been studied in papers including [1, 2, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 25, 26, 31]. Recently, motivated by both a *regular covering space* and a *Deck's transformation group* in algebraic topology [32], their digital versions in digital covering theory have been developed [10, 12, 14, 16], which play important roles in classifying digital covering spaces. The paper studies further their properties.

The paper is organized as follows. Section 2 provides some basic notions. Section 3 investigates some properties of a regular covering space in digital covering theory in relation to a calculation of an automorphism group (or discrete Deck's transformation group) of a digital

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covering space. Finally, Section 4 concludes the paper with a concluding remark and a further work, and proposes an open problem relating to an automorphism group of a digital covering which cannot satisfy a radius $2-(k_0, k_1)$ -isomorphism.

2. Preliminaries

Motivated by the k -adjacency relations of $2D$ and $3D$ digital spaces in [28, 30], the k -adjacency relations of a multi-dimensional digital space in $\mathbf{Z}^n$, induced from the following operator, have been established [4] (see also [7]):

For a natural number m with $1 \leq m \leq n$, two distinct points

$$p = (p_1, p_2, \dots, p_n), q = (q_1, q_2, \dots, q_n) \in \mathbf{Z}^n,$$

are $k(m, n)$ - (briefly, k_m - or k -) adjacent if

- there are at most m indices i such that $|p_i - q_i| = 1$ and
- for all other indices i such that $|p_i - q_i| \neq 1, p_i = q_i$.

Then, we observe that the $k(m, n)$ (briefly k or k_m)-adjacency relations of $\mathbf{Z}^n$ are determined according to the two numbers $m, n \in \mathbf{N}$ [4] (see also [7, 22]), as follows.

Proposition 2.1. [18] $k := k(m, n) = \sum_{i=n-m}^{n-1} 2^{n-i} C_i^n$, where $C_i^n = \frac{n!}{(n-i)! i!}$.

The current k -adjacency relations of $\mathbf{Z}^n$ is often used in studying digital topological properties of a multi-dimensional digital space in $\mathbf{Z}^n$, $n \in \mathbf{N}$.

In general, a pair (X, k) is assumed to be a (binary) digital space with k -adjacency in a quadruple $(\mathbf{Z}^n, k, \bar{k}, X)$, where $(k, \bar{k}) \in \{(k, 2n), (2n, 3^n - 1)\}$, k represents an adjacency relation for X , and $\bar{k}$ represents an adjacency relation for $\mathbf{Z}^n - X$ [28]. Furthermore, owing to the *digital k -connectivity paradox* in [28], we remind $k \neq \bar{k}$ except for the case $(\mathbf{Z}, 2, 2, X)$. For $\{a, b\} \subset \mathbf{Z}$ with $a \leq b$, $[a, b]_{\mathbf{Z}} = \{a \leq n \leq b | n \in \mathbf{Z}\}$ is considered in $(\mathbf{Z}, 2, 2, [a, b]_{\mathbf{Z}})$ [1]. But in this paper we are not concerned with $\bar{k}$ -adjacency between two points in $\mathbf{Z}^n - X$.

We say that two subsets (A, k) and (B, k) of (X, k) are k -adjacent to each other if $A \cap B = \emptyset$ and there are points $a \in A$ and $b \in B$ such that a and b are k -adjacent to each other [28]. We say that a set $X \subset \mathbf{Z}^n$ is k -connected if it is not a union of two disjoint non-empty sets that are not k -adjacent to each other [28]. If a digital space (X, k) is k -connected, then for any two points $x, y \in X$ there is a k -path from x to y in X [28].

For an adjacency relation k of $\mathbf{Z}^n$, a *simple k -path* with l elements in $\mathbf{Z}^n$ is assumed to be a *k -sequence* $(x_i)_{i \in [0, l]_{\mathbf{Z}}} \subset \mathbf{Z}^n$ such that x_i and x_j are k -adjacent and further, $x_0 = x$ and $x_l = y$ if and only if either $j = i + 1$ or $i = j + 1$ [28]. We say that the *length* of the simple k -path, denoted by $l_k(x, y)$, is the number l . A *simple closed k -curve* with l elements in $\mathbf{Z}^n$ is a sequence $(x_i)_{i \in [0, l-1]_{\mathbf{Z}}}$ derived from a simple k -curve $(x_i)_{i \in [0, l]_{\mathbf{Z}}}$ with $x_0 = x_l$, where x_i and x_j are k -adjacent if and only if $j = i + 1(\text{mod } l)$ or $i = j + 1(\text{mod } l)$ [28]. Let $SC_k^{n, l}$ denote a simple closed k -curve with l elements in $\mathbf{Z}^n$ [7].

In order to study digital continuity and various properties of digital geometry including both a digital k -curve and a digital k -surface [9, 11], we have used the following digital k -neighborhood of a point $x \in X$ with radius $\varepsilon \in \mathbf{N}$.

Definition 1. [4] (see also [7]) For a digital space (X, k) in $\mathbf{Z}^n$, the digital k -neighborhood of $x_0 \in X$ with radius ε is defined in X to be the following subset of X

$$N_k(x_0, \varepsilon) = \{x \in X \mid l_k(x_0, x) \leq \varepsilon\} \cup \{x_0\},$$

where $l_k(x_0, x)$ is the length of a shortest simple k -path from x_0 to x and $\varepsilon \in \mathbf{N}$.

Motivated by both the digital continuity in [30] and the (k_0, k_1) -continuity in [1], the following notion of digital continuity in a form has been often used for the study of a digital space in $\mathbf{Z}^n, n \in \mathbf{N}$.

Proposition 2.2. [4] (see also [7]) Let (X, k_0) and (Y, k_1) be digital spaces in $\mathbf{Z}^{n_0}$ and $\mathbf{Z}^{n_1}$, respectively. A function $f : X \rightarrow Y$ is (k_0, k_1) -continuous if and only if for every $x \in X, \varepsilon \in \mathbf{N}$, and $N_{k_1}(f(x), \varepsilon) \subset Y$, there is $\delta \in \mathbf{N}$ such that the corresponding $N_{k_0}(x, \delta) \subset X$ satisfies $f(N_{k_0}(x, \delta)) \subset N_{k_1}(f(x), \varepsilon)$.

Indeed, the current (k_0, k_1) -continuity of $f : (X, k_0) \rightarrow (Y, k_1)$ at a point $x \in X$ is equivalent to the following statement because every point x of a digital space (X, k) always has an $N_k(x, 1)$ in (X, k) .

Proposition 2.3. [16] Let (X, k_0) and (Y, k_1) be digital spaces in $\mathbf{Z}^{n_0}$ and $\mathbf{Z}^{n_1}$, respectively. A function $f : X \rightarrow Y$ is (k_0, k_1) -continuous if and only if for every $x \in X$ $f(N_{k_0}(x, 1)) \subset N_{k_1}(f(x), 1)$.

Unlike the pasting property of classical continuity in topology, (k_0, k_1) -continuity has some intrinsic features [27]: Digital (k_0, k_1) -continuity has the *almost pasting property* instead of the *pasting property* of classical topology.

Since a digital space (X, k) can be recognized to be a digital k -graph, we may use the term a (k_0, k_1) -isomorphism as in [2, 8] rather than a (k_0, k_1) -homeomorphism as in [1]:

Definition 2. [8] For two digital spaces (X, k_0) in $\mathbf{Z}^{n_0}$ and (Y, k_1) in $\mathbf{Z}^{n_1}$, a map $h : X \rightarrow Y$ is called a (k_0, k_1) -isomorphism if h is a (k_0, k_1) -continuous bijection and further, $h^{-1} : Y \rightarrow X$ is (k_1, k_0) -continuous. Then we use the notation $X \approx_{(k_0, k_1)} Y$. If $n_0 = n_1$ and $k_0 = k_1$, then we call it a k_0 -isomorphism and use the notation $X \approx_{k_0} Y$.

For a digital space (X, k) and $A \subset X$, (X, A) is called a *digital space pair* with k -adjacency [9]. Furthermore, if A is a singleton set $\{x_0\}$, then (X, x_0) is called a *pointed digital space* [28]. Motivated by the *pointed digital homotopy* in [1], the following notion of *relative k -homotopy* to a subset $A \subset X$ is often used in studying a k -homotopic thinning and a strong k -deformation retract of a digital space (X, k) in $\mathbf{Z}^n$.

Definition 3. [9] (see also [10, 14]) Let $((X, A), k_0)$ and (Y, k_1) be a digital space pair and a digital space, respectively. Let $f, g : X \rightarrow Y$ be (k_0, k_1) -continuous functions. Suppose there exist $m \in \mathbf{N}$ and a function $F : X \times [0, m]_{\mathbf{Z}} \rightarrow Y$ such that

- for all $x \in X$, $F(x, 0) = f(x)$ and $F(x, m) = g(x)$;
- for all $x \in X$, the induced function $F_x : [0, m]_{\mathbf{Z}} \rightarrow Y$ given by $F_x(t) = F(x, t)$ for all $t \in [0, m]_{\mathbf{Z}}$ is $(2, k_1)$ -continuous;
- for all $t \in [0, m]_{\mathbf{Z}}$, the induced function $F_t : X \rightarrow Y$ given by $F_t(x) = F(x, t)$ for all $x \in X$ is (k_0, k_1) -continuous.

Then we say that F is a (k_0, k_1) -homotopy between f and g [1].

- Furthermore, for all $t \in [0, m]_{\mathbf{Z}}$, then suppose the induced map F_t on A is a constant which is the prescribed function from A to Y . In other words, $F_t(x) = f(x) = g(x)$ for all $x \in A$ and for all $t \in [0, m]_{\mathbf{Z}}$.

Then we call F a (k_0, k_1) -homotopy relative to A between f and g , and we say f and g are (k_0, k_1) -homotopic relative to A in Y . Then we use the notation $f \simeq_{(k_0, k_1)rel A} g$.

In Definition 3, if $A = \{x_0\} \subset X$, then we say that F is a pointed (k_0, k_1) -homotopy at $\{x_0\}$ [1]. When f and g are pointed (k_0, k_1) -homotopic in Y , we use the notation $f \simeq_{(k_0, k_1)} g$. In addition, if $k_0 = k_1$ and $n_0 = n_1$, then we say that f and g are pointed k_0 -homotopic in Y and use the notation $f \simeq_{k_0} g$ and $f \in [g]$ which means the k_0 -homotopy class of g . If, for some $x_0 \in X$, 1_X is k -homotopic to the constant map with space x_0 relative to $\{x_0\}$, then we say that (X, x_0) is *pointed k -contractible* [1]. Indeed, the notion of k -contractibility is slightly different from the contractibility in Euclidean topology [1] (see also [7]).

Motivated by Khalimsky's (digital) k -fundamental group in [25], the paper [1] establishes the k -fundamental group by using the *trivial extension* in [1] and the *Khalimsky operation* in [25]: Precisely, if $m_f \leq m_{f'}$, we can obtain a trivial extension of a loop $f : [0, m_f]_{\mathbf{Z}} \rightarrow X$ to a loop $f' : [0, m_{f'}]_{\mathbf{Z}} \rightarrow X$ via

$$f'(t) = \begin{cases} f(t) & \text{if } 0 \leq t \leq m_f; \\ f(m_f) & \text{if } m_f \leq t \leq m_{f'}. \end{cases}$$

Let us now recall the *Khalimsky operation* in [25]: Let $F^k(X, x_0) = \{f \mid f \text{ is a } k\text{-loop based at } x_0\}$. For members $f : [0, m_f]_{\mathbf{Z}} \rightarrow X$ and $g : [0, m_g]_{\mathbf{Z}} \rightarrow X$ of $F^k(X, x_0)$, the Khalimsky operation $f * g : [0, m_f + m_g]_{\mathbf{Z}} \rightarrow X$ is defined by

$$f * g(t) = \begin{cases} f(t) & \text{if } 0 \leq t \leq m_f; \\ g(t - m_f) & \text{if } m_f \leq t \leq m_f + m_g. \end{cases}$$

For a digital space (X, k) , consider a k -loop f with base point x_0 , we denote $[f]_X$ (briefly, $[f]$) the k -homotopy class of f in X . Then, for a k -loop f_1 with the same base point $x_0 \in X$, $f_0 \in [f]$ means that the two k -loops f and f_0 have trivial extensions that can be joined by a k -homotopy keeping the end point fixed [7] (see also [2]). Furthermore, if $f_1, f_2, g_1, g_2 \in F^k(X, x_0)$, $f_1 \in [f_2]$, and $g_1 \in [g_2]$, then $f_1 * g_1 \in [f_2 * g_2]$, i.e., $[f_1 * g_1] = [f_2 * g_2]$ [1, 25]. Then, we use the notation $\pi^k(X, x_0) = \{[f] \mid f \in F^k(X, x_0)\}$ which is a group [1] with the operation $[f] \cdot [g] = [f * g]$ and called the (*digital*) k -fundamental group of (X, x_0) [1]. The base point of $\pi^k(X, x_0)$ should be taken as a point $x_0 \in X$ which cannot be deleted by a strong k -deformation retract [13, 14, 23]. If X is pointed k -contractible, then $\pi^k(X, x_0)$ is proved trivial [1].

Let $((X, A), k)$ be a digital space pair with k -adjacency. A map $f : ((X, A), k_0) \rightarrow ((Y, B), k_1)$ is called (k_0, k_1) -continuous if f is (k_0, k_1) -continuous and $f(A) \subset B$ [9]. If $A = \{a\}$, $B = \{b\}$, we write $(X, A) = (X, a)$, $(Y, B) = (Y, b)$, and we say that f is a *pointed* (k_0, k_1) -continuous map [28]. A (k_0, k_1) -continuous map $f : ((X, x_0), k_0) \rightarrow ((Y, y_0), k_1)$ induces a group homomorphism $f_* : \pi^{k_0}(X, x_0) \rightarrow \pi^{k_1}(Y, y_0)$ given by $f_*([\alpha]) = [f \circ \alpha]$, where $[\alpha] \in \pi^{k_0}(X, x_0)$ [1]. In addition, a (k_0, k_1) -isomorphism $\phi : ((X, x_0), k_0) \rightarrow ((Y, y_0), k_1)$ induces a group isomorphism $\phi_* : \pi^{k_0}(X, x_0) \rightarrow \pi^{k_1}(Y, y_0)$ [1].

The following notion has been often used in digital k -homotopy theory and digital covering theory.

Definition 4. [7] A pointed k -connected digital space (X, x_0) is called simply k -connected if $\pi^k(X, x_0)$ is a trivial group.

3. Some Properties of a Regular Covering Space in Digital Covering Theory

For the purpose of calculating $\pi^k(X, x_0)$ and classifying digital spaces, we have often used some properties of a (k_0, k_1) -covering [6, 7]. In digital covering theory each digital space (X, k) is assumed to be k -connected.

Definition 5. [7] (see also [10, 16]) Let (E, k_0) and (B, k_1) be digital spaces in $\mathbf{Z}^{n_0}$ and $\mathbf{Z}^{n_1}$, respectively and $p : E \rightarrow B$ a (k_0, k_1) -continuous surjection. Suppose, for any $b \in B$, there exists $\varepsilon \in \mathbf{N}$ such that

- (1) for some index set M , $p^{-1}(N_{k_1}(b, \varepsilon)) = \cup_{i \in M} N_{k_0}(e_i, \varepsilon)$ with $e_i \in p^{-1}(b)$;
- (2) if $i, j \in M$ and $i \neq j$, then $N_{k_0}(e_i, \varepsilon) \cap N_{k_0}(e_j, \varepsilon)$ is an empty set; and
- (3) the restriction map p on $N_{k_0}(e_i, \varepsilon)$ is a (k_0, k_1) -isomorphism for all $i \in M$.

Then, the map p is called a (k_0, k_1) -covering map and (E, p, B) is said to be a (k_0, k_1) -covering.

The k_1 -neighborhood $N_{k_1}(b, \varepsilon)$ in Definition 5 is called an *elementary k_1 -neighborhood* of b with some radius ε and further, E is called a (k_0, k_1) -covering space of p . For instance a $(2, k)$ -covering map, let $(\mathbf{Z}, p, SC_k^{n,l} := (c_t)_{t \in [0, l-1]_{\mathbf{Z}}})$ is a $(2, k)$ -covering map [7], where $p(t) = c_{t(mod l)}$, $t \in \mathbf{Z}$.

For any two points $b_0, b_1 \in B$ of a (k_0, k_1) -covering $(E, e_0), p, (B, b_0)$, since the cardinalities of $p(b_0)$ and $p(b_1)$ are equal [23], we can obtain the following:

Definition 6. [10] (see also [16]) We say that a (k_0, k_1) -covering map $p : (E, e_0) \rightarrow (B, b_0)$ is an n -fold (k_0, k_1) -covering map if the cardinality of the index set M is n .

Namely, Definition 6 can be represented as follows. For a (k_0, k_1) -covering map $p : (E, e_0) \rightarrow (B, b_0)$ if the set $p^{-1}(b_0)$ has n elements (or the number n can be also called the *sheets* of the digital covering (see [29]), then the map p is called an n -fold (k_0, k_1) -covering map [14, 16]. For instance, $(SC_k^{n, ml} := (a_t)_{t \in [0, ml-1]_{\mathbf{Z}}}, p, SC_k^{n, l} := (c_t)_{t \in [0, l-1]_{\mathbf{Z}}})$ is an m -fold (k, k) -covering, where $p(a_i) = c_{i(mod l)}$.

For pointed digital spaces $((E, e_0), k_0)$ and $((B, b_0), k_1)$, if $p : (E, e_0) \rightarrow (B, b_0)$ is a (k_0, k_1) -covering map such that $p(e_0) = b_0$, then the map p

is a *pointed* (k_0, k_1) -covering map [7]. Hereafter, we assume that each digital covering map is a *pointed* one unless stated.

Definition 7. [5] (see also [6, 17]) For two digital spaces (X, k_0) in $\mathbf{Z}^{n_0}$ and (Y, k_1) in $\mathbf{Z}^{n_1}$, a (k_0, k_1) -continuous map $h : X \rightarrow Y$ is called a *local* (k_0, k_1) -isomorphism if for every $x \in X$, h maps $N_{k_0}(x, 1) \subset X$ (k_0, k_1) -isomorphically onto $N_{k_1}(h(x), 1) \subset Y$. If $n_0 = n_1$ and $k_0 = k_1$, then the map h is called a *local* k_0 -isomorphism.

This local (k_0, k_1) -isomorphism can be strongly used in studying the investigation of the preservation of local k_0 -properties of a digital space (X, k_0) into its corresponding k_1 -ones in digital geometry [17]. Since a (k_0, k_1) -isomorphism is equivalent to locally (k_0, k_1) -isomorphic bijection and a restriction map of a (k_0, k_1) -isomorphism is also a (k_0, k_1) -isomorphism [15], we obtain the following property. If $h : (X, k_0) \rightarrow (Y, k_1)$ is a (k_0, k_1) -isomorphism, then the restriction map on $N_{k_0}(x, 1)$, $h|_{N_{k_0}(x, 1)} : N_{k_0}(x, 1) \rightarrow N_{k_1}(h(x), 1)$, is a (k_0, k_1) -isomorphism [5] (see also [11]).

Remark 3.1. [10] Let $h : (X, k_0) \rightarrow (Y, k_1)$ be a (k_0, k_1) -isomorphism. Then, for any $X_0 \subset X$, the restriction $h|_{X_0} : (X_0, k_0) \rightarrow (h(X_0), k_1)$ is also a (k_0, k_1) -isomorphism [11]. Furthermore, since a (k_0, k_1) -isomorphism is equivalent to a local (k_0, k_1) -isomorphic bijection [5], we may take $\varepsilon = 1$ for the (k_0, k_1) -covering of Definition 5 (see also [2]).

In light of Remark 3.1, even though we may take $\varepsilon = 1$ in Definition 5, in relation with Lemma 3.2, and various digital topological properties in Section 3, we may take some $\varepsilon \in \mathbf{N}$ in Definition 5 according to the situation.

Definition 8. [6] For $n \in \mathbf{N}$, a (k_0, k_1) -covering (E, p, B) is a *radius* n *local isomorphism* if the restriction map $p|_{N_{k_0}(e_i, n)} : N_{k_0}(e_i, n) \rightarrow N_{k_1}(b, n)$ is a (k_0, k_1) -isomorphism for all $i \in M$.

Definition 9. [6] (see also [16]) A (k_0, k_1) -covering (E, p, B) is called a *radius* n - (k_0, k_1) -covering if $\varepsilon \geq n$, where the number ε is the same as the ε of Definition 5.

In terms of Definitions 8 and 9, we observe that a (k_0, k_1) -covering satisfying a *radius* n *local isomorphism* is equivalent to a *radius* n - (k_0, k_1) -covering. By Remark 3.1, we observe that a (k_0, k_1) -covering of Definition 5 is clearly a *radius* 1- (k_0, k_1) -covering.

For three digital spaces (E, k_0) in $\mathbf{Z}^{n_0}$, (B, k_1) in $\mathbf{Z}^{n_1}$, and (X, k_2) in $\mathbf{Z}^{n_2}$, let $p : E \rightarrow B$ be a (k_0, k_1) -continuous map. For a (k_2, k_1) -continuous map $f : (X, k_2) \rightarrow (B, k_1)$, as the digital analogue of the

lifting in [29], we say that a *digital lifting* of f is a (k_2, k_0) -continuous map $\tilde{f} : X \rightarrow E$ such that $p \circ \tilde{f} = f$ [7]. We now recall the *unique digital lifting theorem* in [7].

Lemma 3.2. [7] *For pointed digital spaces $((E, e_0), k_0)$ in $\mathbf{Z}^{n_0}$ and $((B, b_0), k_1)$ in $\mathbf{Z}^{n_1}$, let $p : (E, e_0) \rightarrow (B, b_0)$ be a pointed (k_0, k_1) -covering map. Any k_1 -path $f : [0, m]_{\mathbf{Z}} \rightarrow B$ beginning at b_0 has a unique digital lifting to a k_0 -path $\tilde{f}$ in E beginning at e_0 .*

Moreover, the following *digital homotopy lifting theorem* was introduced in [6], which plays an important role in studying digital covering theory.

Lemma 3.3. [6] *Let $((E, e_0), k_0)$ and $((B, b_0), k_1)$ be pointed digital spaces. Let $p : (E, e_0) \rightarrow (B, b_0)$ be a radius 2- (k_0, k_1) -covering map. For k_0 -paths g_0, g_1 in (E, e_0) that start at e_0 , if there is a k_1 -homotopy in B from $p \circ g_0$ to $p \circ g_1$ that holds the endpoints fixed, then g_0 and g_1 have the same terminal point, and there is a k_0 -homotopy in E from g_0 to g_1 that holds the endpoints fixed.*

Definition 10. [16] *For a digital space (B, k) in $\mathbf{Z}^n$, let (E_1, p_1, B) and (E_2, p_2, B) be two (k_1, k) - and (k_2, k) -coverings, respectively. Then we say that a (k_1, k_2) -continuous map $\phi : E_1 \rightarrow E_2$ such that $p_2 \circ \phi = p_1$ is a (k_1, k_2) -homomorphism from (E_1, p_1, B) into (E_2, p_2, B) .*

In order to study a discrete Deck's transformation group of a digital covering map, we recall the following theorem.

Theorem 3.4. [13] *Let $p_1 : (E_1, e_1) \rightarrow (B, b_0)$ be a (k_1, k_0) -covering map and let $p_2 : (E_2, e_2) \rightarrow (B, b_0)$ be a (k_2, k_0) -covering map. If there is a (k_1, k_2) -continuous map $\phi : (E_1, e_1) \rightarrow (E_2, e_2)$ such that $p_2 \circ \phi = p_1$, then the map ϕ is a (k_1, k_2) -covering map.*

Theorem 3.4 has been often used for the study of a discrete Deck's transformation group of a digital (k_0, k_1) -covering map in relation with a classification of digital covering spaces [14, 16].

Corollary 3.5. [20] *Let $p_1 : (E_1, e_1) \rightarrow (B, b_0)$ be a (k_1, k_0) -covering map and $p_2 : (E_2, e_2) \rightarrow (B, b_0)$ a (k_2, k_0) -covering map. If there are both a (k_1, k_2) -continuous map $\phi : (E_1, e_1) \rightarrow (E_2, e_2)$ and a (k_2, k_1) -continuous map $\psi : (E_2, e_2) \rightarrow (E_1, e_1)$ such that $p_2 \circ \phi = p_1$ and $p_1 \circ \psi = p_2$, then the map ϕ is a (k_1, k_2) -isomorphism.*

Definition 11. [10] (see also [12]) *A (k_0, k_1) -covering $((E, e_0), p, (B, b_0))$ is called a regular (k_0, k_1) -covering if $p_*\pi^{k_0}(E, e_0)$ is a normal subgroup of $\pi^{k_1}(B, b_0)$.*

A discrete Deck's transformation group of a digital (k_0, k_1) -covering has some advantages which lie so convenient and efficient for calculating the digital fundamental group of some digital spaces and classifying digital spaces [14, 16]. As the digital version of Deck's transformation group of a covering map in algebraic topology [32], we obtain the following:

Definition 12. [16] (see also [14, 26]) Consider a (k_0, k_1) -covering map $p : ((E, e_0), k_0) \rightarrow ((B, b_0), k_1)$. A self k_0 -isomorphism of the (k_0, k_1) -covering map p , denoted by $h : (E, e_0) \rightarrow (E, e_0)$, is called a k_0 -covering transformation, a discrete Deck's transformation [26], or a discrete automorphism of a digital covering map p if $p = p \circ h$, where $\circ$ means the composition. The set of the discrete automorphisms of a digital covering map with composition operation is obviously a group which is denoted by $\text{Aut}((E, e_0)|(B, b_0))$.

For a (k_0, k_1) -covering map $p : ((E, e_0), k_0) \rightarrow ((B, b_0), k_1)$ in order to study a relation between $p^{-1}(b_0)$ and a coset $\pi^{k_1}(B, b_0)/p_*\pi^{k_0}(E, e_0)$, we often use the following property.

Lemma 3.6. [29] Let E be a set and G a group. If $E \times G \rightarrow E$ is a transitive action, then E is isomorphic to the factor group G/G_{x_0} , where $G_{x_0} := \{g \in G | g(x_0) = x_0\}$ called the isotropy subgroup of G .

Using Massey's program [29], we obtain the following:

Theorem 3.7. Let $((E, e_0), p, (B, b_0))$ be a pointed radius 2- (k_0, k_1) -covering and (E, k_0) k_0 -connected. Then we obtain the following.

- (1) $\pi^{k_1}(B, b_0)$ operates transitively on $p^{-1}(b_0)$ on the right [12].
- (2) The right $\pi^{k_1}(B, b_0)$ -space $p^{-1}(b_0)$ is bijective to $\pi^{k_1}(B, b_0)/p_*\pi^{k_0}(E, e_0)$.

Proof: (1) Consider $e \in p^{-1}(b_0)$ and $\alpha \in \pi^{k_1}(B, b_0)$. Take a k_1 -path $f : [0, m_f]_{\mathbf{Z}} \rightarrow (B, b_0)$ such that $[f] = \alpha$, $f(0) = b_0 = f(m_f)$. Then, by Lemma 3.3, the map f is well-defined. By Lemma 3.2, there is the unique k_0 -path $\tilde{f} : [0, m_f]_{\mathbf{Z}} \rightarrow (E, e_0)$ such that $p_*([\tilde{f}]) = [f]$ and $\tilde{f}(0) = e$. Precisely, define

$$p^{-1}(b_0) \times \pi^{k_1}(B, b_0) \rightarrow p^{-1}(b_0)$$

by

$$(e, \alpha) \rightarrow e \cdot \alpha = \tilde{f}(m_f)$$

and by Lemma 3.3 this action is well-defined because this process does not depend on the choice of the map f , where $[f] = \alpha$. Then, for $e \in p^{-1}(b_0)$ we clearly observe [12]

$$(e, 1) = e \quad \text{and} \quad ((e, \alpha), \beta) = (e, \alpha \cdot \beta), \quad (3.1)$$

where $\alpha, \beta \in \pi^{k_1}(B, b_0)$ and 1 is the identity element. This implies $p^{-1}(b_0)$ admits $\pi^{k_1}(B, b_0)$ as a group of operator (or permutation). The property (3.1) completes the proof.

(2) Take $\alpha \in p_*\pi^{k_0}(E, e_0)$. Then there exists $[\tilde{f}] \in \pi^{k_0}(E, e_0)$ such that $\alpha = [p\tilde{f}]$. Then we obtain $e_0 \cdot \alpha = \tilde{f}(m_{\tilde{f}}) = e_0$ and finally, $\alpha \in (\pi^{k_1}(B, b_0))_{e_0}$.

If $\beta \in (\pi^{k_1}(B, b_0))_{e_0}$ with $\beta = [g]$ and $g : [0, m_g]_{\mathbf{Z}} \rightarrow (B, b_0)$ represents β , i.e., $e_0 \cdot \beta = \tilde{g}(m_g) = e_0$ which is well-defined by Lemma 3.3. Thus $[\tilde{g}] \in \pi^{k_0}(E, e_0)$ and $[p\tilde{g}] = p_*[\tilde{g}] \in p_*\pi^{k_0}(E, e_0)$. Consequently, we obtain $p_*\pi^{k_0}(E, e_0) = (\pi^{k_1}(B, b_0))_{e_0}$. Thus, by Lemma 3.6, $p^{-1}(b_0)$ is proved to be bijective to the factor group $\pi^{k_1}(B, b_0)/p_*\pi^{k_0}(E, e_0)$. $\square$

Remark 3.8. Let us remind that, by Theorem 3.7(2), the group $p^{-1}(b_0)$ in [12, 18, 20] related to the assertion of Theorem 3.7(2) is clearly bijective to the factor group $\pi^{k_1}(B, b_0)/p_*\pi^{k_0}(E, e_0)$ as a right $\pi^{k_1}(B, b_0)$ -space, and the papers [12, 18] tells further the case $p^{-1}(b_0)$ has some group structure such as $(p^{-1}(b_0), +)$ derived from the given digital coverings such as $(\mathbf{Z}, p, SC_k^{n,l})$ and $(\mathbf{Z} \times \mathbf{Z}, p_1 \times p_2, SC_{k_1}^{n_1,l_1} \times SC_{k_2}^{n_2,l_2})$, where the k -adjacency of $(SC_{k_1}^{n_1,l_1} \times SC_{k_2}^{n_2,l_2}, k)$ has the L_S - or L_C -property relative to $SC_{k_i}^{n_i,l_i}, i \in \{1, 2\}$ (see [10, 12, 18, 20]). Of course, in this case we clearly observe that the given digital covering map p should be regular so that $\pi^{k_1}(B, b_0)/p_*\pi^{k_0}(E, e_0)$ is clearly isomorphic to $N(p_*\pi^{k_0}(E, e_0))/p_*\pi^{k_0}(E, e_0)$.

Lemma 3.9. [3] Let $((E, e_0), p, (B, b_0))$ be a pointed radius 2- (k_0, k_1) -covering and (E, k_0) k_0 -connected. For any $\phi \in \text{Aut}(E|B)$, any $e \in p^{-1}(b_0)$ and any $\alpha \in \pi^{k_1}(B, b_0)$ we obtain $\phi(e \cdot \alpha) = \phi(e) \cdot \alpha$.

Proof: Take $\alpha = [f]$, where $f : [0, m_f]_{\mathbf{Z}} \rightarrow (B, b_0)$ represents α . Consider the digital lifting $\tilde{f}$ of f such that $\phi\tilde{f} : [0, m_f]_{\mathbf{Z}} \rightarrow E$ such that $\phi\tilde{f}(0) = \phi(e)$ and $\phi\tilde{f}(m_f) = \phi(e \cdot \alpha)$ and $p\phi\tilde{f} = p\tilde{f} = f$. Thus $\phi\tilde{f}$ is a digital lifting of f . Therefore, we obtain $\phi(e) \cdot \alpha = \phi\tilde{f}(m_f) = \phi(e \cdot \alpha)$.

$\square$

Theorem 3.10. Let $((E, e_0), p, (B, b_0))$ be a pointed radius 2- (k_0, k_1) -covering and (E, k_0) k_0 -connected. Then $\text{Aut}(E|B)$ is isomorphic to $\text{Aut}(p^{-1}(b_0))$, where $\text{Aut}(p^{-1}(b_0))$ is considered as a right $\pi^{k_1}(B, b_0)$ -space.

Proof: Define the map

$$H : \text{Aut}(E|B) \rightarrow \text{Aut}(p^{-1}(b_0))$$

given by

$$\phi \rightarrow H(\phi) := \phi|_{p^{-1}(b_0)} : p^{-1}(b_0) \rightarrow p^{-1}(b_0).$$

First, we can prove that H is injective, as follows. Assume $\phi_1|_{p^{-1}(b_0)} = \phi_2|_{p^{-1}(b_0)}$, then we obtain $\phi_1(e_0) = \phi_2(e_0)$. Thus, $\phi_1 = \phi_2$.

Second, we can prove that H is surjective, as follows. Take any $f \in \text{Aut}(p^{-1}(b_0))$. Then $f : p^{-1}(b_0) \rightarrow p^{-1}(b_0)$ given by $e_0 \rightarrow e_1 = f(e_0)$. Thus, $(\pi^{k_1}(B, b_0))_{e_0} = (\pi^{k_1}(B, b_0))_{e_1}$ with $\pi^{k_1}(B, b_0)_{e_0} = p_*\pi^{k_0}(E, e_0)$ and $\pi^{k_1}(B, b_0)_{e_1} = p_*\pi^{k_0}(E, e_1)$. Therefore, there exists (k_0, k_0) -isomorphism $\phi : E \rightarrow E$ such that $\phi(e_0) = e_1$. Namely, $\phi \in \text{Aut}(E|B)$ such that $\phi(e_0) = e_1$. Consequently, we obtain $H(\phi) := \phi|_{p^{-1}(b_0)} = f$. $\square$

Corollary 3.11. [23] *Let $((E, e_0), p, (B, b_0))$ be a pointed radius 2- (k_0, k_1) -covering and (E, k_0) k_0 -connected. Then we obtain $\text{Aut}(E|B) \simeq N(p_*\pi^{k_0}(E, e_0))/p_*\pi^{k_0}(E, e_0)$.*

Corollary 3.12. [23] *Let $((E, e_0), p, (B, b_0))$ be a pointed radius 2- (k_0, k_1) -covering which is regular and (E, k_0) k_0 -connected. Then we obtain $\text{Aut}(E|B) \simeq \text{Aut}(p^{-1}(b_0)) \simeq \pi^{k_1}(B, b_0)/p_*\pi^{k_0}(E, e_0)$.*

Theorem 3.13. *Let $((E, e_0), p, (B, b_0))$ be a pointed radius 2- (k_0, k_1) -covering and (E, k_0) k_0 -connected. $((E, e_0), p, (B, b_0))$ is regular if and only if $p_*\pi^{k_0}(E, e_0) = p_*\pi^{k_0}(E, e_1)$, where $p(e_0) = p(e_1) = b_0$.*

Proof: If $((E, e_0), p, (B, b_0))$ is regular, then let $\tilde{w} : [0, m]_{\mathbf{Z}} \rightarrow E$ be a k_0 -loop at e_0 . Then $\tilde{w}$ is a digital closed lifting of $p(\tilde{w})$. Thus, there is a closed lifting $\tilde{w}_1$ of $p\tilde{w}$ at $\tilde{e}_1$. Therefore, we obtain

$$p_*[\tilde{w}] = [p\tilde{w}] = [p\tilde{w}_1] = p_*[\tilde{w}_1].$$

Finally, we clearly obtain $p_*\pi^{k_0}(E, e_0) \subset p_*\pi^{k_0}(E, e_1)$ and $p_*\pi^{k_0}(E, e_1) \subset p_*\pi^{k_0}(E, e_0)$, where $p(e_0) = p(e_1) = b_0$.

Conversely, the hypothesis of $p_*\pi^{k_0}(E, e_0) = p_*\pi^{k_0}(E, e_1)$ with $p(e_0) = p(e_1) = b_0$ implies that $p_*\pi^{k_0}(E, e_0)$ and $p_*\pi^{k_0}(E, e_1)$ are conjugates in $\pi^{k_1}(B, b_0)$, which makes the map p regular [29]. $\square$

Corollary 3.14. *Let $((E, e_0), p, (B, b_0))$ be a pointed radius 2- (k_0, k_1) -covering and regular and (E, k_0) k_0 -connected. Then $\text{Aut}(E|B) \simeq \pi^{k_1}(B, b_0)/p_*\pi^{k_0}(E, e_0)$.*

4. Concluding Remark and Further Work

Even though a digital covering space is motivated by a covering space in algebraic topology, a regular covering space in digital covering theory has some intrinsic features. All results have started that given covering maps are required to be radius $2-(k_0, k_1)$ -covering maps because this property is strongly related to Lemma 3.3. Thus, in case a given covering map p is not a radius $2-(k_0, k_1)$ -covering map, the existence problem of a regular covering still remains to be studied. In relation to the automorphism group of a digital covering which cannot satisfy a $2-(k_0, k_1)$ -isomorphism, we can suggest the following open problem.

[Open problem] For a digital covering which cannot satisfy a $2-(k_0, k_1)$ -isomorphism, how can we calculate its automorphism group? Furthermore, by using some properties of various types of homeomorphisms and a function space in computer topology [15, 24], we can study computer topological analog of the results in this paper.

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