ON P - $\mathcal{I}$ -OPEN SETS

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Abstract. The notions of pre-local function, semi-local functions and α -local functions with respect to a topology and an ideal are introduced, and several properties are investigated. Also, the concept of P - $\mathcal{I}$ -open sets and P - $\mathcal{I}$ -closed sets in ideal topological spaces are discussed. Relations between $\mathcal{I}$ -open sets and P - $\mathcal{I}$ -open sets are provided, and several properties related to P - $\mathcal{I}$ -open sets, pre-local functions, semi-local functions and α -local functions with respect to a topology and an ideal are investigated.

1. Introduction

In 1990, D. Janković, and T.R. Hamlett have introduced the notion of $\mathcal{I}$ -open sets in topological spaces. Since then, several kinds of $\mathcal{I}$ -openness, that is, (weakly) semi- $\mathcal{I}$ -open set, δ - $\mathcal{I}$ -open sets, β - $\mathcal{I}$ -open sets, α - $\mathcal{I}$ -open sets, b - $\mathcal{I}$ -open sets, (weakly) pre- $\mathcal{I}$ -open sets, etc. are introduced, and several properties and relations are investigated (see [1, 2, 6, 8, 9, 10, 7, 19, 23]).

In this paper, we first introduce the notions of pre-local function, semi-local function and α -local function with respect to a topology and an ideal, and several properties are investigated. We next introduce the concept of P - $\mathcal{I}$ -open set and P - $\mathcal{I}$ -closed set in ideal topological spaces, and investigates related properties. We discuss relations between $\mathcal{I}$ -open sets and P - $\mathcal{I}$ -open sets. Finally we introduce the notion of P -*-closure, and investigates many properties related to P - $\mathcal{I}$ -open set, pre-local function, semi-local function and α -local function with respect to a topology and an ideal.

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2. Preliminaries

Through this paper, (X, τ) and (Y, κ) (simply X and Y) always mean topological spaces. A subset A of X is said to be *semi-open* [20] (respectively, *α -open* [21] and *pre-open* [18]) if $A \subset \text{Cl}(\text{Int}(A))$ (respectively, $A \subset \text{Int}(\text{Cl}(\text{Int}(A)))$ and $A \subset \text{Int}(\text{Cl}(A))$). The complement of a pre-open set (respectively, an α -open set and a semi-open set) is called a *pre-closed set* (respectively, an *α -closed set* and a *semi-closed set*). The intersection of all pre-closed sets (respectively, α -closed sets and semi-closed sets) containing A is called the *pre-closure* (respectively, *α -closure* and *semi-closure*) of A , denoted by $p\text{Cl}(A)$ (respectively, $\alpha\text{Cl}(A)$ and $s\text{Cl}(A)$). A subset A is also pre-closed (respectively, α -closed and semi-closed) if and only if $A = p\text{Cl}(A)$ (respectively, $A = \alpha\text{Cl}(A)$ and $A = s\text{Cl}(A)$). We denote the family of all pre-open sets (respectively, α -open sets and semi-open sets) of (X, τ) by τ^p (respectively, τ^α and τ^s).

An *ideal* is defined as a nonempty collection $\mathcal{I}$ of subsets of X satisfying the following two conditions.

1. If $A \in \mathcal{I}$ and $B \subset A$, then $B \in \mathcal{I}$. (heredity)
2. If $A \in \mathcal{I}$ and $B \in \mathcal{I}$, then $A \cup B \in \mathcal{I}$. (finite additivity)

An *ideal topological space* is a topological space (X, τ) with an ideal $\mathcal{I}$ on X , and it is denoted by $(X, \tau, \mathcal{I})$. For a subset $A \subset X$, the set

$$A^*(\tau, \mathcal{I}) = \{x \in X : U \cap A \notin \mathcal{I} \text{ for each } U \in \tau(x)\}$$

is called the *local function* of A with respect to τ and $\mathcal{I}$, where

$$\tau(x) = \{U \in \tau : x \in U\}.$$

We will use A^* and/or $A^*(\mathcal{I})$ instead of $A^*(\tau, \mathcal{I})$. Given ideal topological space $(X, \tau, \mathcal{I})$, the **-topology*, written as $\tau^*(\mathcal{I})$ or written simply as τ^* , on X is defined to be a topology with a basis

$$\beta(\mathcal{I}, \tau) = \{U \setminus E \mid U \in \tau, E \in \mathcal{I}\}.$$

Note that τ^* is finer than τ . For a subset $A \subset X$, $\text{Cl}^*(A) = A \cup A^*$ defined a Kuratowski closure operator for a topology τ^* .

Lemma 2.1. [25] *Let (X, τ) be a topological space with ideals $\mathcal{I}$ and $\mathcal{J}$ on X . For subsets A and B of X , we have the following assertions.*

- (i) $A \subset B \Rightarrow A^* \subset B^*$.
- (ii) $\mathcal{I} \subset \mathcal{J} \Rightarrow A^*(\mathcal{J}) \subset A^*(\mathcal{I})$.
- (iii) $A^* = \text{Cl}(A^*) \subset \text{Cl}(A)$ (A^* is a closed subset of $\text{Cl}(A)$).
- (iv) $(A^*)^* \subset A^*$.

- (v) $(A \cup B)^* = A^* \cup B^*$.
- (vi) $A^* \setminus B^* = (A \setminus B)^* \setminus B^* \subset (A \setminus B)^*$.
- (vii) $U \in \tau \Rightarrow U \cap A^* = U \cap (U \cap A)^* \subset (U \cap A)^*$.
- (viii) $B \in \mathcal{I} \Rightarrow (A \cup B)^* = A^* = (A \setminus B)^*$.

Definition 2.2. Let $(X, \tau, \mathcal{I})$ be an ideal topological space. A subset A of X is said to be $\mathcal{I}$ -open [26] if $A \subset \text{Int}(A^*)$.

The set of all $\mathcal{I}$ -open sets in ideal topological space $(X, \tau, \mathcal{I})$ is denoted by $\mathcal{IO}(X, \tau, \mathcal{I})$ or written simply as $\mathcal{IO}(X)$ when there is no chance for confusion.

Remark 2.3. [26] One can deduce that $\mathcal{I}$ -open set $\Rightarrow$ pre-open set, and the converse is not true, in general.

3. Pre(resp. semi and α)-local functions in ideal topological spaces

Definition 3.1. Let A be a subset of an ideal topological space $(X, \tau, \mathcal{I})$. Then the set

$$A_p^*(\tau, \mathcal{I}) = \{x \in X : U \cap A \notin \mathcal{I} \text{ for each } U \in \tau^p(x)\}$$

is called the *pre-local function* with respect to τ and $\mathcal{I}$, where

$$\tau^p(x) = \{U \in \tau^p : x \in U\}.$$

We will use A_p^* and/or $A_p^*(\mathcal{I})$ instead of $A_p^*(\tau, \mathcal{I})$.

Definition 3.2. Let $(X, \tau, \mathcal{I})$ be an ideal topological space. Then the set

$$A_s^*(\tau, \mathcal{I}) = \{x \in X : U \cap A \notin \mathcal{I} \text{ for each } U \in \tau^s(x)\}$$

is called the *semi-local function* with respect to τ and $\mathcal{I}$, where

$$\tau^s(x) = \{U \in \tau^s : x \in U\}.$$

We will use A_s^* and/or $A_s^*(\mathcal{I})$ instead of $A_s^*(\tau, \mathcal{I})$.

Definition 3.3. Let $(X, \tau, \mathcal{I})$ be an ideal topological space. Then the set

$$A_\alpha^*(\tau, \mathcal{I}) = \{x \in X : U \cap A \notin \mathcal{I} \text{ for each } U \in \tau^\alpha(x)\}$$

is called the α -local function with respect to τ and $\mathcal{I}$, where

$$\tau^\alpha(x) = \{U \in \tau^\alpha : x \in U\}.$$

We will use A_α^* and/or $A_\alpha^*(\mathcal{I})$ instead of $A_\alpha^*(\tau, \mathcal{I})$.

Example 3.4. Let $X = \{a, b, c, d\}$, $\tau = \{X, \emptyset, \{c\}, \{a, b\}, \{a, b, c\}\}$ and $\mathcal{I} = \{\emptyset, \{a\}\}$. Then $(X, \tau, \mathcal{I})$ is an ideal topological space. We know that $\tau^\alpha = \tau$, $\tau^p = \{X, \emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}, \{a, c, d\}, \{b, c, d\}\}$ and $\tau^s = \{X, \emptyset, \{c\}, \{a, b\}, \{c, d\}, \{a, b, c\}, \{a, b, d\}\}$. If $A = \{a, b\}$ then $A_\alpha^* = \{a, b, d\}$, $A_s^* = \{a, b\}$ and $A_p^* = \{b\}$. If $B = \{a, c\}$, then $B_\alpha^* = \{c, d\} = B_p^*$ and $A_s^* = \{c\}$.

Theorem 3.5. Let $(X, \tau, \mathcal{I})$ be an ideal topological space and let A be a subset of X . Then

- (i) $A_p^* \subset A_\alpha^* \subset A^*$.
- (ii) $A_s^* \subset A_\alpha^* \subset A^*$.
- (iii) $A_p^* \subset p\text{Cl}(A)$, $A_s^* \subset s\text{Cl}(A)$ and $A_\alpha^* \subset \alpha\text{Cl}(A)$.

Proof. (i) and (ii) are straightforward.

(iii) Let $x \in A_p^*$. Then $x \in \{y \in X : U \cap A \notin \mathcal{I} \text{ for every } U \in \tau^p(y)\}$ where $\tau^p(y) = \{U \in \tau^p : y \in U\}$. If $x \in A$ then $x \in A \cup D_p(A) = p\text{Cl}(A)$ where $D_p(A)$ is the pre-derived set of A . If $x \notin A$ then $U \cap A \notin \mathcal{I}$ because $x \in A_p^*$ for every $U \in \tau^p(x)$. Since $x \notin A$, $(U \setminus \{x\}) \cap A \notin \mathcal{I}$. It follows that $(U \setminus \{x\}) \cap A \neq \emptyset$ because $\{\emptyset\} \subset \mathcal{I}$ so that $x \in D_p(A) \subset A \cup D_p(A) = p\text{Cl}(A)$. Hence $A_p^* \subset p\text{Cl}(A)$. By the similar way, we can obtain the other results. $\square$

Theorem 3.6. Let $(X, \tau, \mathcal{I})$ be an ideal topological space and let A be a subset of X . Then

- (i) If $\mathcal{I} = \{\emptyset\}$, then $A_p^* = p\text{Cl}(A)$, $A_s^* = s\text{Cl}(A)$ and $A_\alpha^* = \alpha\text{Cl}(A)$.
- (ii) If $\mathcal{I} = \mathcal{P}(X)$, then $A_p^* = A_s^* = A_\alpha^* = \emptyset$.

Proof. (i) Let $\mathcal{I} = \{\emptyset\}$. We know that $p\text{Cl}(A) = A \cup D_p(A)$ where $D_p(A)$ is a pre-derived set of A . Let $x \in A \cup D_p(A)$ and let $G_x \in \tau^p$ containing x . Then $x \in A$ or $x \in D_p(A)$. If $x \in A$ then $x \in G_x \cap A$, and so $G_x \cap A \neq \emptyset$. If $x \in D_p(A)$ then $\emptyset \neq (G_x \setminus \{x\}) \cap A \subset G_x \cap A$ and thus $G_x \cap A \neq \emptyset$. Hence $p\text{Cl}(A) = A \cup D_p(A) \subset A_p^*$. We know from Theorem 3.5 that $A_p^* \subset p\text{Cl}(A)$. Therefore $A_p^* = p\text{Cl}(A)$. Similarly, we have $A_s^* = s\text{Cl}(A)$ and $A_\alpha^* = \alpha\text{Cl}(A)$.

(ii) Straightforward. $\square$

Lemma 3.7. [19] Let (X, τ) be a topological space and let A, B be subsets of X . If $A \in \tau^\alpha$ and $B \in \tau^p$ then $A \cap B \in \tau^p$.

Lemma 3.8. Let (X, τ) be a topological space with ideals $\mathcal{I}$ and $\mathcal{J}$ on X , and let A, B be subsets of X . Then

- (i) $A \subset B \Rightarrow A_p^* \subset B_p^*$.
- (ii) $\mathcal{I} \subset \mathcal{J} \Rightarrow A_p^*(\mathcal{J}) \subset A_p^*(\mathcal{I})$.

- (iii) $A_p^* = p\text{Cl}(A_p^*) \subset p\text{Cl}(A)$ (A_p^* is a pre-closed subset of $p\text{Cl}(A)$).
- (iv) $(A_p^*)_p^* \subset A_p^*$.
- (v) $B \in \mathcal{I} \Rightarrow B_p^* = \emptyset$.
- (vi) $U \in \tau^\alpha \Rightarrow U \cap A_p^* = U \cap (U \cap A)_p^* \subset (U \cap A)_p^*$.
- (vii) $B \in \mathcal{I} \Rightarrow (A \cup B)_p^* = A_p^* = (A \setminus B)_p^*$.
- (viii) $A_p^*(\mathcal{I} \cap \mathcal{J}) \supset A_p^*(\mathcal{I}) \cup A_p^*(\mathcal{J})$.

Proof. (i) Suppose that $A \subset B$. Let $x \in A_p^*$. Then $G_x \cap A \notin \mathcal{I}$ for every pre-open G_x containing x . Since $A \subset B$, we have $G_x \cap A \subset G_x \cap B$. Then $G_x \cap B \notin \mathcal{I}$ by the heredity of ideal. Hence $x \in B_p^*$, and therefore $A_p^* \subset B_p^*$.

(ii) Suppose that $\mathcal{I} \subset \mathcal{J}$. Let $x \in A_p^*(\mathcal{J})$. Then $G_x \cap A \notin \mathcal{J}$ for every pre-open G_x containing x . Since $\mathcal{I} \subset \mathcal{J}$, we get $G_x \cap A \notin \mathcal{I}$. Hence $x \in A_p^*(\mathcal{I})$, and so $A_p^*(\mathcal{J}) \subset A_p^*(\mathcal{I})$.

(iii) We will show that A_p^* is a pre-closed set. Let $x \in (A_p^*)^c$. Then $x \notin A_p^*$ and so there exist a pre-open set U_x containing x such that $U_x \cap A \in \mathcal{I}$. It follows that $y \notin A_p^*$ for all $y \in U_x$ so that $y \in (A_p^*)^c$. Then $x \in U_x \subset (A_p^*)^c$. Hence $(A_p^*)^c$ is pre-open. Finally, we know that A_p^* is pre-closed. Therefore $A_p^* = p\text{Cl}(A_p^*) \subset p\text{Cl}(A)$ by Theorem 3.5.

(iv) By (iii), we obtain $(A_p^*)_p^* \subset p\text{Cl}(A_p^*) = A_p^*$.

(v) Suppose that $B \in \mathcal{I}$. Since $G \cap B \subset B$ for every subset G of X , $G \cap B \in \mathcal{I}$ by the heredity of ideal. Hence $B_p^* = \emptyset$.

(vi) Since $U \cap A \subset A$, (i) implies that $(U \cap A)_p^* \subset A_p^*$. Hence $U \cap (U \cap A)_p^* \subset U \cap A_p^*$. Let $x \in U \cap A_p^*$. Then $x \in U$ and $x \in A_p^*$. It follows that $G \cap A \notin \mathcal{I}$ for every $G \in \tau^p(x) = \{H \in \tau^p : x \in H\}$. Since U is an α -open set, $U \cap G$ is pre-open containing x by Lemma 3.7 and so $U \cap G \in \tau^p(x)$. Thus $(G \cap U) \cap A \notin \mathcal{I}$, and hence $G \cap (U \cap A) \notin \mathcal{I}$. Then $x \in (U \cap A)_p^*$. Hence $x \in U \cap (U \cap A)_p^*$. Finally, we have $U \cap A_p^* = U \cap (U \cap A)_p^* \subset (U \cap A)_p^*$.

(vii) By (i), we have that $(A \cup B)_p^* \supset A_p^*$. Let $x \in (A \cup B)_p^*$. Then $U \cap (A \cup B) \notin \mathcal{I}$ for every $U \in \tau^p(x)$. This implies that $(U \cap A) \cup (U \cap B) \notin \mathcal{I}$. Since $U \cap B \subset B$, $U \cap B \in \mathcal{I}$ by the heredity of ideal and so $U \cap A \notin \mathcal{I}$. Hence $x \in A_p^*$. Therefore $(A \cup B)_p^* \subset A_p^*$. Consequently $(A \cup B)_p^* = A_p^*$. Since $A_p^* = ((A \setminus B) \cup (A \cap B))_p^*$, we obtain $A_p^* = ((A \setminus B) \cup (A \cap B))_p^* = (A \setminus B)_p^*$ by the first result of (vii).

(viii) It is straightforward by (ii). □

In Lemma 3.8, the reverse inclusions of (iii) and (viii) are not valid as seen in the following example.

Example 3.9. Consider ideal topological spaces $(X, \tau, \mathcal{I})$, $(X, \tau, \mathcal{J})$ and $(X, \kappa, \mathcal{I})$ where $X = \{a, b, c, d\}$, $\tau = \{X, \emptyset, \{a, b\}, \{a, b, c\}\}$,

$$\kappa = \{X, \emptyset, \{c\}, \{a, b\}, \{a, b, c\}\},$$

$\mathcal{I} = \{\emptyset, \{a\}\}$ and $\mathcal{J} = \{\emptyset, \{b\}\}$. If $A = \{a, c\}$ in $(X, \kappa, \mathcal{I})$ then

$$A_p^*(\kappa, \mathcal{I}) = \{c, d\} \not\supset \{a, c, d\} = p\text{Cl}(A).$$

If $A = \{a, b\}$ then $A_p^*(\tau, \mathcal{I}) = \{b\}$, $A_p^*(\tau, \mathcal{J}) = \{a\}$ and $A_p^*(\tau, \mathcal{I} \cap \mathcal{J}) = \{a, b, c, d\}$. Hence we know that $A_p^*(\tau, \mathcal{I} \cap \mathcal{J}) \not\subset A_p^*(\tau, \mathcal{I}) \cup A_p^*(\tau, \mathcal{J})$.

The following example shows that the identity $A_p^* \setminus B_p^* = (A \setminus B)_p^*$ and the inclusions $A_p^* \setminus B_p^* \subset (A \setminus B)_p^*$ and $A_p^* \setminus B_p^* \supset (A \setminus B)_p^*$ are not valid.

Example 3.10. Let $X = \{a, b, c, d\}$, $\tau = \{X, \emptyset, \{a, b\}, \{a, b, c\}\}$ and $\mathcal{I} = \{\emptyset\}$. Then $(X, \tau, \mathcal{I})$ is an ideal topological space. Take $A = \{a, b\}$, $B = \{a, d\}$. Then $A_p^* = \{a, b, c, d\}$, $B_p^* = \{a, d\}$ and $(A \setminus B)_p^* = \{b\}$. Thus $A_p^* \setminus B_p^* = \{b, c\} \neq \{b\} = (A \setminus B)_p^* \setminus B_p^*$.

If $A = \{a, b\}$, $B = \{a\}$ then $A_p^* = \{a, b, c, d\}$, $B_p^* = \{a\}$. Hence $A_p^* \setminus B_p^* = \{b, c, d\} \not\subset \{b\} = \{b\}_p^* = (A \setminus B)_p^*$.

If $A = \{a, b, c, d\}$, $B = \{a, b, c\}$ then $A_p^* = \{a, b, c, d\}$, $B_p^* = \{a, b, c, d\}$. Hence $A_p^* \setminus B_p^* = \emptyset \not\supset \{d\} = \{d\}_p^* = (A \setminus B)_p^*$.

The converses of (i), (ii), (vi) and (vii) in Lemma 3.8 may not be true as seen in the the following example.

Example 3.11. Let $X = \{a, b, c, d\}$, $\tau = \{X, \emptyset, \{c\}, \{a, b\}, \{a, b, c\}\}$, $\mathcal{I} = \{\emptyset, \{a\}\}$ and $\mathcal{J} = \{\emptyset, \{c\}\}$. Then $(X, \tau, \mathcal{I})$ and $(X, \tau, \mathcal{J})$ are ideal topological spaces.

(i) If $A = \{b, c\}$ and $B = \{a, b, d\}$ then $A_p^*(\mathcal{I}) = \{b, c, d\}$ and $B_p^*(\mathcal{I}) = \{b, d\}$. Hence we know that $A_p^*(\mathcal{I}) \supset B_p^*(\mathcal{I})$ but $A \not\supset B$.

(ii) If $A = \{b, c\}$ then $A_p^*(\mathcal{I}) = \{b, c, d\}$ and $A_p^*(\mathcal{J}) = \{b\}$. Thus $A_p^*(\mathcal{J}) \subset A_p^*(\mathcal{I})$ but $\mathcal{I} \not\subset \mathcal{J}$.

(vi) If $A = \{a, b, c\}$ and $U = \{a, c, d\}$ then $(U \cap A)_p^*(\mathcal{I}) = U \cap (U \cap A)_p^*(\mathcal{I}) = \{c, d\}$ and $U \cap A_p^*(\mathcal{I}) = \{c, d\}$. Thus

$$U \cap A_p^*(\mathcal{I}) = U \cap (U \cap A)_p^*(\mathcal{I}) \subset (U \cap A)_p^*(\mathcal{I})$$

but $U \not\subset \tau^\alpha$.

(vii) If $A = \{b, c, d\}$ and $B = \{d\}$ then $(A \cup B)_p^*(\mathcal{I}) = A_p^*(\mathcal{I}) = \{b, c, d\}$ and $(A \setminus B)_p^*(\mathcal{I}) = \{b, c, d\}$. Thus $(A \cup B)_p^*(\mathcal{I}) = A_p^*(\mathcal{I}) = (A \setminus B)_p^*(\mathcal{I})$ but $B \not\subset \mathcal{I}$.

Lemma 3.12. For a class $\{U_i \mid i \in \Lambda\}$ of subsets of an ideal topological space $(X, \tau, \mathcal{I})$, we have

- (i) $\bigcup_{i \in \Lambda} ((U_i)_p^*) \subset (\bigcup_{i \in \Lambda} (U_i))^*_p$.
- (ii) $\bigcap_{i \in \Lambda} ((U_i)_p^*) \supset (\bigcap_{i \in \Lambda} (U_i))^*_p$.

Proof. Straightforward by Lemma 3.8. $\square$

The reverse inclusion of (i) in Lemma 3.12 is not valid as seen in the following example.

Example 3.13. Let $X = \{a, b, c, d\}$, $\tau = \{X, \emptyset, \{a, b\}, \{a, b, c\}\}$ and $\mathcal{I} = \{\emptyset\}$. Then $(X, \tau, \mathcal{I})$ is an ideal topological space. We note that $\tau^p = \{X, \emptyset, \{a\}, \{b\}, \{a, b\}, \{a, c\}, \{a, d\}, \{b, c\}, \{b, d\}, \{a, b, c\}, \{a, b, d\}, \{a, c, d\}, \{b, c, d\}\}$. If $A = \{a\}$ and $B = \{b\}$ then $A_p^* = \{a\}$, $B_p^* = \{b\}$ and $(A \cup B)_p^* = \{a, b, c, d\}$. Hence $A_p^* \cup B_p^* \not\supset (A \cup B)_p^*$.

Lemma 3.14. For any subset A of an ideal topological space $(X, \tau, \mathcal{I})$, if $A \subset A_p^*$ then $A_p^* = p\text{Cl}(A_p^*) = p\text{Cl}(A)$.

Proof. Straightforward. $\square$

Theorem 3.15. Let $(X, \tau, \mathcal{I})$ be an ideal topological space. Then

$$A_p^* \supset A \setminus \bigcup \{U \subset X : U \in \mathcal{I}\}$$

for all $A \subset X$.

Proof. Put $B := \bigcup \{U \subset X : U \in \mathcal{I}\}$ and let $x \in A \setminus B$. Then $x \in A$ and $x \notin B$. This implies that $x \notin U$ for all $U \in \mathcal{I}$ so that $\{x\} = \{x\} \cap A \notin \mathcal{I}$ because $x \in A$. For every $G \in \tau^p(x)$, we have $\{x\} \cap A \subset G \cap A \notin \mathcal{I}$ by the heredity of ideal. Hence $x \in A_p^*$. This completes the proof. $\square$

The reverse inclusion of Theorem 3.15 is not valid as seen in the following example.

Example 3.16. Consider an ideal topological space $(X, \tau, \mathcal{I})$ where $X = \{a, b, c, d\}$, $\tau = \{X, \emptyset, \{c\}, \{a, b\}, \{a, b, c\}\}$ and $\mathcal{I} = \{\emptyset, \{a\}\}$. Then $B := \bigcup \{U \subset X : U \in \mathcal{I}\} = \{a\}$. If $A = \{a, c\}$ then $A_p^* = \{c, d\}$. Hence $A_p^* = \{c, d\} \not\supset \{c\} = A \setminus B$.

Theorem 3.17. Let $(X, \tau, \mathcal{I})$ be an ideal topological space. Put

$$B := \bigcup \{U \subset X : U \in \mathcal{I}\}.$$

If $B \in \mathcal{I}$ then $(A_p^*)_p^* = A_p^*$ for all $A \subset X$.

Proof. Let A be a subset of X . Then $(A_p^*)^* \subset A_p^*$ by Lemma 3.8(iv). Furthermore $A_p^* \supset A \setminus B$ by Theorem 3.15. It follows from Lemma 3.8(i) that $(A_p^*)^* \supset (A \setminus B)_p^*$. Since $B \in \mathcal{I}$, Lemma 3.8(vii) implies that $(A_p^*)^* \supset (A \setminus B)_p^* = A_p^*$. Therefore $(A_p^*)^* = A_p^*$. $\square$

Theorem 3.18. *Let $(X, \tau, \mathcal{I})$ be an ideal topological space in which the cardinality of $\mathcal{I}$ is finite. Then $(A_p^*)^* = A_p^*$ for all $A \subset X$.*

Proof. Let $B := \cup\{U \subset X : U \in \mathcal{I}\}$. Since the cardinality of $\mathcal{I}$ is finite, $B \in \mathcal{I}$ by the finite additivity of ideal. Therefore $(A_p^*)^* = A_p^*$ for all $A \subset X$ by Theorem 3.17. $\square$

Corollary 3.19. *Let $(X, \tau, \mathcal{I})$ be an ideal topological space in which the cardinality of X is finite. Then $(A_p^*)^* = A_p^*$ for all $A \subset X$.*

Proof. Straightforward. $\square$

Theorem 3.20. *Let $(X, \tau, \mathcal{I})$ be an ideal topological space in which $\tau^p = \mathcal{P}(X)$. Then*

$$A_p^* = A \setminus \cup\{U \subset X : U \in \mathcal{I}\}$$

for all $A \subset X$.

Proof. Put $B := \cup\{U \subset X : U \in \mathcal{I}\}$ and let $x \in A_p^*$. Then $\{x\} \cap A \notin \mathcal{I}$ because $\{x\} \in \tau^p = \mathcal{P}(X)$. Since ideal $\mathcal{I}$ always contain $\emptyset$, $\{x\} \cap A \neq \emptyset$ and so $x \in A$. It follows that $\{x\} = \{x\} \cap A \notin \mathcal{I}$ so that $x \notin U$ for all $U \in \mathcal{I}$. Hence $x \notin B$, and therefore $x \in A \setminus B$. Hence $A_p^* \subset A \setminus B$. The reverse inclusion is obvious by Theorem 3.15. $\square$

Corollary 3.21. *Let $(X, \tau, \mathcal{I})$ be an ideal topological space in which every member of τ is clopen. Then*

$$A_p^* = A \setminus \cup\{U \subset X : U \in \mathcal{I}\}$$

for all $A \subset X$. or all $A \subset X$.

Proof. Put $B := \cup\{U \subset X : U \in \mathcal{I}\}$ and let $A \in \mathcal{P}(X)$. Then $A \subset \text{Cl}(A) = \text{Int}(\text{Cl}(A))$ because every member of τ is clopen. Hence $A \in \tau^p$, which means that $\mathcal{P}(X) \subset \tau^p$ so that $\mathcal{P}(X) = \tau^p$. By Theorem 3.20, $A_p^* = A \setminus B$. $\square$

Lemma 3.22. *Let (X, τ) be a topological space. Then the set*

$$\{U \subset X : U \cap G \neq \emptyset \text{ for all } G \in \tau \setminus \{\emptyset\}\}$$

is contained in τ^p .

Proof. Let $H \in \{U \subset X : U \cap G \neq \emptyset \text{ for all } G \in \tau \setminus \{\emptyset\}\}$. For any $a \in X$, if $a \in H$ then clearly $a \in \text{Cl}(H)$. If $a \notin H$ then $H \cap (F \setminus \{a\}) = H \cap F \neq \emptyset$ for any $F \in \tau \setminus \{\emptyset\}$ containing a because $H \in \{U \subset X : U \cap G \neq \emptyset \text{ for all } G \in \tau \setminus \{\emptyset\}\}$. Thus $a \in D(H) \subset \text{Cl}(H)$ where $D(H)$ is the derived set of H . It follows that $\text{Cl}(H) = X$ so that $H \subset X = \text{Int}(\text{Cl}(H))$. Hence $H \in \tau^p$. Therefore the result is valid. $\square$

Lemma 3.23. *Let (X, τ) be a topological space. If τ is a chain under the set inclusion then*

$$\tau^p = \{U \subset X : U \cap G \neq \emptyset \text{ for all } G \in \tau \setminus \{\emptyset\}\} \cup \{\emptyset\}.$$

Proof. Since $\emptyset \in \tau^p$, $\tau^p \supset \{U \subset X : U \cap G \neq \emptyset \text{ for all } G \in \tau \setminus \{\emptyset\}\} \cup \{\emptyset\}$ by Lemma 3.22.

Conversely, let $H \in \tau^p$. Then $H \subseteq \text{Int}(\text{Cl}(H))$. If $H = \emptyset$ then clearly $H \in \{U \subset X : U \cap G \neq \emptyset \text{ for all } G \in \tau \setminus \{\emptyset\}\} \cup \{\emptyset\}$. Suppose that $H \neq \emptyset$. If there exist $F \in \tau \setminus \{\emptyset\}$ such that $H \cap F = \emptyset$, then $H \subset F^c$ and so $\text{Int}(\text{Cl}(H)) \subset \text{Int}(\text{Cl}(F^c)) = \text{Int}(F^c)$. Since τ is a chain under the set inclusion, $I \cap F \neq \emptyset$ for all $I \in \tau \setminus \{\emptyset\}$. It follows that $I \not\subset F^c$ so that $\text{Int}(F^c) = \emptyset$. Thus $H \subseteq \text{Int}(\text{Cl}(H)) \subset \text{Int}(\text{Cl}(F^c)) = \text{Int}(F^c) = \emptyset$. This is a contradiction. Hence $H \cap F \neq \emptyset$ for all $F \in \tau \setminus \{\emptyset\}$. Therefore $\tau^p = \{U \subset X : U \cap G \neq \emptyset \text{ for all } G \in \tau \setminus \{\emptyset\}\} \cup \{\emptyset\}$. $\square$

Theorem 3.24. *Let $(X, \tau, \mathcal{I})$ be an ideal topological space. If τ is a chain under the set inclusion, then either $G_p^* = X$ or $G_p^* \subset G$ for $G \subset X$.*

Proof. Let G be a subset of X . Suppose that $G_p^* \neq X$. We will show that $G \subseteq G_p^*$. Let $x \notin G$. Since $G_p^* \neq X$, there exists $a \in X$ such that $a \notin G_p^*$. This implies that there exists $H \in \tau^p(a)$ such that $H \cap G \in \mathcal{I}$. Since τ is a chain under the set inclusion, $H \cap F \neq \emptyset$ for each $F \in \tau \setminus \{\emptyset\}$ by Lemma 3.23. Then $(H \cup \{x\}) \cap F \neq \emptyset$ for each $F \in \tau \setminus \{\emptyset\}$ and so, $x \in H \cup \{x\} \in \tau^p$ by Lemma 3.23. Furthermore $(H \cup \{x\}) \cap G = H \cap G \in \mathcal{I}$. Hence $x \notin G_p^*$. Therefore we obtain the desired result $G_p^* \subseteq G$. $\square$

Theorem 3.25. *Let $(X, \tau, \mathcal{I})$ be an ideal topological space. Assume that τ is a chain under the set inclusion in which there exists the smallest element A of $(\tau \setminus \{\emptyset\}, \subset)$. If $A \cap B = \emptyset$ for every $B \in \mathcal{I}$, then $G_p^* = X$ for $G \subset X$ containing A .*

Proof. Let $A \subset G$. Assume that there exists $x \in X$ such that $x \notin G_p^*$. Then there exists $H \in \tau^p(x) = \{U \in \tau^p : x \in U\}$ such that $H \cap G \in \mathcal{I}$. Since $H \in \tau^p(x)$, $H \cap A \neq \emptyset$ by Lemma 3.23. Since $H \cap G \in \mathcal{I}$, $A \cap (H \cap G) = \emptyset$ by the hypothesis. It follows that $A \cap H = A \cap (H \cap G) = \emptyset$ because $A \subset G$. This is a contradiction. Therefore $x \in G_p^*$. $\square$

4. $P\mathcal{I}$ -open sets and $P\mathcal{I}$ -closed sets

Definition 4.1. Let $(X, \tau, \mathcal{I})$ be an ideal topological space. A subset A of X is said to be $P\mathcal{I}$ -open if $A \subset p\text{Int}(A_p^*)$. A subset B of X is said to be $P\mathcal{I}$ -closed if the complement of B is $P\mathcal{I}$ -open.

The set of all $P\mathcal{I}$ -open sets in $(X, \tau, \mathcal{I})$ is denoted by $P\mathcal{IO}(X, \tau, \mathcal{I})$. Simply $P\mathcal{IO}(X, \tau, \mathcal{I})$ is written as $P\mathcal{IO}(X)$ or $P\mathcal{IO}(X, \tau)$ when there is no chance for confusion.

Lemma 4.2. [24] Let S be a set in (X, τ) . Then

- (i) $p\text{Cl}(S) = S \cup \text{Cl}(\text{Int}(S))$, $p\text{Int}(S) = S \cap \text{Int}(\text{Cl}(S))$.
- (ii) $\text{Int}(s\text{Cl}(S)) = p\text{Int}(\text{Cl}(S)) = p\text{Int}(s\text{Cl}(S)) = s\text{Cl}(p\text{Int}(S)) = \text{Int}(\text{Cl}(S))$.
- (iii) $\text{Int}(p\text{Cl}(S)) = s\text{Cl}(\text{Int}(S)) = \text{Int}(\text{Cl}(\text{Int}(S)))$.

Proposition 4.3. For any set A in $(X, \tau, \mathcal{I})$, we have

$$p\text{Int}(A_p^*) \subseteq \text{Int}(A^*).$$

Proof. We have

$$\begin{aligned} p\text{Int}(A_p^*) &= A_p^* \cap \text{Int}(\text{Cl}(A_p^*)) \quad (\text{by Lemma 4.2}) \\ &\subset \text{Int}(\text{Cl}(A_p^*)) \\ &\subset \text{Int}(\text{Cl}(A^*)) \quad (\text{by Theorem 3.5}) \\ &= \text{Int}(A^*) \quad (\text{by Lemma 2.1}) \end{aligned}$$

This completes the proof. $\square$

Theorem 4.4. Let $A \in P\mathcal{IO}(X)$. Then A is $\mathcal{I}$ -open.

Proof. It is straightforward by Proposition 4.3. $\square$

The converse of Theorem 4.4 may not be true as seen in the following example.

Example 4.5. Let $X = \{a, b, c, d\}$, $\tau = \{X, \emptyset, \{c\}, \{a, b\}, \{a, b, c\}\}$ and $\mathcal{I} = \{\emptyset, \{a\}\}$. Then $(X, \tau, \mathcal{I})$ be an ideal topological space. We have that

$\tau^p = \{X, \emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}, \{a, c, d\}, \{b, c, d\}\}$. Since $\text{Int}(\{a, b\}^*) = \text{Int}(\{a, b, d\}) = \{a, b\}$, we have $\{a, b\} \in \mathcal{IO}(X)$. But $\{a, b\} \notin P\mathcal{IO}(X)$ because $p\text{Int}(\{a, b\}_p^*) = p\text{Int}(\{b\}) = \{b\}$.

Lemma 4.6. [27] Let A be a subset of a topological space (X, τ) . Then the following assertions are satisfied.

- (i) $(p\text{Int}(A))^c = p\text{Cl}(A^c)$.

(ii) $(pCl(A))^c = pInt(A^c)$.

Theorem 4.7. *If A is P - $\mathcal{I}$ -closed in an ideal topological space $(X, \tau, \mathcal{I})$, then $A \supset (pInt(A))_p^*$.*

Proof. Since A is P - $\mathcal{I}$ -closed, A^c is P - $\mathcal{I}$ -open. Thus

$$\begin{aligned} A^c &\subset pInt((A^c)_p^*) \\ &\subset pInt(pCl(A^c)) \\ &= (pCl(pInt(A)))^c \quad (\text{by Lemma 4.6}). \end{aligned}$$

Hence $A \supset pCl(pInt(A)) \supset (pInt(A))_p^*$ by Theorem 3.5. $\square$

Theorem 4.8. *Let $(X, \tau, \mathcal{I})$ be an ideal topological space. Let A be a subset of X such that $((pInt(A))_p^*)^c = pInt((A^c)_p^*)$. Then A is P - $\mathcal{I}$ -closed if and only if $A \supset (pInt(A))_p^*$.*

Proof. Let A be a subset of X such that $((pInt(A))_p^*)^c = pInt((A^c)_p^*)$. Suppose that A is P - $\mathcal{I}$ -closed. Then $A \supset (pInt(A))_p^*$ by Theorem 4.7.

Conversely, suppose that $A \supset (pInt(A))_p^*$. Then $A^c \subset ((pInt(A))_p^*)^c = pInt((A^c)_p^*)$. Hence A is P - $\mathcal{I}$ -closed. $\square$

Theorem 4.9. *For any P - $\mathcal{I}$ -open set A in an ideal topological space $(X, \tau, \mathcal{I})$, we have $A_p^* = (pInt(A_p^*))_p^*$.*

Proof. Let A be a P - $\mathcal{I}$ -open set. Then $A \subset pInt(A_p^*)$. By Lemma 3.8, we know that $A_p^* \subset (pInt(A_p^*))_p^*$. In general, since $A_p^* \supset pInt(A_p^*)$, it follows from Lemma 3.8 that $A_p^* \supset (A_p^*)_p^* \supset (pInt(A_p^*))_p^*$. Therefore we obtain the desired result. $\square$

Theorem 4.10. *Let $\{U_i \in PIO(X) : i \in \Lambda\}$ be a class of P - $\mathcal{I}$ -open sets in an ideal topological space $(X, \tau, \mathcal{I})$. Then $\bigcup_{i \in \Lambda} \{U_i \in PIO(X) : i \in \Lambda\}$ is P - $\mathcal{I}$ -open.*

Proof. Note that $U_i \subset pInt((U_i)_p^*)$ for every $i \in \Lambda$. It follows that

$$\begin{aligned} \bigcup_{i \in \Lambda} U_i &\subset \bigcup_{i \in \Lambda} pInt((U_i)_p^*) \\ &\subset pInt\left(\bigcup_{i \in \Lambda} (U_i)_p^*\right) \\ &\subset pInt\left(\left(\bigcup_{i \in \Lambda} (U_i)\right)_p^*\right) \quad (\text{by Lemma 3.12}) \end{aligned}$$

so that $\bigcup_{i \in \Lambda} \{U_i \in PIO(X) : i \in \Lambda\}$ is P - $\mathcal{I}$ -open. $\square$

The intersection of two $P\mathcal{I}$ -open sets need not be $P\mathcal{I}$ -open as seen in following example.

Example 4.11. Consider an ideal topological space $(X, \tau, \mathcal{I})$ where $X = \{a, b, c, d\}$, $\tau = \{X, \emptyset, \{a, b\}, \{a, b, c\}\}$, $\mathcal{I} = \{\emptyset\}$. Then $\tau^p = \{X, \emptyset, \{a\}, \{b\}, \{a, b\}, \{a, c\}, \{a, d\}, \{b, c\}, \{b, d\}, \{a, b, c\}, \{a, b, d\}, \{a, c, d\}, \{b, c, d\}\}$. Note that $\{a, c\}, \{b, c, d\} \in PIO(X)$, but $\{a, c\} \cap \{b, c, d\} = \{c\} \notin PIO(X)$.

Theorem 4.12. Let $(X, \tau, \mathcal{I})$ be an ideal topological space. If W is a $P\mathcal{I}$ -open and τ^p is a topology, then $pCl(V) \cap W \subset (V \cap W)_p^*$ for every $V \in \tau^\alpha$.

Proof. Suppose that W is a $P\mathcal{I}$ -open and τ^p is a topology. Let $V \in \tau^\alpha$. Then $V \subset \text{Int}(\text{Cl}(\text{Int}(V)))$. It follows that

$$\begin{aligned} pCl(V) \cap W &\subset pCl(V) \cap pInt(W_p^*) \\ &\subset pCl(V \cap pInt(W_p^*)) \quad (\text{by [27, Theorem 4.17]}) \\ &\subset pCl(V \cap W_p^*) \\ &\subset pCl(V \cap W)_p^* \quad (\text{by Lemma 3.8}) \\ &= (V \cap W)_p^*. \end{aligned}$$

This completes the proof. $\square$

Lemma 4.13. Let (X, τ) be a topological space. Let A be a subset of X and let $B \in \tau^\alpha$. Then $pInt(A) \cap B = pInt(A \cap B)$.

Proof. Since $pInt(A \cap B) \subset pInt(A)$ and $pInt(A \cap B) \subset B$, we have

$$pInt(A \cap B) \subset pInt(A) \cap B.$$

Conversely, let $x \in pInt(A) \cap B$. Then $x \in pInt(A)$ and $x \in B$. Since $x \in pInt(A)$, there exists pre-open set G_x containing x such that $G_x \subset A$. Since $x \in B$ and $B \in \tau^\alpha$, $G_x \cap B \in \tau^p$ by Lemma 3.7 and $x \in G_x \cap B \subset A \cap B$. Hence $x \in pInt(A \cap B)$. Therefore $pInt(A) \cap B = pInt(A \cap B)$. $\square$

Theorem 4.14. Let $(X, \tau, \mathcal{I})$ be an ideal topological space and let A, B be subsets of X . If $A \in PIO(X, \tau)$ and $B \in \tau^\alpha$ then $A \cap B \subset pInt(B \cap (A \cap B)_p^*)$.

Proof. Let $A \in PIO(X, \tau)$ and $B \in \tau^\alpha$. Then

$$\begin{aligned} A \cap B &\subset pInt(A_p^*) \cap B \\ &= pInt(A_p^* \cap B) \quad (\text{by Lemma 4.13}) \\ &= pInt(B \cap (A \cap B)_p^*). \quad (\text{by Lemma 3.8}) \end{aligned}$$

This completes the proof. $\square$

If A is not P - $\mathcal{I}$ -open but $\mathcal{I}$ -open in an ideal topological space $(X, \tau, \mathcal{I})$ then is the above theorem valid? The answer is negative as seen in the following example.

Example 4.15. Consider the ideal topological space $(X, \tau, \mathcal{I})$ which is presented in Example 3.4. Let $A = \{a, b\}$ and $B = \{a, b, c\}$. Then A is $\mathcal{I}$ -open which is not P - $\mathcal{I}$ -open, and $B \in \tau^\alpha$. But $A \cap B = \{a, b\} \not\subseteq \{b\} = p\text{Int}(B \cap (A \cap B)_p^*)$.

Theorem 4.16. Let $(X, \tau, \mathcal{I})$ be an ideal topological space and let A, B be subsets of X . If $A \in P\mathcal{IO}(X, \tau)$ and $B \in \tau^\alpha$ then $A \cap B \in P\mathcal{IO}(X, \tau)$.

Proof. Let $A \in P\mathcal{IO}(X, \tau)$ and $B \in \tau^\alpha$. Then

$$\begin{aligned} A \cap B &\subset p\text{Int}(A_p^*) \cap B \\ &= p\text{Int}(A_p^* \cap B) \quad (\text{by Lemma 4.13}) \\ &\subset p\text{Int}((A \cap B)_p^*), \quad (\text{by Lemma 3.8}) \end{aligned}$$

and therefore $A \cap B \in P\mathcal{IO}(X, \tau)$. $\square$

If $A \in P\mathcal{IO}(X, \tau)$ and $B \in \tau^p$ then is $A \cap B$ P - $\mathcal{I}$ -open? The answer is negative as seen in the following example.

Example 4.17. Consider an ideal topological space $(X, \tau, \mathcal{I})$ where $X = \{a, b, c, d\}$, $\tau = \{X, \emptyset, \{c\}, \{a, b\}, \{a, b, c\}\}$ and $\mathcal{I} = \{\emptyset, \{a\}\}$. Then $\tau^p = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}, \{a, c, d\}, \{b, c, d\}, X\}$. Let $A = \{b, c, d\}$ and $B = \{a, c, d\}$. Then $A \in P\mathcal{IO}(X, \tau)$ and $B \in \tau^p$. Since $(A \cap B)_p^* = \{c, d\}$, $p\text{Int}((A \cap B)_p^*) = p\text{Int}(\{c, d\}) = \{c\}$. Hence $A \cap B = \{c, d\} \not\subseteq p\text{Int}((A \cap B)_p^*) = \{c\}$. Therefore $A \cap B \notin P\mathcal{IO}(X, \tau)$.

If $A \in \mathcal{IO}(X, \tau)$ and $B \in \tau^\alpha$ then is $A \cap B$ P - $\mathcal{I}$ -open? The answer is negative as seen in the following example.

Example 4.18. Consider the ideal topological space $(X, \tau, \mathcal{I})$ in Example 4.17. Take $A = \{a, b, c\} \in \mathcal{IO}(X) \setminus P\mathcal{IO}(X)$ and $B = \{a, b\} \in \tau \subset \tau^\alpha$. Then $A \cap B = \{a, b\} \not\subseteq \{b\} = p\text{Int}(\{a, b\}_p^*) = p\text{Int}((A \cap B)_p^*)$. Hence $A \cap B$ is not P - $\mathcal{I}$ -open.

Corollary 4.19. The union of P - $\mathcal{I}$ -closed set and α -closed set is P - $\mathcal{I}$ -closed set.

Proof. Let A be a P - $\mathcal{I}$ -closed set and B be an α -closed set. Then A^c is a P - $\mathcal{I}$ -open set and B^c is an α -open set. By Theorem 4.16, $(A \cup B)^c = A^c \cap B^c \in P\mathcal{IO}(X, \tau)$. Hence $A \cup B$ is P - $\mathcal{I}$ -closed. $\square$

Theorem 4.20. Let $(X, \tau, \mathcal{I})$ be an ideal topological space and let A be a subset of X . If A is $P\mathcal{I}$ -open and semi-closed, then $A = p\text{Int}(A_p^*)$.

Proof. Since A is $P\mathcal{I}$ -open, $A \subset p\text{Int}(A_p^*)$. Since A is semi-closed, $A^c \subset \text{Cl}(\text{Int}(A^c))$. Thus

$$\begin{aligned} A^c &\subset \text{Cl}(\text{Int}(A^c)) \\ &= p\text{Cl}(\text{Int}(A^c)) \quad (\text{by Lemma 4.2}) \\ &\subset p\text{Cl}(p\text{Int}(A^c)) \\ &= (p\text{Int}(p\text{Cl}(A)))^c. \quad (\text{by Lemma 4.6}) \end{aligned}$$

Hence $A \supset p\text{Int}(p\text{Cl}(A)) \supset p\text{Int}(A_p^*)$, and therefore $A = p\text{Int}(A_p^*)$. $\square$

If $A \in \mathcal{IO}(X, \tau)$ is semi-closed then is $A = p\text{Int}(A_p^*)$ valid? The answer is negative as seen in the following example.

Example 4.21. Consider the ideal topological space $(X, \tau, \mathcal{I})$ in Example 4.17. Take $A = \{a, b\}$. Then A is both semi-closed and $\mathcal{I}$ -open but not $P\mathcal{I}$ -open. We know that $p\text{Int}(A_p^*) = \{b\} \neq A$.

Definition 4.22. A set S in (X, τ) is called *b-open* [24] if

$$S \subseteq \text{Cl}(\text{Int}(S)) \cup \text{Int}(\text{Cl}(S)).$$

Theorem 4.23. If $A \subset W \subset p\text{Cl}(A)$ and $A \in \mathcal{IO}(X, \tau)$, then W is *b-open*.

Proof. Let $A \subset W \subset p\text{Cl}(A)$ and $A \in \mathcal{IO}(X, \tau)$. Then $A \subset \text{Int}(A^*)$ and $\text{Cl}(W) = \text{Cl}(A)$. This implies that

$$\begin{aligned} W &\subset p\text{Cl}(A) \\ &= A \cup \text{Cl}(\text{Int}(A)) \quad (\text{by Lemma 4.2}) \\ &\subset \text{Int}(A^*) \cup \text{Cl}(\text{Int}(A)) \\ &\subset \text{Int}(\text{Cl}(A)) \cup \text{Cl}(\text{Int}(A)) \\ &\subset \text{Int}(\text{Cl}(W)) \cup \text{Cl}(\text{Int}(W)). \end{aligned}$$

Therefore W is *b-open*. $\square$

Corollary 4.24. If $A \subset W \subset p\text{Cl}(A)$ and $A \in P\mathcal{IO}(X, \tau)$, then W is *b-open*.

Proof. Let $A \in P\mathcal{IO}(X, \tau)$. Then by Theorem 4.4, $A \in \mathcal{IO}(X, \tau)$. Hence W is *b-open* by Theorem 4.23. $\square$

Is every superset of a $P\mathcal{I}$ -open set $P\mathcal{I}$ -open? The answer is negative as seen in the example.

Example 4.25. Consider the ideal topological space $(X, \tau, \mathcal{I})$ in Example 3.4. Let $A = \{b, c\}$ and $W = \{a, b, c\}$. Then A is a P - $\mathcal{I}$ -open set and $A \subset W$. But W is not a P - $\mathcal{I}$ -open set.

Theorem 4.26. Let A be a subset of an ideal topological space $(X, \tau, \mathcal{I})$. If $A \subset W \subset p\text{Int}(A_p^*)$ then W is P - $\mathcal{I}$ -open.

Proof. Let A be a subset of X . Suppose that $A \subset W \subset p\text{Int}(A_p^*)$. Then $W \subset p\text{Int}(A_p^*) \subset p\text{Int}(W_p^*)$ by Lemma 3.8. Hence W is P - $\mathcal{I}$ -open. $\square$

5. P -*-closure in the ideal topological spaces

Throughout this chapter $D(A)$ (resp. $D_\alpha(A)$, $D_p(A)$, $D_s(A)$) always mean a derived set (resp. α -derived set, pre-derived set, semi-derived set) of A .

Definition 5.1. Let $(X, \tau, \mathcal{I})$ be an ideal topological space. Let A be a subset of X . Then $\text{Cl}_p^*(A, \tau, \mathcal{I})$ is defined as $\text{Cl}_p^*(A, \tau, \mathcal{I}) = A \cup A_p^*(\tau, \mathcal{I})$. $\text{Cl}_p^*(A, \tau, \mathcal{I})$ is written simply as $\text{Cl}_p^*(A)$ when there is no chance for confusion.

In an ideal topological space $(X, \tau, \mathcal{I})$, $\tau_p^*(\mathcal{I})$ is the collection of the complement of all sets satisfying the following condition :

$$\text{Cl}_p^*(A) = A \text{ for } A \in \mathcal{P}(X).$$

That is, $\tau_p^*(\mathcal{I}) := \{A^c : \text{Cl}_p^*(A) = A \text{ for } A \in \mathcal{P}(X)\}$. $\tau_p^*(\mathcal{I})$ is written simply as τ_p^* when there is no chance for confusion.

Example 5.2. Consider the ideal topological space $(X, \tau, \mathcal{I})$ in Example 3.4. If $A = \{a, c\}$ then $A_p^* = \{c, d\}$. Hence $\text{Cl}_p^*(A, \tau, \mathcal{I}) = A \cup A_p^* = \{a, c, d\}$.

Theorem 5.3. Let $(X, \tau, \mathcal{I})$ be an ideal topological space. Then the following assertions are valid.

- (i) If $\mathcal{I} = \{\emptyset\}$ then $\text{Cl}_p^*(A) = p\text{Cl}(A)$.
- (ii) If $\mathcal{I} = \mathcal{P}(X)$ then $\text{Cl}_p^*(A) = A$.
- (iii) If A is P - $\mathcal{I}$ -open then $\text{Cl}_p^*(A) = A_p^*$.
- (iv) If $A \in \mathcal{I}$ then $\text{Cl}_p^*(A) = A$.

Proof. Straightforward. $\square$

Let $(X, \tau, \mathcal{I})$ be an ideal topological space. If $A \subset W \subset \text{Cl}_p^*(A)$ and $A \in P\mathcal{IO}(X, \tau)$, then is W a P - $\mathcal{I}$ -open set? The answer to this question is negative as seen in the following example.

Example 5.4. Let $X = \{a, b, c, d\}$, $\tau = \{X, \emptyset, \{c\}, \{a, b\}, \{a, b, c\}\}$ and $\mathcal{I} = \{\emptyset\}$. Then $(X, \tau, \mathcal{I})$ is an ideal topological space. Let $A = \{c\}$ and $W = \{c, d\}$. Then $A_p^* = \{c, d\}$, $\text{Int}(A_p^*) = \{c\}$. We know that A is P - $\mathcal{I}$ -open and $A \subset W \subset \text{Cl}_p^*(A)$. But $W \not\subset \text{Int}(W_p^*) = \{c\}$. Hence $W \notin \text{PTIO}(X, \tau)$.

Lemma 5.5. For any subsets A and B of an ideal topological space $(X, \tau, \mathcal{I})$, we have the following assertions:

- (i) $\text{Cl}_p^*(\emptyset) = \emptyset$, $\text{Cl}_p^*(X) = X$.
- (ii) If $A \subset B$ then $\text{Cl}_p^*(A) \subset \text{Cl}_p^*(B)$.
- (iii) $\text{Cl}_p^*(\text{Cl}_p^*(A)) \supset \text{Cl}_p^*(A)$.
- (iv) $\text{Cl}_p^*(A \cup B) \supset \text{Cl}_p^*(A) \cup \text{Cl}_p^*(B)$.
- (v) $\text{Cl}_p^*(A \cap B) \subset \text{Cl}_p^*(A) \cap \text{Cl}_p^*(B)$.
- (vi) $\text{Cl}_p^*(A) \subset \text{Cl}^*(A)$.
- (vii) $\text{Cl}_p^*(A) \subset p\text{Cl}(A)$.
- (viii) Let $\mathcal{J}$ be a superset of $\mathcal{I}$. If $\mathcal{J}$ is an ideal, then $\text{Cl}_p^*(A, \tau, \mathcal{I}) \supset \text{Cl}_p^*(A, \tau, \mathcal{J})$.

Proof. It is easy to verify the desired results by using Lemma 3.8, Theorem 3.5 and Definition 5.1. $\square$

In Lemma 5.5, the reverse inclusions of (iv), (v), (vi), (vii) and the converse of (ii) and (viii) are not valid as seen in the following example.

Example 5.6. Consider an ideal topological space $(X, \tau, \mathcal{I})$ where $X = \{a, b, c, d\}$, $\tau = \{X, \emptyset, \{c\}, \{a, b\}, \{a, b, c\}\}$ and $\mathcal{I} = \{\emptyset, \{c\}\}$. Let $A = \{a, d\}$ and $B = \{a, b\}$. Then $\text{Cl}_p^*(A) = \{a, d\} \subset \{a, b, d\} = \text{Cl}_p^*(B)$. But $A \not\subset B$. Let $A = \{a\}$ and $B = \{b\}$. Then $\text{Cl}_p^*(A \cup B) = \{a, b, d\} \not\subset \{a, b\} = \text{Cl}_p^*(A) \cup \text{Cl}_p^*(B)$. Let $A = \{a, d\}$ and $B = \{a, b\}$. Then $\text{Cl}_p^*(A \cap B) = \text{Cl}_p^*(\{a\}) = \{a\} \not\subset \{a, d\} = \text{Cl}_p^*(A) \cap \text{Cl}_p^*(B)$. Let $A = \{a, c\}$. Then $\text{Cl}_p^*(A) = \{a, c\} \not\subset X = \text{Cl}^*(A)$. Let $A = \{a, c\}$. Then $\text{Cl}_p^*(A) = \{a, c\} \not\subset \{a, c, d\} = p\text{Cl}(A)$. Let $\mathcal{J} = \{\emptyset, \{a\}\}$. Then $\mathcal{J}$ is an ideal. Let $A = \{a, b\}$. Then $\text{Cl}_p^*(A, \tau, \mathcal{I}) = \{a, b, d\} \supset \{a, b\} = \text{Cl}_p^*(A, \tau, \mathcal{J})$. But $\mathcal{I} \not\subset \mathcal{J}$.

In an ideal topological space $(X, \tau, \mathcal{I})$, since

$$A \cup A_p^* = \text{Cl}_p^*(A) \subset p\text{Cl}(A) = A \cup D_p(A),$$

we can guess that $A_p^* \subset D_p(A)$. But this is wrong as seen in the following example.

Example 5.7. Let $X = \{a, b, c, d\}$, $\tau = \{\emptyset, X, \{a, b\}, \{a, b, c\}\}$ and $\mathcal{I} = \{\emptyset, \{a\}\}$. Then $(X, \tau, \mathcal{I})$ is an ideal topological space. If $A =$

$\{a, b, c\}$, then $A_p^* = \{b, c\}$ and $D_p(A) = \{c, d\}$. Hence $A_p^* \not\subset D_p(A)$ and $A_p^* \not\supset D_p(A)$.

Theorem 5.8. *Let $(X, \tau, \mathcal{I})$ be an ideal topological space. For any subset A of X , the following assertions are valid:*

- (i) $A_p^* \setminus A \subset p\text{Cl}(A) \setminus A \subset D_p(A)$.
- (ii) If $\mathcal{I} = \{\emptyset\}$, then $A_p^* \setminus A = p\text{Cl}(A) \setminus A \subset D_p(A)$.
- (iii) If $\mathcal{I} = \mathcal{P}(X)$, then $A_p^* \subset D_p(A)$.

Proof. (i) By Theorem 3.5, $A_p^* \subset p\text{Cl}(A)$. Then $A_p^* \setminus A \subset p\text{Cl}(A) \setminus A$. Since $p\text{Cl}(A) = A \cup D_p(A)$, $p\text{Cl}(A) \setminus A \subset D_p(A)$. It follows that $A_p^* \setminus A \subset p\text{Cl}(A) \setminus A \subset D_p(A)$.

(ii) and (iii) are straightforward by (i) and Theorem 3.6. $\square$

Theorem 5.9. *Let $(X, \tau, \mathcal{I})$ be an ideal topological space. If τ is a chain under the set inclusion, then either $\text{Cl}_p^*(G) = X$ or $\text{Cl}_p^*(G) = G$ for $G \subset X$.*

Proof. It follows from Theorem 3.24. $\square$

Theorem 5.10. *Let $(X, \tau, \mathcal{I})$ be an ideal topological space. Then $\tau^p \subset \tau_p^*$.*

Proof. Let $A \in \tau^p$. Then A^c is pre-closed (i.e. $A^c = p\text{Cl}(A^c)$). This implies that $A^c \subset \text{Cl}_p^*(A^c) \subset p\text{Cl}(A^c) = A^c$. Hence $A^c = \text{Cl}_p^*(A^c)$ (i.e. $A \in \tau_p^*$.) Therefore $\tau^p \subset \tau_p^*$. $\square$

Remark 5.11. The above theorem shows that every open set, pre-open set, P - $\mathcal{I}$ -open set and $\mathcal{I}$ -open set in an ideal topological space $(X, \tau, \mathcal{I})$ is a member of τ_p^* .

The reverse inclusion of Theorem 5.10 may not hold as seen in the following example

Example 5.12. Let $X = \{a, b, c, d\}$, $\tau = \{\emptyset, X, \{a\}, \{b, c\}, \{a, b, c\}\}$ and $\mathcal{I} = \{\emptyset, \{c\}\}$. Then $(X, \tau, \mathcal{I})$ is an ideal topological space and

$\tau^p = \{\emptyset, X, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}, \{a, b, d\}, \{a, c, d\}\}$
 $\tau_p^* = \{\emptyset, X, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{a, d\}, \{b, c\}, \{a, b, c\}, \{a, b, d\}, \{a, c, d\}\}$. We know that $\{a, d\} \in \tau_p^*$ but $\{a, d\} \notin \tau^p$.

Theorem 5.13. *Let $(X, \tau, \mathcal{I})$ be an ideal topological space. Then*

- (i) If $\mathcal{I} = \{\emptyset\}$ then $\tau_p^* = \tau^p$.
- (ii) If $\mathcal{I} = \mathcal{P}(X)$ then $\tau_p^* = \mathcal{P}(X)$.

Proof. (i) Since $\mathcal{I} = \{\emptyset\}$, $A_p^* = p\text{Cl}(A)$ for all $A \in \mathcal{P}(X)$ by Theorem 3.6. Thus

$$\begin{aligned}\tau_p^* &= \{B^c : \text{Cl}_p^*(B) = B \text{ for } B \in \mathcal{P}(X)\} \\ &= \{B^c : B \cup B_p^* = B \text{ for } B \in \mathcal{P}(X)\} \\ &= \{B^c : B \cup p\text{Cl}(B) = B \text{ for } B \in \mathcal{P}(X)\} \\ &= \{B^c : p\text{Cl}(B) = B \text{ for } B \in \mathcal{P}(X)\} \\ &= \tau^p.\end{aligned}$$

(ii) Suppose that $\mathcal{I} = \mathcal{P}(X)$. Let $A \in \mathcal{P}(X)$. Then $\text{Cl}_p^*(A^c) = A^c \cup (A^c)_p^* = A^c$ by Theorem 3.6. Hence $A \in \tau_p^*$. Therefore $\mathcal{P}(X) \subset \tau_p^*$, and hence $\mathcal{P}(X) = \tau_p^*$. $\square$

Theorem 5.14. *Let $(X, \tau, \mathcal{I})$ be an ideal topological space. Then $\tau^* \subset \tau_p^*$.*

Proof. Let $A \in \tau^*$. Then $\text{Cl}^*(A^c) = A^c$, which implies that $A^c \subset \text{Cl}_p^*(A^c) \subset \text{Cl}^*(A^c) = A^c$. Hence $\text{Cl}_p^*(A^c) = A^c$ (i.e. $A \in \tau_p^*$). Therefore $\tau^* \subset \tau_p^*$. $\square$

The reverse inclusion of Theorem 5.14 does not hold as seen in the following example

Example 5.15. Let $X = \{a, b, c, d\}$, $\tau = \{\emptyset, X, \{c\}, \{a, b\}, \{a, b, c\}\}$ and $\mathcal{I} = \{\emptyset\}$. Then $(X, \tau, \mathcal{I})$ is an ideal topological space and

$$\tau_p^* = \{\emptyset, X, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}, \{a, c, d\}, \{b, c, d\}\}$$

and

$$\tau^* = \{\emptyset, X, \{c\}, \{a, b\}, \{a, b, c\}\}.$$

We know that $\{b\} \in \tau_p^*$ but $\{b\} \notin \tau^*$.

Note that τ_p^* contains both τ^p and τ^* . The following example shows that τ^p and τ^* are independent.

Example 5.16. Let $X = \{a, b, c, d\}$, $\tau = \{\emptyset, X, \{c\}, \{a, b\}, \{a, b, c\}\}$ and $\mathcal{I} = \{\emptyset, \{c\}\}$. Then $(X, \tau, \mathcal{I})$ is an ideal topological space and

$$\tau^p = \{\emptyset, X, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}, \{a, c, d\}, \{b, c, d\}\}$$

and

$$\tau^* = \{\emptyset, X, \{c\}, \{a, b\}, \{a, b, c\}, \{a, b, d\}\}.$$

We know that τ^p and τ^* are independent.

Theorem 5.17. *Let $(X, \tau, \mathcal{I})$ be an ideal topological space. If every member of τ is clopen then $\tau_p^* = \mathcal{P}(X)$.*

Proof. Let $A \in \mathcal{P}(X)$. Since every member of τ is clopen, $A \subset \text{Cl}(A) = \text{Int}(\text{Cl}(A))$ and so $A \in \tau^p$. This means that $\mathcal{P}(X) = \tau^p$. By Theorem 5.10, $\mathcal{P}(X) = \tau^p \subset \tau_p^* \subset \mathcal{P}(X)$. Hence $\tau_p^* = \mathcal{P}(X)$. $\square$

In an ideal topological space $(X, \tau, \mathcal{I})$, is τ_p^* a topology? The answer to this question is negative as seen in the following example.

Example 5.18. Consider the ideal topological space $(X, \tau, \mathcal{I})$ in Example 5.15. Then $\{a, c, d\}, \{b, c, d\} \in \tau_p^*$. But $\{a, c, d\} \cap \{b, c, d\} = \{c, d\} \notin \tau_p^*$.

Theorem 5.19. Let $(X, \tau, \mathcal{I})$ be an ideal topological space. If

$$\text{Cl}_p^*(B \cup C) \subset \text{Cl}_p^*(B) \cup \text{Cl}_p^*(C)$$

for any subsets B and C of X , then τ_p^* is a topology.

Proof. Obviously, $\emptyset, X \in \tau_p^*$ by Lemma 5.5(i). Suppose that $G_i \in \tau_p^*$ for every $i \in \Lambda$. Then $\text{Cl}_p^*(G_i^c) = G_i^c$ for every $i \in \Lambda$ and so

$$\text{Cl}_p^*\left(\bigcup_{i \in \Lambda} (G_i)^c\right) = \text{Cl}_p^*\left(\bigcap_{i \in \Lambda} G_i^c\right) \subset \bigcap_{i \in \Lambda} \text{Cl}_p^*(G_i^c) = \bigcap_{i \in \Lambda} G_i^c = \left(\bigcup_{i \in \Lambda} G_i\right)^c.$$

Furthermore, $\left(\bigcup_{i \in \Lambda} G_i\right)^c \subset \text{Cl}_p^*\left(\left(\bigcup_{i \in \Lambda} G_i\right)^c\right)$ by Definition 5.1. Hence

$$\left(\bigcup_{i \in \Lambda} G_i\right)^c = \text{Cl}_p^*\left(\left(\bigcup_{i \in \Lambda} G_i\right)^c\right),$$

and so $\bigcup_{i \in \Lambda} G_i \in \tau_p^*$. Let $G, H \in \tau_p^*$. Then $\text{Cl}_p^*(G^c) = G^c$ and $\text{Cl}_p^*(H^c) = H^c$. It follows from Lemma 5.5(iv) and hypothesis that

$$\text{Cl}_p^*((G \cap H)^c) = \text{Cl}_p^*(G^c \cup H^c) = \text{Cl}_p^*(G^c) \cup \text{Cl}_p^*(H^c) = G^c \cup H^c = (G \cap H)^c$$

so that $G \cap H \in \tau_p^*$. $\square$

Corollary 5.20. Let (X, τ) be a topological space with ideal $\mathcal{I} = \mathcal{P}(X)$. Then τ_p^* is a topology.

Proof. By Theorem 3.6(ii), $A_p^* = \emptyset$ for all subset A of X . This implies that $\text{Cl}_p^*(B \cup C) = (B \cup C) \cup (B \cup C)_p^* = B \cup C = (B \cup B_p^*) \cup (C \cup C_p^*) = \text{Cl}_p^*(B) \cup \text{Cl}_p^*(C)$ for any subsets B and C of X . Hence by Theorem 5.19, τ_p^* is a topology. $\square$

Corollary 5.21. Let $(X, \tau, \mathcal{I})$ be an ideal topological space. If

$$(B \cup C)_p^* \subset B_p^* \cup C_p^*$$

for any subsets B and C of X , then τ_p^* is a topology.

Proof. For any subsets B and C of X , $(B \cup C)_p^* \supset B_p^* \cup C_p^*$ by Lemma 3.8. Using hypothesis, we get $(B \cup C)_p^* \subset B_p^* \cup C_p^*$. Hence $(B \cup C)_p^* = B_p^* \cup C_p^*$. we can easily check that $\text{Cl}_p^*(B \cup C) = \text{Cl}_p^*(B) \cup \text{Cl}_p^*(C)$. It follows from Theorem 5.19 that τ_p^* is a topology. $\square$

Theorem 5.22. Let $(X, \tau, \mathcal{I})$ and $(X, \tau, \mathcal{J})$ be ideal topological spaces. If $\mathcal{I} \subset \mathcal{J}$ then $\tau_p^*(\mathcal{I}) \subset \tau_p^*(\mathcal{J})$.

Proof. Suppose that $\mathcal{I} \subset \mathcal{J}$ and let $A \in \tau_p^*(\mathcal{I})$. Using Lemma 5.5(viii), we have $A^c = \text{Cl}^*(A^c, \tau, \mathcal{I}) \supset \text{Cl}^*(A^c, \tau, \mathcal{J}) \supset A^c$. Hence $\text{Cl}^*(A^c, \tau, \mathcal{J}) = A^c$, i.e., $A \in \tau_p^*(\mathcal{J})$. Therefore $\tau_p^*(\mathcal{I}) \subset \tau_p^*(\mathcal{J})$. $\square$

Let $(X, \tau, \mathcal{I})$ and $(X, \kappa, \mathcal{I})$ be ideal topological spaces. If $\tau \subset \kappa$ then is the inclusion $\tau_p^* \subset \kappa_p^*$ valid? The answer is negative as seen in the following example.

Example 5.23. Consider two ideal topological spaces $(X, \tau, \mathcal{I})$ and $(X, \kappa, \mathcal{I})$ where $X = \{a, b, c, d\}$, $\tau = \{\emptyset, X, \{c\}, \{a, b\}, \{a, b, c\}\}$, $\kappa = \{\emptyset, X, \{a, b\}, \{a, b, c\}\}$ and $\mathcal{I} = \{\emptyset\}$. Then

$\tau_p^* = \{\emptyset, X, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}, \{a, c, d\}, \{b, c, d\}\}$
and $\kappa_p^* = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}, \{a, c\}, \{a, d\}, \{b, c\}, \{b, d\}, \{a, b, c\}, \{a, b, d\}, \{a, c, d\}, \{b, c, d\}\}$. It follows that $\tau_p^* \not\subset \kappa_p^*$ and $\tau_p^* \not\supset \kappa_p^*$.

Let $(X, \tau, \mathcal{I})$ be an ideal topological space. If $\mathcal{I} \subset \tau$ then is τ^p equal to τ_p^* ? The answer is negative as seen in the following example.

Example 5.24. Let $X = \{a, b, c, d\}$, $\tau = \{\emptyset, X, \{c\}, \{a, b\}, \{a, b, c\}\}$ and $\mathcal{I} = \{\emptyset, \{c\}\}$. Then $(X, \tau, \mathcal{I})$ is an ideal topological spaces with $\mathcal{I} \subset \tau$, and

$\tau^p = \{\emptyset, X, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}, \{a, c, d\}, \{b, c, d\}\}$,
and $\tau_p^* = \{\emptyset, X, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{a, d\}, \{b, c\}, \{b, d\}, \{a, b, c\}, \{a, b, d\}, \{a, c, d\}, \{b, c, d\}\}$. Clearly $\{a, d\} \in \tau_p^*$ but $\{a, d\} \notin \tau^p$. Hence $\tau^p \neq \tau_p^*$.

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