EMPTY CONVEX 5-GONS IN PLANAR POINT SETS

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Abstract. Erdős posed the problem of determining the minimum number $g(n)$ such that any set of $g(n)$ points in general position in the plane contains an empty convex n -gon. In 1978, Harborth proved that $g(5) = 10$. We reprove the result in a geometric approach.

1. Introduction

Due to the importance in computer graphics, geometric modellings, motion planing, and other areas, the study of the convex polygons has a renewed interest, recently. In this paper, we reprove a known result related to the Erdős and Szekeres problem which is a classic problem counting empty convex k -gons in planar points sets.

In 1935, Erdős and Szekeres posed the problem of determining the smallest positive integer $f(n)$ so that any set of at least $f(n)$ points in opened position in the plane contains a convex n -gon and also conjectured that $f(n) = 2^{n-2} + 1$ for all $n \geq 3$ [1]. Until now, the problem has been solved for the values $n = 3, 4$, and 5 only ([4]). For larger values of n , the best known upper bound $f(n) \leq \binom{2n-5}{n-2} + 1$ was recently proved by Tóth and Valtr[7]. Later, Erdős posed a similar problem on empty convex polygons, which is of determining a smallest positive integer $g(n)$ such that any set of at least $g(n)$ points in general position in the plane contains an empty convex n -gon. For $n = 3$ and 4 , it is easy to show that $g(3) = 3$ and $g(4) = 5$. For $n = 5$, Harborth proved that $g(5) = 10$ [2]. But Horton showed that $g(n)$ does not exist for all $n \geq 7$ [3]. The only open case is for $n = 6$. Even the existence of $g(6)$ still

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remains as a question. Here we reprove that $g(5) = 10$ by visualizing the method of Harborth.

Theorem 1.1. *Any set X of at least 10 points in general position in the plane contains the vertex set of an empty convex pentagon.*

2. Proof of Theorem 1

Let X be a set of $n \geq 10$ points in general position in the plane. Since $f(5) = 9$, among the points in X , there are 5 points in convex position.

FIGURE 1.

If there are $m \geq 2$ points of X inside the convex pentagon, we construct another pentagon with two of m points inside the pentagon and three of 5 points on the pentagon. The pentagon reconstructed has points inside less than m . In this way, we reduce the number of points inside a pentagon and eventually will get a convex pentagon P with zero or one point inside it. If P is non-empty, denote the points of X on P by P_1, P_2, P_3, P_4, P_5 in a counterclockwise order and the point inside P by C . If C is inside one of triangles $\triangle P_1P_2P_3$, $\triangle P_2P_3P_4$, $\triangle P_3P_4P_5$, $\triangle P_4P_5P_1$, and $\triangle P_5P_1P_2$, then we can construct an empty convex pentagon with C and four of five points P_1, P_2, P_3, P_4, P_5 .

Suppose that C is not inside either of the triangles and so is in the small pentagon inside P (see the figure 2). We divide cases according to place of points outside P and will get an empty convex pentagon from $C, P_1, P_2, P_3, P_4, P_5$ and points outside P . We denote (P_i, P_j) the open half-plane lying to the left of the line $\overline{P_iP_j}$ directed from P_i to P_j . Then we consider the region $R_i = (C, P_i) \cap (P_{i+1}, P_i) \cap (P_{i+1}, P_{i+2})$, $i = 1, 2, 3, 4, 5$ and $P_6 = P_1$ and $P_7 = P_2$ (see the figure 3).

FIGURE 2.

FIGURE 3.

If there is a point of X in the region R_i , then we can construct an empty convex pentagon with C and P_i, P_{i+1}, P_{i+2} and the innermost one q_5 of points in R_5 . Similarly, for the region $R'_i = (P_i, C) \cap (P_i, P_{i-1}) \cap (P_{i-2}, P_{i-1})$, $i = 1, 2, 3, 4, 5$ and $P_0 = P_5$ and $P_{-1} = P_4$, we can also get an empty convex pentagon in the case that there is a point in the region R'_i (see the figure 4).

Now we suppose that there are no points of X in either R_i or R'_i , $i = 1, 2, 3, 4, 5$. Then at least four points of X are in the region $S = S_1 \cup S_2 \cup S_3 \cup S_4 \cup S_5$, where $S_i = (P_i, P_{i-1}) \cap (P_{i+2}, P_{i+1})$, $i = 1, 2, 3, 4, 5$ and $P_0 = P_5, P_6 = P_1$, and $P_7 = P_2$ (see the figure 5). We first consider the case that each region of S_i , $i = 1, 2, 3, 4, 5$, has at most one point. In this case, there are three consecutive regions S_i, S_{i+1}, S_{i+2} each of

FIGURE 4.

which has one point of X , and so that the three points of $S_i \cup S_{i+1} \cup S_{i+2}$ with P_{i+1} and P_{i+2} form an empty convex pentagon (see the figure 5).

FIGURE 5.

Now we suppose that there are two or more points in a region S_i . We may assume the region as S_5 . Let v_1 be the innermost point of X in S_5 so that there is no point of X inside the triangle $\Delta v_1 P_1 P_5$. Here we divide two cases according to the position of the points of X remaining in S_5 .

Case 1. There are points of X in the region $T_0 = ((P_5, v_1) \cup (v_1, P_1)) \cap S_5$, then choose the innermost one t_0 so that v_1, t_0, P_1, C, P_5 form an empty convex pentagon (see the figure 6).

FIGURE 6.

Case 2. If there is no point of X in T_0 , the region $T_1 = (P_1, v_1) \cap (v_1, P_5)$ has at least one point of X . Choose the innermost one t_1 .

Case 2.1. If there is a point of X in $S_1 \cup S_2$, we can get an empty convex pentagon with t_1, v_1, P_1, P_2 and one point of $S_1 \cup S_2$.

Case 2.2. If there is a point of X in $S_3 \cup S_4$, we can get an empty convex pentagon with t_1, v_1, P_5, P_4 and one point of $S_3 \cup S_4$ (see the figure 7).

FIGURE 7.

Case 2.3. Suppose that there is no point in $S_1 \cup S_2 \cup S_3 \cup S_4$. Then there is at least two more points of X in T_1 . Choose the innermost one t_2 of X in T_1 except for t_1 . If t_2 is in (t_1, v_1) , then t_2, t_1, v_1, P_5, P_4 form an empty convex pentagon. Otherwise, t_2, t_1, v_1, P_1, P_2 form an empty convex pentagon (see the figure 8).

FIGURE 8.

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