FUZZIFYING FILTERS IN LATTICE IMPLICATION ALGEBRAS

SEON YU KIM AND EUN HWAN ROH*

Abstract. We introduce the concepts of a fuzzifying filter and a fuzzifying implicative filter in lattice implication algebras.

1. Introduction

In 1993, Y. Xu [9] proposed the concept of lattice implication algebras, and discussed some properties. Also, in [10], Y. Xu and K. Y. Qin introduced the notions of a filter and implicative filter in a lattice implication algebra, and investigated properties. The present authors investigated properties of lattice implication algebras([3, 6, 8, 11, 12]). After the introduction of the concept of fuzzy set by L. A. Zadeh [15] several researchers were conducted on the generalizations of the notion of fuzzy set. The concepts of “fuzzifying” was published by M. S. Ying [13], as elementally developed so call fuzzifying topology with the semantic method of continuous valued logic. J. Z. Shen [14] further established the theory of fuzzifying groups based on complete residuated lattice-valued logic. Recently, many mathematical papers have been written investigating the fuzzifying concepts, and applying the fuzzifying concepts to various mathematics, for example, topology and algebra and so on (see [1, 2, 16, 17]). In this paper, using the Ying’s idea, we introduce fuzzifying filters and fuzzifying implicative filters in lattice implication algebras, and study some properties.

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*Corresponding author. Tel.: +82 55 740 1232.

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2. Preliminaries

We recall a few definitions and properties. Lattice implication algebras have been studied in details(see [4, 5, 9]).

By a *lattice implication algebra* we mean a bounded lattice $L = (L, \vee, \wedge, 0, 1)$ with order-reversing involution “ $\prime$ ” and a binary operation “ $\rightarrow$ ” satisfying the following axioms for all $x, y, z \in L$:

- (I1) $x \rightarrow (y \rightarrow z) = y \rightarrow (x \rightarrow z)$,
- (I2) $x \rightarrow x = 1$,
- (I3) $x \rightarrow y = y' \rightarrow x'$,
- (I4) $x \rightarrow y = y \rightarrow x = 1 \Rightarrow x = y$,
- (I5) $(x \rightarrow y) \rightarrow y = (y \rightarrow x) \rightarrow x$,
- (L1) $(x \vee y) \rightarrow z = (x \rightarrow z) \wedge (y \rightarrow z)$,
- (L2) $(x \wedge y) \rightarrow z = (x \rightarrow z) \vee (y \rightarrow z)$.

We can define a partial ordering “ $\leq$ ” on a lattice implication algebra L by $x \leq y$ if and only if $x \rightarrow y = 1$.

In a lattice implication algebra L , the following hold: for all $x, y, z \in L$,

- (1) $0 \rightarrow x = 1, 1 \rightarrow x = x$ and $x \rightarrow 1 = 1$,
- (2) $x' = x \rightarrow 0, x = x' \rightarrow 0$,
- (3) $x \leq y \rightarrow x$
- (4) $(x \rightarrow y) \rightarrow ((y \rightarrow z) \rightarrow (x \rightarrow z)) = 1$.

Let F be a subset of a lattice implication algebra L . F is called a *filter* of L if it satisfies for all $x, y \in L$:

- (F1) $1 \in F$,
- (F2) $x \in F$ and $x \rightarrow y \in F$ imply $y \in F$.

F is called an *implicative filter* of L , if it satisfies (F1) and

- (F3) $x \rightarrow (y \rightarrow z) \in F$ and $x \rightarrow y \in F$ imply $x \rightarrow z \in F$ for all $x, y, z \in L$.

In a broad sense, fuzzy logic is viewed as a system of concepts, principles and methods for dealing with modes of reasoning that are approximate rather than exact. There is a connection between degrees of membership in fuzzy sets and degrees of truth of fuzzy statements which is explained as follows.

Given a fuzzy set A , the membership degree $A(x)$ for any element x in the underlying universal set X may be interpreted as the degree of

truth of the fuzzy statement “ x is a member of A ”. Conversely, given an arbitrary statement “ x is F ” where x is from the set X and F is a fuzzy linguistic expression, its degree of truth may be interpreted as the membership degree $A(x)$ by which a fuzzy set A characterized by the linguistic expression F is defined in a given context.

In the beginning, Lukasiewicz study the three valued logic (the truth values are 0, 1 and $\frac{1}{2}$). The classical definitions of the five connectives ($\wedge, \vee, \rightarrow, \leftrightarrow$ and $\neg$) in terms of truth values 0 and 1 are preserved in three valued logic.

In second step Lukasiewicz generalized the three valued logic to what is called *n-valued logic*. The n truth values (or degrees of truth) in this generalized logic are assumed to be relation numbers in the closed unit interval $[0, 1]$. Allowing n to go to infinity and the truth value to be any real number in $[0, 1]$, which called *continuous valued logics*.

In continuous valued logic, the behaviors of the five logical connectives are given by the following equations:

$$(5) \quad \begin{aligned} [\neg p] &:= 1 - [p], \\ [p \wedge q] &:= \min([p], [q]), \\ [p \vee q] &:= \max([p], [q]), \\ [p \rightarrow q] &:= \min(1, 1 - [p] + [q]), \\ [p \leftrightarrow q] &:= [(p \rightarrow q) \wedge (q \rightarrow p)], \end{aligned}$$

where $[p]$ is the degree of truth of a statement p .

Using the degrees of truth explained above one may deduce the following. For any fuzzy sets $\mathfrak{A}$ and $\mathfrak{B}$ of a nonempty set X :

$$(6) \quad \begin{aligned} [A \in \mathfrak{A}] &:= \mathfrak{A}(A), \\ [\forall A(A \in \mathfrak{A})] &:= \inf_{A \subseteq X} \mathfrak{A}(A), \\ \exists A(\mathfrak{A}(A)) &:= \neg \forall A \neg \mathfrak{A}(A), \\ \mathfrak{A} \subseteq \mathfrak{B} &:= \forall A(A \in \mathfrak{A} \rightarrow A \in \mathfrak{B}), \\ \mathfrak{A} \equiv \mathfrak{B} &:= (\mathfrak{A} \subseteq \mathfrak{B}) \wedge (\mathfrak{B} \subseteq \mathfrak{A}), \end{aligned}$$

$$(7) \quad [a] := a \text{ if } a \in [0, 1].$$

Let p be any statement. By $\models p$ we mean that p is a *tautology* i.e., $[p] = 1$.

Through this paper, the binary operation “ $\rightarrow$ ” in a lattice implication algebra will be denoted by juxtaposition, and for any nonempty set

X we denote $\mathcal{P}(X)$ and $\mathcal{F}(X)$ are the family of all subsets of X and the fuzzy power set of X , respectively.

3. Fuzzifying (implicative) filters

Definition 3.1. Let L be a lattice implication algebra and $A \in \mathcal{F}(L)$. We set

$$\begin{aligned} F_1(A) &:= (\forall x)(x \in A \rightarrow 1 \in A), \\ F_2(A) &:= (\forall x)(\forall y)((xy \in A) \wedge (x \in A) \rightarrow y \in A), \\ F_3(A) &:= (\forall x)(\forall y)(\forall z)((x(yz) \in A) \wedge (xy \in A) \rightarrow xz \in A). \end{aligned}$$

The unary fuzzy predicate $F, IF \in \mathcal{F}(\mathcal{P}(L))$ are interpreted as the fuzzy properties to be a *fuzzifying filter* in L and to be a *fuzzifying implicative filter* in L respectively, and are given as follows:

$$F(A) := F_1(A) \wedge F_2(A) \text{ and } IF(A) := F_1(A) \wedge F_3(A).$$

The following Proposition is simple, but sometimes this results is very useful.

Proposition 3.2. Let L be a lattice implication algebra and $\models F_1(A)$. Then for any $x \in L$, we have

$$[x \in A] \leq [1 \in A].$$

Theorem 3.3. Let L be a lattice implication algebra and $\models F(A)$. If $x \leq y$ in L , then $[x \in A] \leq [y \in A]$.

Proof. Suppose that $[x \in A] > [y \in A]$. Then we have

$$\begin{aligned} & \inf_{x,y} \min(1, 1 - \min([xy \in A], [x \in A]) + [y \in A]) \\ &= \inf_{x,y} \min(1, 1 - \min([1 \in A], [x \in A]) + [y \in A]) \quad [\text{by (1)}] \\ &= \inf_{x,y} \min(1, 1 - [x \in A] + [y \in A]) \quad [\text{by Proposition 3.2}] \\ &< 1. \end{aligned}$$

This is a contradiction. Therefore we get $[x \in A] \leq [y \in A]$. $\square$

Theorem 3.4. Let L be a lattice implication algebra. Then we have $\models IF \subseteq F$.

Proof. We prove that $[\forall A(A \in F_2)] \geq [\forall A(A \in F_3)]$.

$$\begin{aligned} & [\forall A(A \in F_2)] \\ &= \inf_A \inf_{x,y} \min(1, 1 - \min([xy \in A], [x \in A]) + [y \in A]) \\ &= \inf_A \inf_{x,y} \min(1, 1 - \min([1(xy) \in A], [1x \in A]) + [1y \in A]) \\ &\geq \inf_A \inf_{x,y,z} \min(1, 1 - \min([z(xy) \in A], [zx \in A]) + [zy \in A]) \\ &= [\forall A(A \in F_3)]. \end{aligned}$$

□

Remark 3.5. In fuzzifying setting the statement $\models F \subseteq IF$ may not be true as illustrated by the following example.

Example 3.6. Let $L := \{0, a, b, c, 1\}$. Define the partially ordered relation on L as $0 < a < b < c < 1$, and define $x \wedge y := \min\{x, y\}$, $x \vee y := \max\{x, y\}$ for all $x, y \in L$ and “ $\cdot$ ” and “ $\rightarrow$ ” (also we will use the juxtaposition) as follows:

x	x'		0	a	b	c	1
0	1	0	1	1	1	1	1
a	c	a	c	1	1	1	1
b	b	b	b	b	1	1	1
c	a	c	a	b	b	1	1
1	0	1	0	a	b	c	1

Then L is a lattice implication algebra. Define the degree of truth of the fuzzy statement “ x is a member of A ” by

$$A(x) := \begin{cases} 0.8 & \text{if } x = 1, \\ 0.6 & \text{if } x = c, \\ 0 & \text{otherwise,} \end{cases}$$

for all $x \in L$. Then it is easy check that A is a fuzzifying filter of L , but A is not a fuzzifying implicative filter of L since

$$\min(1, 1 - \min([a(b0) \in A], [ab \in A]) + [a0 \in A]) = 0.8.$$

Therefore we get $\not\models F \subseteq IF$.

Theorem 3.7. Let L be a lattice implication algebra. Then for any $A_\lambda (\lambda \in \Lambda) \in \mathcal{F}(L)$, we have

$$\models (\forall \lambda)((\lambda \in \Lambda) \rightarrow IF(A_\lambda)) \rightarrow IF(\bigcap_{\lambda \in \Lambda} A_\lambda).$$

Proof.

$$\begin{aligned} [F_1(\bigcap_{\lambda \in \Lambda} A_\lambda)] &= \inf_x \min(1, 1 - [x \in \bigcap_{\lambda \in \Lambda} A_\lambda] + [1 \in \bigcap_{\lambda \in \Lambda} A_\lambda]) \\ &= \inf_x \min(1, 1 - \inf_{\lambda \in \Lambda} [x \in A_\lambda] + \inf_{\lambda \in \Lambda} [1 \in A_\lambda]) \\ &\geq \inf_x \inf_{\lambda \in \Lambda} \min(1, 1 - [x \in A_\lambda] + [1 \in A_\lambda]) \\ &= \inf_{\lambda \in \Lambda} [F_1(A_\lambda)], \text{ and} \end{aligned}$$

$$\begin{aligned}
& [F_3(\cap_{\lambda \in \Lambda} A_\lambda)] \\
&= \inf_{x,y,z} \min(1, 1 - \min([x(yz) \in \cap_{\lambda \in \Lambda} A_\lambda], [xy \in \cap_{\lambda \in \Lambda} A_\lambda]) \\
&\quad + [xz \in \cap_{\lambda \in \Lambda} A_\lambda]) \\
&= \inf_{x,y,z} \min(1, 1 - \min(\inf_{\lambda \in \Lambda} [x(yz) \in A_\lambda], \inf_{\lambda \in \Lambda} [xy \in A_\lambda]) \\
&\quad + \inf_{\lambda \in \Lambda} [xz \in A_\lambda]) \\
&\geq \inf_{x,y,z} \min(1, 1 - \inf_{\lambda \in \Lambda} \min([x(yz) \in A_\lambda], [xy \in A_\lambda]) \\
&\quad + \inf_{\lambda \in \Lambda} [xz \in A_\lambda]) \\
&\geq \inf_{x,y,z} \inf_{\lambda \in \Lambda} \min(1, 1 - \min([x(yz) \in A_\lambda], [xy \in A_\lambda]) + [xz \in A_\lambda]) \\
&= \inf_{\lambda \in \Lambda} [F_3(A_\lambda)].
\end{aligned}$$

Therefore we have

$$\begin{aligned}
[IF(\cap_{\lambda \in \Lambda} A_\lambda)] &= \min([F_1(\cap_{\lambda \in \Lambda} A_\lambda)], [F_3(\cap_{\lambda \in \Lambda} A_\lambda)]) \\
&\geq \inf_{\lambda \in \Lambda} \min([F_1(A_\lambda)], [F_3(A_\lambda)]) \\
&= \inf_{\lambda \in \Lambda} [IF(A_\lambda)].
\end{aligned}$$

□

By Theorems 3.4 and 3.7, we have the following result.

Corollary 3.8. *Let L be a lattice implication algebra. Then for any $A_\lambda (\lambda \in \Lambda) \in \mathcal{F}(L)$, we have*

$$\models (\forall \lambda)((\lambda \in \Lambda) \rightarrow F(A_\lambda)) \rightarrow F(\cap_{\lambda \in \Lambda} A_\lambda).$$

Theorem 3.9. *Let L be a lattice implication algebra and $\models F(A)$. Then we have*

$$\models A \in IF \rightarrow (\forall x)(\forall y)(x(xy) \in A \rightarrow xy \in A).$$

Proof.

$$\begin{aligned}
& \inf_{x,y} \min(1, 1 - [x(xy) \in A] + [xy \in A]) \\
&= \inf_{x,y} \min(1, 1 - \min([x(xy) \in A], [xx \in A]) \\
&\quad + [xy \in A]) \quad [\text{by Proposition 3.2}] \\
&\geq \inf_{x,y,z} \min(1, 1 - \min([x(yz) \in A], [xy \in A]) + [xz \in A]) \\
&= [A \in F_3].
\end{aligned}$$

□

Theorem 3.10. *Let L be a lattice implication algebra and $a \in L$. Then we have*

$$\models A \in IF \rightarrow A_a \in F,$$

where $A_a := \{t \in L \mid A(at) = 1\}$.

Proof.

$$\begin{aligned}
 [A_a \in F_1] &= \inf_x \min(1, 1 - [x \in A_a] + [1 \in A_a]) \\
 &= \inf_x \min(1, 1 - [x \in A_a] + [a1 \in A]) \\
 &= \inf_x \min(1, 1 - [ax \in A] + [1 \in A]) \\
 &= 1, \quad [\text{by Proposition 3.2}]
 \end{aligned}$$

and

$$\begin{aligned}
 [A_a \in F_2] &= \inf_{x,y} \min(1, 1 - \min([xy \in A_a], [x \in A_a]) + [y \in A_a]) \\
 &= \inf_{x,y} \min(1, 1 - \min([a(xy) \in A], [ax \in A]) + [ay \in A]) \\
 &= [A \in F_3]
 \end{aligned}$$

Thus we have $\models A \in IF \rightarrow A_a \in F$. $\square$

4. Cartesian product of fuzzifying implicative filters

Definition 4.1. Let L be a lattice implication algebra, $A \in \mathcal{F}(L \times L)$ and $B \in \mathcal{F}(L)$. A binary fuzzy predicate $R_B \in \mathcal{F}(\mathcal{P}(L \times L))$, called *fuzzifying relation* on B , is given as follows:

$$A \in R_B := (\forall x)(\forall y)((x, y) \in A \rightarrow (x \in B) \wedge (y \in B)).$$

Lemma 4.2. Let L be a lattice implication algebra and $A, B \in \mathcal{F}(L)$. Then

$$\models (A \times B) \in R_L \leftrightarrow (\forall x)(\forall y)((x, y) \in A \times B \leftrightarrow (x \in B) \wedge (y \in B)).$$

Theorem 4.3. Let L be a lattice implication algebra and $A, B \in \mathcal{F}(L)$. Then we have

$$\models (A \in IF) \wedge (B \in IF) \rightarrow (A \times B \in IF).$$

Proof.

$$\begin{aligned}
& [F_1(A \times B)] \\
&= \inf_{x,y} \min(1, 1 - [(x, y) \in A \times B] + [(1, 1) \in A \times B]) \\
&= \inf_{x,y} \min(1, 1 - \min([x \in A], [y \in B]) + \min([1 \in A], [1 \in B])) \\
&\geq \min(\inf_x \min(1, 1 - [x \in A] + [1 \in A]), \\
&\quad \inf_y \min(1, 1 - [y \in B] + [1 \in B])) \\
&= [F_1(A) \wedge F_1(B)], \text{ and} \\
& [F_3(A \times B)] \\
&= \inf_{x_1, y_1, z_1, x_2, y_2, z_2} \min(1, 1 - \min([(x_1, x_2)((y_1, y_2)(z_1, z_2)) \in A \times B], \\
&\quad [(x_1, x_2)(y_1, y_2) \in A \times B] + [(x_1, x_2)(z_1, z_2) \in A \times B]) \\
&= \inf_{x_1, y_1, z_1, x_2, y_2, z_2} \min(1, 1 - \min([(x_1(y_1 z_1), x_2(y_2 z_2)) \in A \times B], \\
&\quad [(x_1 y_1, x_2 y_2) \in A \times B] + [(x_1 z_1, x_2 z_2) \in A \times B]) \\
&= \inf_{x_1, y_1, z_1, x_2, y_2, z_2} \min(1, 1 - \min(\min([x_1(y_1 z_1) \in A], [x_2(y_2 z_2) \in B]), \\
&\quad \min([x_1 y_1 \in A], [x_2 y_2 \in B])) + \min([x_1 z_1 \in A], [x_2 z_2 \in B])) \\
&\geq \min(\inf_{x_1, y_1, z_1} \min(1, 1 - \min([x_1(y_1 z_1) \in A], [x_1 y_1 \in A]) \\
&\quad + [x_1 z_1 \in A]), (\inf_{x_2, y_2, z_2} \min(1, 1 - \min([x_2(y_2 z_2) \in A], \\
&\quad [x_2 y_2 \in A]) + [x_2 z_2 \in A]) \\
&= [F_3(A) \wedge F_3(B)].
\end{aligned}$$

Therefore we have $[IF(A) \wedge IF(B)] \leq [IF(A \times B)]$. $\square$

Theorem 4.4. *Let L be a lattice implication algebra and $A, B \in \mathcal{F}(L)$. Then we have*

- (a) $\models (A \times B \in IF) \rightarrow (\forall x)((x \in A \rightarrow 1 \in A) \vee (x \in B \rightarrow 1 \in B))$.
- (b) *If $\models (\forall x)((x \in A \rightarrow 1 \in B)$, then $\models A \times B \in IF \rightarrow B \in IF$.*

Proof. (a) Since $[IF(A \times B)] \leq [F_1(A \times B)]$, we only need to prove that $[F_1(A \times B)] \leq \inf_x \max(\min(1, 1 - [x \in A] + [1 \in A]), \min(1, 1 - [x \in B] + [1 \in B]))$. Suppose that $[F_1(A \times B)] \geq t$ for $t \in [0, 1]$. If there exists a $x \in L$ and a $y \in L$ such that $\min(1, 1 - [x \in A] + [1 \in A]) < t$ and $\min(1, 1 - [x \in B] + [1 \in B]) < t$ respectively, then there exists a $(x, y) \in L \times L$ such that $[x \in A] > 1 - t + [1 \in A]$ and $[x \in B] > 1 - t + [1 \in B]$.

Thus we have

$$\begin{aligned}
 (A \times B)(x, y) &= \min([x \in A], [y \in B]) \\
 &> \min(1 - t + [1 \in A], 1 - t + [1 \in B]) \\
 &= 1 - t + \min([1 \in A], [1 \in B]) \\
 &= 1 - t + (A \times B)(1, 1).
 \end{aligned}$$

On the other hand, such a some $(x, y) \in L \times L$, we obtain

$$\begin{aligned}
 &\min(1, 1 - [(x, y) \in A \times B] + [(1, 1) \in A \times B]) \\
 &\leq 1 - [(x, y) \in A \times B] + [(1, 1) \in A \times B] \\
 &< 1 - (1 - t + [(1, 1) \in A \times B]) + [(1, 1) \in A \times B] \\
 &= t,
 \end{aligned}$$

it is a contradiction.

(b) By (a), we can suppose that $\max(\inf_x \min(1, 1 - [x \in A] + [1 \in A]), \inf_x \min(1, 1 - [x \in B] + [1 \in B])) = \inf_x \min(1, 1 - [x \in A] + [1 \in A])$. Then $[IF(A \times B)] \leq [(\forall x)(x \in A \rightarrow 1 \in A)] = [F_1(A)]$. If $\models (\forall x)((x \in A \rightarrow 1 \in B))$, we have $[x \in A] \leq [1 \in B]$ for any $x \in L$. Thus we have

$$\begin{aligned}
 &[IF(A \times B)] \leq [F_3(A \times B)] \\
 &= \inf_{x_1, y_1, z_1, x_2, y_2, z_2} \min(1, 1 - \min([(x_1, x_2)((y_1, y_2)(z_1, z_2)) \in A \times B], \\
 &\quad [(x_1, x_2)(y_1, y_2) \in A \times B]) + [(x_1, x_2)(z_1, z_2) \in A \times B]) \\
 &= \inf_{x_1, y_1, z_1, x_2, y_2, z_2} \min(1, 1 - \min([(x_1(y_1 z_1), x_2(y_2 z_2)) \in A \times B], \\
 &\quad [(x_1 y_1, x_2 y_2) \in A \times B]) + [(x_1 z_1, x_2 z_2) \in A \times B]) \\
 &\leq \inf_{x_1, y_1, z_1} \min(1, 1 - \min([(x_1(y_1 z_1), 1) \in A \times B], \\
 &\quad [(x_1 y_1, 1) \in A \times B]) + [(x_1 z_1, 1) \in A \times B]) \\
 &= \inf_{x_1, y_1, z_1} \min(1, 1 - \min(\min([x_1(y_1 z_1) \in A], [1 \in B]), \\
 &\quad \min([x_1 y_1 \in A], [1 \in B])) + \min([x_1 z_1 \in A], [1 \in B])) \\
 &= \inf_{x_1, y_1, z_1} \min(1, 1 - \min([x_1(y_1 z_1) \in A], [x_1 y_1 \in A]) + [x_1 z_1 \in A]) \\
 &= [F_3(A)].
 \end{aligned}$$

Therefore we get $[IF(A \times B)] \leq \min([F_1(A)], [F_3(A)]) = [IF(A)]$. $\square$

Theorem 4.5. Let L be a lattice implication algebra and $B \in \mathcal{F}(L)$. If $B^2 \in (L \times L)$ which is defined as follows: for all $x, y \in L$,

$$(x, y) \in B^2 := (x \in B) \wedge (y \in B).$$

Then $\models B \in IF \leftrightarrow B^2 \in IF$.

Proof. First we prove that $\models B \in IF \rightarrow B^2 \in IF$. $[F_1(B^2)] = \inf_{x,y} \min(1, 1 - [(x, y) \in B^2] + [(1, 1) \in B^2]) = \inf_{x,y} \min(1, 1 - \min([x \in B], [y \in B]) + [1 \in B]) \geq \inf_{x,y} \min(1, 1 - [x \in B] + [1 \in B]) = \inf_x \min(1, 1 - [x \in B] + [1 \in B]) = [F_1(B)]$.

$$\begin{aligned}
& [F_3(B^2)] \\
&= \inf_{x_1, y_1, z_1, x_2, y_2, z_2} \min(1, 1 - \min([(x_1, x_2)((y_1, y_2)(z_1, z_2)) \in B^2], \\
&\quad [(x_1, x_2)(y_1, y_2) \in B^2]) + [(x_1, x_2)(z_1, z_2) \in B^2]) \\
&= \inf_{x_1, y_1, z_1, x_2, y_2, z_2} \min(1, 1 - \min([(x_1(y_1 z_1), x_2(y_2 z_2)) \in B^2], \\
&\quad [(x_1 y_1, x_2 y_2) \in B^2]) + [(x_1 z_1, x_2 z_2) \in B^2]) \\
&= \inf_{x_1, y_1, z_1, x_2, y_2, z_2} \min(1, 1 - \min(\min([x_1(y_1 z_1) \in B], [x_1 y_1 \in B]), \\
&\quad \min([x_2(y_2 z_2) \in B], [x_2 y_2 \in B]))) \\
&\quad + \min([x_1 z_1 \in B], [x_2 z_2 \in B])) \\
&\geq \inf_{x, y, z} \min(1, 1 - \min([x(yz) \in B], [xy \in B]) + [xz \in B]) \\
&= [F_3(B)].
\end{aligned}$$

Conversely, $[F_1(B^2)] = \inf_{x,y} \min(1, 1 - [(x, y) \in B^2] + [(1, 1) \in B^2]) \leq \inf_x \min(1, 1 - [(x, x) \in B^2] + [(1, 1) \in B^2]) = \inf_x \min(1, 1 - [x \in B] + [1 \in B]) = [F_1(B)]$

$$\begin{aligned}
& [F_3(B^2)] \\
&= \inf_{x_1, y_1, z_1, x_2, y_2, z_2} \min(1, 1 - \min([(x_1, x_2)((y_1, y_2)(z_1, z_2)) \in B^2], \\
&\quad [(x_1, x_2)(y_1, y_2) \in B^2]) + [(x_1, x_2)(z_1, z_2) \in B^2]) \\
&= \inf_{x_1, y_1, z_1, x_2, y_2, z_2} \min(1, 1 - \min([(x_1(y_1 z_1), x_2(y_2 z_2)) \in B^2], \\
&\quad [(x_1 y_1, x_2 y_2) \in B^2]) + [(x_1 z_1, x_2 z_2) \in B^2]) \\
&= \inf_{x_1, y_1, z_1, x_2, y_2, z_2} \min(1, 1 - \min(\min([x_1(y_1 z_1) \in B], [x_2(y_2 z_2) \in B]), \\
&\quad \min([x_1 y_1 \in B], [x_2 y_2 \in B]))) + \min([x_1 z_1 \in B], [x_2 z_2 \in B])) \\
&\leq \inf_{x, y, z} \min(1, 1 - \min(\min([x(yz) \in B], [1 \in B]), \min([xy \in B], \\
&\quad [1 \in B]))) + \min([xz \in B], [1 \in B])) \\
&\leq \inf_{x, y, z} \min(1, 1 - \min([x(yz) \in B], [xy \in B]) + [xz \in B]) \\
&= [F_3(B)].
\end{aligned}$$

□

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Seon Yu Kim and Eun Hwan Roh
 Department of Mathematics Education
 and Institute of Mathematics Education,
 Chinju National University of Education,
 Junju 660-756, Korea
 e-mail: sykim@cue.ac.kr
 e-mail: ehroh@cue.ac.kr or ehroh0923@gmail.com