A NOTE ON GENOCCHI-ZETA FUNCTIONS

KYOUNG HO PARK

Abstract. In this paper, we study the Genocchi-zeta functions which are entire functions in whole complex s -plane these zeta functions have the values of the Genocchi numbers and the Genocchi polynomials at negative integers respectively. That is $\zeta_G(1-k) = \frac{G_k}{k}$ and $\zeta_G(1-k, x) = \frac{G_k(x)}{k}$.

1. Introduction

Let p be a fixed odd prime number. Throughout this paper $\mathbb{Z}_p, \mathbb{Q}_p, \mathbb{C}$, and $\mathbb{C}_p$ will respectively denote the ring of p -adic rational integers, the field of p -adic rational numbers, the complex number field, and the completion of the algebraic closure of $\mathbb{Q}_p$. Let v_p be the normalized exponential valuation of $\mathbb{C}_p$ with $|p|_p = p^{-v_p(p)} = 1/p$. When one talks of q -extension, q is variously considered as an indeterminate, a complex $q \in \mathbb{C}$, or a p -adic number $q \in \mathbb{C}_p$. If $q \in \mathbb{C}$, one normally assumes that $|q| < 1$. If $q \in \mathbb{C}_p$, then we always assumes $|q - 1|_p < 1$. For a fixed positive integer d with $(p, d) = 1$, set

$$X = X_d = \varprojlim_{N} \mathbb{Z}/dP^N\mathbb{Z}, \quad X_1 = \mathbb{Z}_p,$$

$$X^* = \bigcup_{\substack{0 \leq a < dp^N, \\ (a, p) = 1}} (a + dp\mathbb{Z}_p),$$

$$a + dp^N\mathbb{Z}_p = \{x \in X \mid x \equiv a \pmod{dp^N}\}, \quad \text{cf. [1-2, 4-31]},$$

where $a \in \mathbb{Z}$ satisfies the condition $0 \leq a < dp^N$. We say that f is uniformly differentiable function at $a \in \mathbb{Z}_p$, and we write $f \in UD(\mathbb{Z}_p)$,

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if the difference quotients,

$$F_f(x, y) = \frac{f(x) - f(y)}{x - y}$$

have a finite limit $f'(a)$ as $(x, y) \longrightarrow (a, a)$. Throughout this paper, we use the following notation:

$$[x]_q = \frac{1 - q^x}{1 - q} = 1 + q + q^2 + \cdots + q^{x-1}, \quad \text{and} \quad [x]_{-q} = \frac{1 - (-q)^x}{1 + q}.$$

For a function $f \in UD(\mathbb{Z}_p)$, the fermionic p -adic invariant q -integral on $\mathbb{Z}_p$ is defined as

$$I_{-q}(f) = \int_{\mathbb{Z}_p} f(x) d\mu_{-q}(x) = \lim_{N \rightarrow \infty} \frac{1}{[p^N]_{-q}} \sum_{x=0}^{p^N-1} f(x) (-q)^x,$$

see [1-2, 4-31].

Note that

$$I_{-1}(f) = \lim_{q \rightarrow 1} I_{-q}(f) = \int_{\mathbb{Z}_p} f(x) d\mu_{-1}(x).$$

In this study, we investigate some interesting integral equations related to $I_{-1}(f)$. From these integral equations related to $I_{-1}(f)$, we can derive some properties of the Genocchi-zeta function and Hurwitz's type Genocchi-zeta function at negative integers.

2. On Genocchi-zeta Functions

For any complex number t , it is known that the Genocchi polynomials $G_n(x)$ are defined by means of the following generating function:

$$(1) \quad \frac{2t}{e^t + 1} = \sum_{n=0}^{\infty} G_n \frac{t^n}{n!}, \quad |t| < \pi.$$

They satisfy that $G_0 = 0, G_1 = 1, G_2 = -1, G_3 = 0, G_4 = 1, \dots, G_{2k+1} = 0$, and even indexed coefficients are given by

$$G_n = 2(1 - 2^n)B_n,$$

where B_n are Bernoulli numbers. The Genocchi polynomials $G_n(x)$ are defined by means of the following function:

$$(2) \quad F(t, x) = \frac{2t}{e^t + 1} e^{xt} = \sum_{n=0}^{\infty} G_n(x) \frac{t^n}{n!}, \quad |t| < \pi.$$

Let $f_1(x)$ be translation with $f_1(x) = f(x+1)$. Then we have

$$(3) \quad I_{-1}(f_1) + I_{-1}(f) = 2f(0).$$

If we take $f(x) = e^{xt}$, then we can derive the Euler numbers from the integral equation of $I_{-1}(f)$ as follows:

$$\int_{\mathbb{Z}_p} e^{xt} d\mu_{-1}(x) = \frac{2}{e^t + 1}.$$

Then we can derive

$$t \int_{\mathbb{Z}_p} e^{xt} d\mu_{-1}(x) = \sum_{n=0}^{\infty} G_n \frac{t^n}{n!} = t \sum_{n=0}^{\infty} \frac{G_{n+1}}{n+1} \frac{t^n}{n!},$$

and

$$t \int_{\mathbb{Z}_p} e^{(x+y)t} d\mu_{-1}(y) = \sum_{n=0}^{\infty} G_n(x) \frac{t^n}{n!} = t \sum_{n=1}^{\infty} \frac{G_{n+1}(x)}{n+1} \frac{t^n}{n!}.$$

Thus, we obtain

$$(4) \quad \int_{\mathbb{Z}_p} x^n d\mu_{-1}(x) = \frac{G_{n+1}}{n+1}, \quad \text{and} \quad \int_{\mathbb{Z}_p} (x+y)^n d\mu_{-1}(y) = \frac{G_{n+1}(x)}{n+1}.$$

For $n \in \mathbb{N}$, we have

$$(5) \quad \int_{\mathbb{Z}_p} f(x+n) d\mu_{-1}(x) = (-1)^n \int_{\mathbb{Z}_p} f(x) d\mu_{-1}(x) + 2 \sum_{\ell=0}^{n-1} (-1)^{n-1-\ell} f(\ell),$$

see [19 - 31].

By (4) and (5), if we take $f(x) = x^k (k \in \mathbb{Z}^+)$, then we easily see that

$$(6) \quad \int_{\mathbb{Z}_p} (x+n)^k d\mu_{-1}(x) - \int_{\mathbb{Z}_p} x^k d\mu_{-1}(x) = 2 \sum_{\ell=0}^{n-1} (-1)^{\ell-1} \ell^k, \quad \text{if } n \equiv 0 \pmod{2},$$

and

$$(7) \quad \int_{\mathbb{Z}_p} (x+n)^k d\mu_{-1}(x) + \int_{\mathbb{Z}_p} x^k d\mu_{-1}(x) = 2 \sum_{\ell=0}^{n-1} (-1)^{\ell} \ell^k, \quad \text{if } n \equiv 1 \pmod{2}.$$

From (6) and (7), we have the following proposition.

PROPOSITION 1. For $n \in \mathbb{N}$, we have

$$(i) \quad \frac{G_{k+1}(n)}{k+1} - \frac{G_{k+1}}{k+1} = 2 \sum_{\ell=0}^{n-1} (-1)^{\ell-1} \ell^k, \quad \text{if } n \equiv 0 \pmod{2},$$

$$(ii) \quad \frac{G_{k+1}(n)}{k+1} + \frac{G_{k+1}}{k+1} = 2 \sum_{\ell=0}^{n-1} (-1)^\ell \ell^k, \quad \text{if } n \equiv 1 \pmod{2}.$$

For a $s \in \mathbb{C}$, the Genocchi-zeta function and Hurwitz's type Genocchi-zeta function are defined by

$$(8) \quad \zeta_G(s) = 2 \sum_{n=1}^{\infty} \frac{(-1)^n}{n^s}, \quad \text{and} \quad \zeta_G(s, x) = 2 \sum_{n=0}^{\infty} \frac{(-1)^n}{(n+x)^s}.$$

Thus, we note that Genocchi-zeta functions which are entire functions in whole complex s -plane and these zeta functions have the values of the Genocchi numbers and Genocchi polynomials at negative integers respectively. The gamma function is defined by

$$(9) \quad \Gamma(s) = \int_0^{\infty} t^{s-1} e^{-t} dt, \quad \text{for } s \in \mathbb{C}.$$

From (2) and (9), we can derive that

$$\begin{aligned} \frac{-1}{\Gamma(s)} \int_0^{\infty} t^{s-2} F(-t, x) dt &= \frac{-1}{\Gamma(s)} \int_0^{\infty} t^{s-2} \frac{-2te^{-xt}}{1+e^{-t}} dt \\ &= 2 \sum_{n=0}^{\infty} \frac{(-1)^n}{\Gamma(s)} \int_0^{\infty} t^{s-1} e^{-(n+x)t} dt \\ (10) \quad &= 2 \sum_{n=0}^{\infty} (-1)^n \frac{1}{(n+x)^s} \\ &= \zeta_G(s, x). \end{aligned}$$

For $s \in \mathbb{C}$, by (2) and (10), we easily see that

$$\begin{aligned} \zeta_G(s, x) &= \frac{-1}{\Gamma(s)} \int_0^{\infty} t^{s-2} F(-t, x) dt \\ &= \frac{-1}{\Gamma(s)} \int_0^{\infty} t^{s-2} \sum_{n=0}^{\infty} G_n(x) \frac{-t^n}{n!} dt \\ &= 2 \sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{n!} G_n(x) \frac{1}{\Gamma(s)} \int_0^{\infty} t^{s-2+n} dt. \end{aligned}$$

Thus we can derive the following theorem.

THEOREM 2. For $k \in \mathbb{N}$, we have

- (i) $\zeta_G(1-k, x) = \frac{G_k(x)}{k}$.
 (ii) $\zeta_G(1-k, 1) = \frac{G_k(1)}{k} = -\frac{G_k}{k} = -\zeta_G(1-k)$.

Let $f(x) = \sin ax$. From (3), we see that

$$\begin{aligned} 0 &= \int_{\mathbb{Z}_p} \sin a(x+1) d\mu_{-1}(x) + \int_{\mathbb{Z}_p} \sin ax d\mu_{-1}(x) \\ &= (1 + \cos a) \int_{\mathbb{Z}_p} \sin ax d\mu_{-1}(x) + \sin a \int_{\mathbb{Z}_p} \cos ax d\mu_{-1}(x). \end{aligned}$$

If we take $f(x) = \cos ax$, then we obtain

$$2 = (1 + \cos a) \int_{\mathbb{Z}_p} \cos ax d\mu_{-1}(x) - \sin a \int_{\mathbb{Z}_p} \sin ax d\mu_{-1}(x).$$

Then we can derive

$$\int_{\mathbb{Z}_p} \cos ax d\mu_{-1}(x) = 1, \quad \text{and} \quad \int_{\mathbb{Z}_p} \sin ax d\mu_{-1}(x) = -\frac{\sin a}{1 + \cos a}.$$

Thus we obtain the following theorem.

THEOREM 3. *Let the notation and assumptions be as above. Then we have*

$$\tan \frac{a}{2} = - \int_{\mathbb{Z}_p} \sin ax d\mu_{-1}(x) = \sum_{n=0}^{\infty} \frac{(-1)^{n+1} a^{2n+1}}{(2n+2)!} G_{2n+2}.$$

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Kyoung Ho Park
 Department of Mathematics,
 Sogang University,
 Seoul 121-742, South Korea
E-mail: bk.ghpark@yahoo.co.kr, sagamath@yahoo.co.kr