NEW ALGEBRAS USING ADDITIVE ABELIAN GROUPS I

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1. Abstract

The simple non-associative algebra $N(e^{As}, q, n, t)_k$ and its simple subalgebras are defined in the papers [1], [3], [4], [5], [6], [12]. We define the non-associative algebra $\overline{WN}_{(g_n, \mathfrak{A}), m, s_B}$ and its antisymmetrized algebra $\overline{WN}_{(g_n, \mathfrak{A}), m, s_B}^-$. We also prove that the algebras are simple in this work. There are various papers on finding all the derivations of an associative algebra, a Lie algebra, and a non-associative algebra (see [3], [5], [6], [9], [12], [14], [15]). We also find all the derivations $Der_{anti}(WN(e^{\pm x^r}, 0, 2)_B^-)$ of the antisymmetrized algebra $WN(e^{\pm x^r}, 0, 2)_B^-$ and every derivation of the algebra is outer in this paper.

2. Preliminaries

Let $\mathbb{F}$ be a field of characteristic zero (not necessarily algebraically closed). Throughout this paper, $\mathbb{N}$ and $\mathbb{Z}$ will denote the non-negative integers and the integers, respectively. Throughout the paper $\mathfrak{A}$ will denote an abelian group containing $\mathbb{Z}$. Let $\mathbb{F}[x_1, \dots, x_{m+s}]$ be the polynomial ring with the variables $x_1, \dots, x_{m+s}$. Let $g_1, \dots, g_n$ be given polynomials in $\mathbb{F}[x_1, \dots, x_{m+s}]$. For $n, m, s \in \mathbb{N}$, let us define the commutative, associative $\mathbb{F}$ -algebra $\mathbb{F}_{(g_n, \mathfrak{A}), m, s} = \mathbb{F}[e^{\pm g_1}, \dots, e^{\pm g_n}, x_1^{\pm 1}, \dots, x_m^{\pm 1}, x_{m+1}, \dots, x_{m+s}]$ which is called a stable algebra in the paper (see [10])

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and [13]) with the standard basis

$$\{e^{a_1 g_1} \cdots e^{a_n g_n} x_1^{i_1} \cdots x_{m+s}^{i_{m+s}} | a_1, \dots, a_n \in \mathfrak{A}, i_1, \dots, i_m \in \mathbb{Z}, i_{m+1}, \dots, i_{m+s} \in \mathbb{N}\}$$

and with the obvious addition and the multiplication (see [10], [?], [14], [16], [17]) where $m + s \geq n$. ∂_w , $1 \leq w \leq m + s$, denotes the usual partial derivative with respect to x_w on $\mathbb{F}_{(g_n, \mathfrak{A}), m, s}$ formally. For partial derivatives $\partial_u, \dots, \partial_v$ of $\mathbb{F}_{(g_n, \mathfrak{A}), m, s}$, the composition $\partial_u^{j_u} \circ \cdots \circ \partial_v^{j_v}$ of them is denoted by $\partial_u^{j_u} \cdots \partial_v^{j_v}$ where $j_u, \dots, j_v \in \mathbb{N}$. Let A be the set

$$\{\partial_u^{j_u} \cdots \partial_v^{j_v} | j_u, \dots, j_v \in \mathbb{N}, 1 \leq u, \dots, v \leq m + s\}.$$

Let us define the vector space $WN((g_n, \mathfrak{A}), m, s) = WN((g_n, \mathfrak{A}), m, s)_A$ over $\mathbb{F}$ which is spanned by the standard basis

$$\{e^{a_1 g_1} \cdots e^{a_n g_n} x_1^{i_1} \cdots x_{m+s}^{i_{m+s}} \partial_u^{j_u} \cdots \partial_v^{j_v} | a_1, \dots, a_n \in \mathfrak{A}, i_1, \dots, i_m \in \mathbb{Z}, \\ (1) \ i_{m+1}, \dots, i_{m+s} \in \mathbb{N}, j_u, \dots, j_v \in \mathbb{N}, 1 \leq u, \dots, v \leq m + s\}$$

Thus we may define the multiplication $*$ on $WN((g_n, \mathfrak{A}), m, s)$ as follows:

$$e^{a_{11} g_1} \cdots e^{a_{1n} g_n} x_1^{i_{11}} \cdots x_{m+s}^{i_{1, m+s}} \partial_u^{j_u} \cdots \partial_v^{j_v} * e^{a_{21} g_1} \cdots e^{a_{2n} g_n} x_1^{i_{21}} \cdots x_{m+s}^{i_{2, m+s}} \partial_h^{j_h} \cdots \partial_w^{j_w} \\ = e^{a_{11} g_1} \cdots e^{a_{1n} g_n} x_1^{i_{11}} \cdots x_{m+s}^{i_{1, m+s}} \partial_u^{j_u} \cdots \partial_v^{j_v} (e^{a_{21} g_1} \cdots e^{a_{2n} g_n} x_1^{i_{21}} \cdots x_{m+s}^{i_{2, m+s}}) \partial_h^{j_h} \cdots \partial_w^{j_w} \\ (2)$$

for any basis elements $e^{a_{11} g_1} \cdots e^{a_{1n} g_n} x_1^{i_{11}} \cdots x_{m+s}^{i_{1, m+s}} \partial_u^{j_u} \cdots \partial_v^{j_v}$ and $e^{a_{21} g_1} \cdots e^{a_{2n} g_n} x_1^{i_{21}} \cdots x_{m+s}^{i_{2, m+s}} \partial_h^{j_h} \cdots \partial_w^{j_w} \in WN((g_n, \mathfrak{A}), m, s)$. Thus we can define the Weyl-type non-associative algebra $\overline{WN}_{(g_n, \mathfrak{A}), m, s}$ with the multiplication $*$ in (2) and with the set $WN((g_n, \mathfrak{A}), m, s)$ (see [12], [14], [18]). For $B \subset A$, let us define the non-associative subalgebra $\overline{WN}_{(g_n, \mathfrak{A}), m, s_B}$ of the non-associative algebra $\overline{WN}_{(g_n, \mathfrak{A}), m, s}$ spanned by

$$\{e^{a_1 g_1} \cdots e^{a_n g_n} x_1^{i_1} \cdots x_s^{i_s} \partial_u^{j_u} \cdots \partial_v^{j_v} | a_1, \dots, a_n, i_1, \dots, i_m \in \mathbb{Z}, \\ (3) \ i_{m+1}, \dots, i_s, j_u, \dots, j_v \in \mathbb{N}, \partial_u^{j_u} \cdots \partial_v^{j_v} \in B, 1 \leq u, \dots, v \leq m + s\}$$

If we take $B = \{\partial_u^0 \cdots \partial_v^0\}$, then the algebra $\overline{WN}_{(g_n, \mathfrak{A}), m, s_B}$ is the $\mathbb{F}$ -algebra $F_{(g_n, \mathfrak{A}), m, s}$. This implies that the algebra $\overline{WN}_{(g_n, \mathfrak{A}), m, s_B}$ contains the polynomial ring, naturally. The simplicity of $\overline{WN}_{(g_n, \mathfrak{A}), m, s_B}$ is depending on the set B . It is well known that the non-associative algebra $WN_{(g_n, \mathfrak{A}), m, s}$ is simple, even though it has the right annihilator (see [7] and [8]). A non-associative algebra $\overline{WN}_{(g_n, \mathfrak{A}), m, s_B}$ is symmetric if there is a isomorphism induced by change of variables (see [3]). Throughout

the paper we put $\partial^1 = \partial$ and $\partial^k = \partial(\partial^{k-1})$. The algebra $\overline{WN}_{(g_n, \mathfrak{A}), m, s_B}$ is $\mathfrak{A}^n$ -graded as follows:

$$(4) \quad \overline{WN}_{(g_n, \mathfrak{A}), m, s_B} = \bigoplus_{(a_1, \dots, a_n) \in \mathfrak{A}^n} N_{(a_1, \dots, a_n)}$$

where $N_{(a_1, \dots, a_n)}$ is the vector subspace of $\overline{WN}_{(g_n, \mathfrak{A}), m, s_B}$ spanned by

$$\{e^{a_1 g_1} \dots e^{a_n g_n} x_1^{i_1} \dots x_s^{i_s} \partial_u^{j_u} \dots \partial_v^{j_v} | i_1, \dots, i_m \in \mathbb{Z}, i_{m+1}, \dots, i_s, \\ j_u, \dots, j_v \in \mathbb{N}, \partial_u^{j_u} \dots \partial_v^{j_v} \in B, 1 \leq u, \dots, v \leq m + s\}.$$

3. Simplicities and Isomorphisms

Note 1. Since the algebra $\overline{WN}_{(g_n, \mathfrak{A}), m, s_B}$ is $\mathfrak{A}^n$ -graded by (4), the $(0, \dots, 0)$ -homogeneous component $N_{(0, \dots, 0)}$ of the algebra $\overline{WN}_{(g_n, \mathfrak{A}), m, s_B}$ is the simple subalgebra $\overline{WN}_{0, m, s_B}$. $\square$

Lemma 1. Every non-zero element of the subalgebra $\overline{WN}_{0, m, s_B}$ of the algebra $\overline{WN}_{(g_n, \mathfrak{A}), m, s_B}$ is full, i.e., an ideal which contains the element is the algebra $\overline{WN}_{(g_n, \mathfrak{A}), m, s_B}$.

Proof. Since the subalgebra $\overline{WN}_{0, m, s_B}$ of the algebra $\overline{WN}_{(g_n, \mathfrak{A}), m, s_B}$ is simple, every non-zero element of $\overline{WN}_{0, m, s_B}$ is full in the algebra $\overline{WN}_{0, m, s_B}$. Thus it is easy to prove that the element is full in the algebra $\overline{WN}_{(g_n, \mathfrak{A}), m, s_B}$. This completes the proof of the lemma. $\square$

Theorem 1. The algebra $\overline{WN}_{(g_n, \mathfrak{A}), m, s_B}$ is simple and the matrix ring $M_{m+s}(\mathbb{F})$ is a subalgebra of $\overline{WN}_{(g_n, \mathfrak{A}), m, s_B}$ (see [12]).

Proof. By Lemma 1, every non-zero element of the subalgebra $\overline{WN}_{0, m, s_B}$ of $\overline{WN}_{(g_n, \mathfrak{A}), m, s_B}$ is full in the algebra $\overline{WN}_{(g_n, \mathfrak{A}), m, s_B}$. This implies that since the algebra is $\mathfrak{A}^n$ -graded, it is easy to prove that every non-zero element of the algebra $\overline{WN}_{(g_n, \mathfrak{A}), m, s_B}$ is full. This implies that the algebra is simple. This completes the proof of the theorem. $\square$

Theorem 2. The antisymmetrized algebra $\overline{WN}_{(g_n, \mathfrak{A}), m, s_B}^-$ of the algebra $\overline{WN}_{(g_n, \mathfrak{A}), m, s_B}$ is simple and the Lie algebra $sl_{m+s}(\mathbb{F})$ is a subalgebra of $\overline{WN}_{(g_n, \mathfrak{A}), m, s_B}^-$ (see [12]) where $B = \{\partial_u | 1 \leq u \leq m + s\}$.

Proof. Since the algebra is $\mathfrak{A}^n$ -graded and the derived algebra $[\overline{WN}_{(g_n, \mathfrak{A}), m, s_B}^-, \overline{WN}_{(g_n, \mathfrak{A}), m, s_B}^-]$ is itself, by Lemma 1, it is easy to

prove that every non-zero element of the algebra $\overline{WN_{(g_n, \mathfrak{A}), m, s_B}}^-$ is full. This implies that the algebra is simple. This completes the proof of the theorem. $\square$

Corollary 1. *The Lie algebra $\overline{WN_{(g_n, \mathfrak{A}), m, s_B}}^-$ is simple and the Lie algebra $sl_{m+s}(\mathbb{F})$ is a subalgebra of $\overline{WN_{(g_n, \mathfrak{A}), m, s_B}}^-$ (see [12]) where $B = \{\partial_u | 1 \leq u \leq m + s\}$.*

Proof. Since it is easy to prove that the algebra $\overline{WN_{(g_n, \mathfrak{A}), m, s_B}}^-$ is a Lie algebra. So the proof of the corollary is straightforward by Theorem 2. This completes the proof of the corollary. $\square$

Proposition 1. *Let B_1 and B_2 be subsets of $\{\partial_u^{j_u} \cdots \partial_v^{j_v} | j_u, \dots, j_v \in \mathbb{N}, 1 \leq u, \dots, v \leq m + s\}$. If the sets B_1 and B_2 have different cardinalities, then the algebras $\overline{WN_{(g_n, \mathfrak{A}), m, s_{B_1}}}$ and $\overline{WN_{(g_n, \mathfrak{A}), m, s_{B_2}}}$ are not isomorphic.*

Proof. If the algebras of the proposition are isomorphic, then they have isomorphic right annihilators. But they are not isomorphic by assumption. This implies that the algebras are not isomorphic. Therefore we have proven the proposition. $\square$

Theorem 3. *If $\mathfrak{A}_1$ and $\mathfrak{A}_2$ are isomorphic groups, then the algebras $\overline{WN_{(g_n, \mathfrak{A}_1), m, s_B}}$ and $\overline{WN_{(g_n, \mathfrak{A}_2), m, s_B}}$ are isomorphic.*

Proof. Let θ be an isomorphism from the group $\mathfrak{A}_1$ to $\mathfrak{A}_2$. For a standard basis element $e^{a_1 g_1} \cdots e^{a_n g_n} x_1^{i_1} \cdots x_{m+s}^{i_{m+s}} \partial_u^{j_u} \cdots \partial_v^{j_v}$ of $\overline{WN_{(g_n, \mathfrak{A}_1), m, s_B}}$, if we define $\mathbb{F}$ -linear map $\bar{\theta}$ from $\overline{WN_{(g_n, \mathfrak{A}_1), m, s_B}}$ to $\overline{WN_{(g_n, \mathfrak{A}_2), m, s_B}}$ as follows:

$$\begin{aligned} \bar{\theta}(e^{a_1 g_1} \cdots e^{a_n g_n} x_1^{i_1} \cdots x_{m+s}^{i_{m+s}} \partial_u^{j_u} \cdots \partial_v^{j_v}) &= e^{\theta(a_1) g_1} \cdots e^{\theta(a_n) g_n} x_1^{i_1} \cdots \\ & x_{m+s}^{i_{m+s}} \partial_u^{j_u} \cdots \partial_v^{j_v}, \end{aligned}$$

then $\bar{\theta}$ can be linearly extended to an algebra homomorphism from $\overline{WN_{(g_n, \mathfrak{A}_1), m, s_B}}$ to $\overline{WN_{(g_n, \mathfrak{A}_2), m, s_B}}$. Since θ is onto and one to one, $\bar{\theta}$ is also bijective. This implies that $\bar{\theta}$ is an isomorphism. Therefore we have proven the theorem. $\square$

Theorem 4. *If $\mathfrak{A}_1$ and $\mathfrak{A}_2$ are isomorphic groups, then the anti-symmetrized algebras $\overline{WN_{(g_n, \mathfrak{A}_1), m, s_B}}^-$ and $\overline{WN_{(g_n, \mathfrak{A}_2), m, s_B}}^-$ are isomorphic.*

Proof. Since the following equalities $[\overline{WN_{(g_n, \mathfrak{A}_1), m, s_B}^-}, \overline{WN_{(g_n, \mathfrak{A}_1), m, s_B}^-}]$
 $= \overline{WN_{(g_n, \mathfrak{A}_1), m, s_B}^-}$ and $[\overline{WN_{(g_n, \mathfrak{A}_2), m, s_B}^-}, \overline{WN_{(g_n, \mathfrak{A}_2), m, s_B}^-}]$
 $= \overline{WN_{(g_n, \mathfrak{A}_2), m, s_B}^-}$ hold, the proof of the theorem is similar to the proof
of Theorem 3. This completes the proof of the theorem. $\square$

Corollary 2. *If $\mathfrak{A}_1$ and $\mathfrak{A}_2$ are isomorphic groups and $B = \{\partial_u | 1 \leq u \leq m+s\}$, then the Lie algebras $\overline{WN_{(g_n, \mathfrak{A}_1), m, s_B}^-}$ and $\overline{WN_{(g_n, \mathfrak{A}_2), m, s_B}^-}$ are isomorphic.*

Proof. The proof of the corollary is straightforward by Theorem 4. Let us omit its proof. $\square$

4. Derivations of the antisymmetrized algebra $WN(e^{\pm x^r}, 0, 2)_B^-$

For this section, let us put $B = \{\partial_1, \partial_2, \partial_1^2, \partial_2^2\}$, $\mathfrak{A} = \mathbb{Z}$, and $g = x^r$.

Note 2. For any basis elements $e^{px^r} e^{qy^s} x^i y^j \partial_1$, $e^{px^r} e^{qy^s} x^i y^j \partial_1^2$, $e^{px^r} e^{qy^s} x^i y^j \partial_2$, and $e^{px^r} e^{qy^s} x^i y^j \partial_2^2$ of $WN(e^{\pm x^r}, 0, 2)_B^-$, and for given $c_1, c_2 \in \mathbb{F}$, if we define an $\mathbb{F}$ -linear map D_{c_1, c_2} from the algebra $WN(e^{\pm x^r}, 0, 2)_B^-$ to itself as follows:

$$\begin{aligned}
 D_{c_1, c_2}(\partial_u^v) &= 0, \\
 D_{c_1, c_2}(e^{px^r} e^{qy^s} x^i y^j \partial_u^v) \\
 &= \delta_{1,u} \delta_{1,v} c_1 p e^{px^r} e^{qy^s} x^i y^j \partial_1 + \delta_{1,u} \delta_{1,v} c_2 q e^{px^r} e^{qy^s} x^i y^j \partial_1 \\
 &\quad + \delta_{1,u} \delta_{2,v} c_1 p e^{px^r} e^{qy^s} x^i y^j \partial_1^2 + \delta_{1,u} \delta_{2,v} c_2 q e^{px^r} e^{qy^s} x^i y^j \partial_1^2 \\
 &\quad + \delta_{2,u} \delta_{1,v} c_1 p e^{px^r} e^{qy^s} x^i y^j \partial_2 + \delta_{2,u} \delta_{1,v} c_2 q e^{px^r} e^{qy^s} x^i y^j \partial_2 \\
 (5) \quad &\quad + \delta_{2,u} \delta_{2,v} c_1 p e^{px^r} e^{qy^s} x^i y^j \partial_2^2 + \delta_{2,u} \delta_{2,v} c_2 q e^{px^r} e^{qy^s} x^i y^j \partial_2^2
 \end{aligned}$$

then the map D_{c_1, c_2} can be linearly extended to the antisymmetrized algebra derivation of

$WN(e^{\pm x^r}, 0, 2)_B^-$ where $1 \leq u, v \leq 2$, $\delta_{1,j}$ and $\delta_{2,j}$ are Kronecker delta and $1 \leq j \leq 2$ (see [6], [7], [9]). $\square$

Lemma 2. *For any derivation $D \in Der_{anti}(WN(e^{\pm x^r}, 0, 2)_B^-)$, we have that*

$$D(\partial_u^v) = 0, \quad D(x^i \partial_u^v) = 0, \quad D(y^j \partial_u^v) = 0, \quad D(x^i y^j \partial_u^v) = 0$$

hold where $1 \leq u, v \leq 2$.

Proof. Let D be the derivation in the lemma. Since the algebra $WN(e^{\pm x^r}, 0, 2)_B^-$ is $\mathbb{Z}$ -graded, $D(\partial_1)$ and $D(\partial_1^2)$ are the sum of terms in different homogeneous components of $WN(e^{\pm x^r}, 0, 2)_B^-$ in (4). Thus $D(\partial_1)$ and $D(\partial_1^2)$ can be written as follows:

$$\begin{aligned} D(\partial_1) &= \sum_{i,j \geq 0} a_{i,j,1} e^{px^r} e^{qx^s} x^i y^j \partial_1 + \sum_{i,j \geq 0} a_{i,j,2} e^{px^r} e^{qx^s} x^i y^j \partial_2 + \\ &\quad \sum_{i,j \geq 0} a_{i,j,3} e^{px^r} e^{qx^s} x^i y^j \partial_1^2 + \sum_{i,j \geq 0} a_{i,j,4} e^{px^r} e^{qx^s} x^i y^j \partial_2^2, \\ D(\partial_1^2) &= \sum_{i,j \geq 0} b_{i,j,1} e^{px^r} e^{qx^s} x^i y^j \partial_1 + \sum_{i,j \geq 0} b_{i,j,2} e^{px^r} e^{qx^s} x^i y^j \partial_2 + \\ &\quad \sum_{i,j \geq 0} b_{i,j,3} e^{px^r} e^{qx^s} x^i y^j \partial_1^2 + \sum_{i,j \geq 0} b_{i,j,4} e^{px^r} e^{qx^s} x^i y^j \partial_2^2 \end{aligned}$$

with appropriate coefficients. Since ∂_1 centralizes ∂_1^2 , we have that

$$\begin{aligned} & \left[\sum_{i,j \geq 0} a_{i,j,1} e^{px^r} e^{qx^s} x^i y^j \partial_1 + \sum_{i,j \geq 0} a_{i,j,2} e^{px^r} e^{qx^s} x^i y^j \partial_2 \right. \\ & + \sum_{i,j \geq 0} a_{i,j,3} e^{px^r} e^{qx^s} x^i y^j \partial_1^2 + \sum_{i,j \geq 0} a_{i,j,4} e^{px^r} e^{qx^s} x^i y^j \partial_2^2, \partial_1^2 \Big] \\ & + [\partial_1, \sum_{i,j \geq 0} b_{i,j,1} e^{px^r} e^{qx^s} x^i y^j \partial_1 + \sum_{i,j \geq 0} b_{i,j,2} e^{px^r} e^{qx^s} x^i y^j \partial_2 \\ & + \sum_{i,j \geq 0} b_{i,j,3} e^{px^r} e^{qx^s} x^i y^j \partial_1^2 + \sum_{i,j \geq 0} b_{i,j,4} e^{px^r} e^{qx^s} x^i y^j \partial_2^2] \\ & = - \sum_{i,j \geq 0} p^2 r^2 a_{i,j,1} e^{px^r} e^{qy^s} x^{i+2r-2} y^j \partial_1 - \sum_{i,j \geq 0} pr(i+r-1) a_{i,j,1} e^{px^r} \\ & e^{qy^s} x^{i+r-2} y^j \partial_1 - \sum_{i \geq 1} pr i a_{i,j,1} e^{px^r} e^{qy^s} x^{i+r-2} y^j \partial_1 - \sum_{i \geq 2} i(i-1) a_{i,j,1} \\ & e^{px^r} e^{qy^s} x^{i-2} y^j \partial_1 - \sum_{i,j \geq 0} p^2 r^2 a_{i,j,2} e^{px^r} e^{qy^s} x^{i+2r-2} y^j \partial_2 - \sum_{i,j \geq 0} pr(i+r \\ & -1) a_{i,j,2} e^{px^r} e^{qy^s} x^{i+r-2} y^j \partial_2 - \sum_{i \geq 1} pr i a_{i,j,2} e^{px^r} e^{qy^s} x^{i+r-2} y^j \partial_2 \\ & - \sum_{i \geq 2} i(i-1) a_{i,j,2} e^{px^r} e^{qy^s} x^{i-2} y^j \partial_2 - \sum_{i,j \geq 0} p^2 r^2 a_{i,j,3} e^{px^r} e^{qy^s} x^{i+2r-2} y^j \partial_1^2 \\ & - \sum_{i,j \geq 0} pr(i+r-1) a_{i,j,3} e^{px^r} e^{qy^s} x^{i+r-2} y^j \partial_1^2 \end{aligned}$$

$$\begin{aligned}
& - \sum_{i \geq 1} pr i a_{i,j,3} e^{px^r} e^{qy^s} x^{i+r-2} y^j \partial_1^2 - \sum_{i \geq 2} i(i-1) a_{i,j,3} e^{px^r} e^{qy^s} x^{i-2} y^j \partial_1^2 \\
& - \sum_{i,j \geq 0} p^2 r^2 a_{i,j,4} e^{px^r} e^{qy^s} x^{i+2r-2} y^j \partial_2^2 \\
& - \sum_{i,j \geq 0} pr(i+r-1) a_{i,j,4} e^{px^r} e^{qy^s} x^{i+r-2} y^j \partial_2^2 \\
& - \sum_{i \geq 1} pr i a_{i,j,4} e^{px^r} e^{qy^s} x^{i+r-2} y^j \partial_2^2 - \sum_{i \geq 2} i(i-1) a_{i,j,4} e^{px^r} e^{qy^s} x^{i-2} y^j \partial_2^2 \\
& + \sum_{i,j \geq 0} pr b_{i,j,1} e^{px^r} e^{qy^s} x^{i+r-1} y^j \partial_1 + \sum_{i \geq 1} i b_{i,j,1} e^{px^r} e^{qy^s} x^{i-1} y^j \partial_1 \\
& + \sum_{i,j \geq 0} pr b_{i,j,2} e^{px^r} e^{qy^s} x^{i+r-1} y^j \partial_2 + \sum_{i \geq 1} i b_{i,j,2} e^{px^r} e^{qy^s} x^{i-1} y^j \partial_2 \\
& + \sum_{i,j \geq 0} pr b_{i,j,3} e^{px^r} e^{qy^s} x^{i+r-1} \partial_1^2 + \sum_{i \geq 1} b_{i,j,3} e^{px^r} e^{qy^s} x^{i-1} \partial_1^2 \\
(6) \quad & + \sum_{i,j \geq 0} pr b_{i,j,4} e^{px^r} e^{qy^s} x^{i+r-1} y^j \partial_2^2 + \sum_{i \geq 1} i b_{i,j,4} e^{px^r} e^{qy^s} x^{i-1} y^j \partial_2^2 = 0
\end{aligned}$$

with appropriate coefficients. Note that since the algebra is $\mathbb{Z}$ -graded, it is enough to assume that non-zero terms of $D(\partial_1)$ and $D(\partial_1^2)$ are in the homogeneous components N_0 or N_p . By (6), we have that $a_{i,j,1}$, $a_{i,j,2}$, $a_{i,j,3}$, $a_{i,j,4}$, $b_{i,j,1}$, $b_{i,j,2}$, $b_{i,j,3}$, and $b_{i,j,4}$ are zeroes, $i, j \geq 0$, and $D(\partial_1)$ and $D(\partial_1^2)$ are also zeroes. Similarly, we can prove that $D(\partial_2)$ and $D(\partial_2^2)$ are also zeroes. Since ∂_1 centralizes $x\partial_1$, we are able to prove that $D(x\partial_1) = c_{0,0,1}\partial_1 + c_{0,0,2}\partial_2 + c_{0,0,3}\partial_1^2 + c_{0,0,4}\partial_2^2$. Similarly, we also prove that $D(y\partial_1) = d_{0,0,1}\partial_1 + d_{0,0,2}\partial_2 + d_{0,0,3}\partial_1^2 + d_{0,0,4}\partial_2^2$. Since $D([\partial_1, x^2\partial_1]) = [\partial_1, D(x^2\partial_1)]$, we are also able to prove that $D(x^2\partial_1) = 2c_{0,0,1}x\partial_1 + 2c_{0,0,2}x\partial_2 + 2c_{0,0,3}x\partial_1^2 + 2c_{0,0,4}x\partial_2^2 + t_1\partial_1 + t_2\partial_2 + t_3\partial_1^2 + t_4\partial_2^2$. Since $x\partial_1$ is an ad-diagonal element with respect to the element $x^2\partial_1$, we have that $c_{0,0,3}$, t_1 , t_2 , t_3 , and t_4 are zeroes. This implies that $D(x\partial_1) = c_{0,0,1}\partial_1 + c_{0,0,2}\partial_2 + c_{0,0,4}\partial_2^2$ and $D(x^2\partial_1) = 2c_{0,0,1}x\partial_1 + 2c_{0,0,2}x\partial_2 + 2c_{0,0,4}x\partial_2^2$ hold with appropriate coefficients. By $D([y\partial_1, x\partial_1]) = D(y\partial_1)$, we have that $c_{0,0,2} = d_{0,0,2} = d_{0,0,3} = d_{0,0,4} = 0$. This implies that

$$\begin{aligned}
(7) \quad & D(x\partial_1) = c_{0,0,1}\partial_1 + c_{0,0,4}\partial_2^2, \quad D(y\partial_1) = d_{0,0,1}\partial_1 \\
& D(x^2\partial_1) = 2c_{0,0,1}x\partial_1 + 2c_{0,0,4}x\partial_2^2
\end{aligned}$$

By $D([\partial_2, y^2\partial_1]) = 2D(y\partial_1)$ and $D([y^2\partial_1, x\partial_1]) = D(y^2\partial_1)$, we have that $c_{0,0,4} = 0$ and $D(y^2\partial_1) = 2d_{0,0,1}y\partial_1 + s_1\partial_1$, $s_1 \in \mathbb{F}$. By $D([y\partial_1, x^2\partial_1]) = 2D(xy\partial_1)$, we have that $D(xy\partial_1) = d_{0,0,1}x\partial_1 + c_{0,0,1}y\partial_1$. Since $D([y\partial_1, xy$

$\partial_1]) = D(y^2\partial_1)$, we have that $D(y^2\partial_1) = 2d_{0,0,1}y\partial_1$. By (7), we have that

$$(8) \quad \begin{aligned} D(x\partial_1) &= c_{0,0,1}\partial_1, \quad D(y\partial_1) = d_{0,0,1}\partial_1, \quad D(x^2\partial_1) = 2c_{0,0,1}x\partial_1 \\ D(y^2\partial_1) &= 2d_{0,0,1}y\partial_1, \quad D(xy\partial_1) = d_{0,0,1}x\partial_1 + c_{0,0,1}y\partial_1 \end{aligned}$$

Since $D([x^i\partial_1, x^{i+1}\partial_1]) = D(x^{2i}\partial_1)$, by induction on i of $x^i\partial_1$, we prove that $D(x^{2i}\partial_1) = 2ic_{0,0,1}x^{2i-1}\partial_1$. Similarly, we also prove that $D(x^{2i-1}\partial_1) = (2i-1)c_{0,0,1}x^{2i-2}\partial_1$. This implies that $D(x^i\partial_1) = ic_{0,0,1}x^{i-1}\partial_1$ for all i .

Since $D([y^{i-1}\partial_1, xy\partial_1]) = D(y^i\partial_1)$, by induction on i of $y^i\partial_1$, we also prove that $D(y^i\partial_1) = id_{0,0,1}y^{i-1}\partial_1$. By $D([\partial_1, x\partial_1^2]) = 0$, we have that $D(x\partial_1^2) = g_{0,0,1}\partial_1 + g_{0,0,2}\partial_2 + g_{0,0,3}\partial_1^2 + g_{0,0,4}\partial_2^2$. By $D([\partial_1, x^2\partial_1^2]) = 2D(x\partial_1^2)$ and $D([x\partial_1, x^2\partial_1^2]) = 2D(x^2\partial_1^2)$, we can prove that $c_{0,0,1} = g_{0,0,1} = g_{0,0,2} = g_{0,0,3} = g_{0,0,4} = 0$. Thus, this means that $D(x^i\partial_1) = 0$ and $D(x\partial_1^2) = D(x^2\partial_1^2) = 0$. By $D([x^2\partial_1, x^{i-1}\partial_1^2]) = (i-1)D(x^i\partial_1^2) - 2D(x^{i-1}\partial_1)$ and by induction on i of $x^i\partial_1^2$, we can prove that $D(x^i\partial_1^2) = 0$ for all i . Similarly, we have that $D(y^i\partial_2) = D(y^i\partial_2^2) = 0$. Since $[y\partial_2, y\partial] = D(y\partial_1) = d_{0,0,1}\partial_1$, we can prove that $d_{0,0,1} = 0$. This implies that $D(y^i\partial_1) = 0$. Similarly, we have that $D(x^i\partial_2) = 0$. By $D([\partial_2, y\partial_1^2]) = 0$ and $D([y\partial_2, y^2\partial_1^2]) = 2D(y^2\partial_1^2)$, we can prove that $D(y\partial_1^2) = D(y^2\partial_1^2) = 0$. By induction on i of $y^i\partial_1^2$, we have that $D(y^i\partial_1^2) = 0$. Similarly, we have that $D(x^i\partial_2^2) = 0$. Therefore we have proven the lemma. $\square$

Lemma 3. For any derivation $D \in \text{Der}_{anti}(WN(e^{\pm x^r}, 0, 2)_B^-)$, we have that

$$D(e^{px^r}x^i\partial_u^v) = c_1pe^{px^r}x^i\partial_u^v \quad \text{and} \quad D(e^{qy^s}y^j\partial_u^v) = c_2qe^{qy^s}y^j\partial_u^v$$

hold where $1 \leq u, v \leq 2$ and $c_1, c_2 \in \mathbb{F}$.

Proof. Let D be the derivation in the lemma. Assume that

$$\begin{aligned} D(e^{x^r}x^k\partial_1) &= \sum_{i,j \geq 0} u_{i,j,1}e^{px^r}e^{qy^s}x^i y^j \partial_1 + \sum_{i,j \geq 0} u_{i,j,2}e^{px^r}e^{qy^s}x^i y^j \partial_2 \\ &\quad + \sum_{i,j \geq 0} u_{i,j,3}e^{px^r}e^{qy^s}x^i y^j \partial_1^2 + \sum_{i,j \geq 0} u_{i,j,4}e^{px^r}e^{qy^s}x^i y^j \partial_2^2, \end{aligned}$$

and

$$\begin{aligned} D(e^{x^r} x^{k+1} \partial_1) &= \sum_{i,j \geq 0} w_{i,j,1} e^{px^r} e^{qy^s} x^i y^j \partial_1 + \sum_{i,j \geq 0} w_{i,j,2} e^{px^r} e^{qy^s} x^i y^j \partial_2 \\ &\quad + \sum_{i,j \geq 0} w_{i,j,3} e^{px^r} e^{qy^s} x^i y^j \partial_1^2 + \sum_{i,j \geq 0} w_{i,j,4} e^{px^r} e^{qy^s} x^i y^j \partial_2^2 \end{aligned}$$

hold with appropriate coefficients. By $D([x\partial_1, e^{x^r} x^k \partial_1]) = rD(e^{x^r} x^{r+k} \partial_1) + (k-1)D(e^{x^r} x^k \partial_1)$ and $D([\partial_1, e^{x^r} x^{k+1} \partial_1]) = rD(e^{x^r} x^{r+k} \partial_1) + (k+1)D(e^{x^r} x^k \partial_1)$, we have that $D([x\partial_1, e^{x^r} x^k \partial_1]) - (k-1)D(e^{x^r} x^k \partial_1) = D([\partial_1, e^{x^r} x^{k+1} \partial_1]) - (k+1)D(e^{x^r} x^k \partial_1)$. This implies that $w_{i+1,j,1} = u_{i,j,1}$, $i, j \geq 0$, $w_{0,j,1} = 0$, $j \geq 0$, and $u_{i,j,l} = w_{i,j,l} = 0$, $i, j \geq 0$, $2 \leq l \leq 4$. The followings

$$\begin{aligned} D(e^{x^r} x^{k+1} \partial_1) &= \sum_{i > 1, j \geq 0} w_{i,j,1} e^{px^r} e^{qy^s} x^i y^j \partial_1 \\ &= x \left(\sum_{i,j \geq 0} u_{i,j,1} e^{px^r} e^{qy^s} x^i y^j \partial_1 \right) = xD(e^{x^r} x^k \partial_1) \end{aligned}$$

hold. We can write that $D(e^{x^r} x^k \partial_1) = xD(e^{x^r} x^{k-1} \partial_1) = x^2 D(e^{x^r} x^{k-2} \partial_1) = \dots = x^k D(e^{x^r} \partial_1)$, $u_{0,j,1} = \dots = u_{k-1,j,1} = 0$, $j \geq 0$, and $D(e^{x^r} \partial_1) = \sum_{i \geq k, j \geq 0} u_{i,j,1} e^{px^r} e^{qy^s} x^{i-k} y^j \partial_1$. By $D([\partial_1, e^{x^r} \partial_1]) = rD(e^{x^r} x^{r-1} \partial_1) = rx^{r-1} D(e^{x^r} \partial_1)$, we can prove that $p = 1$ and $u_{i,j,1} = 0$, $i \geq k+1$, $j \geq 0$. So we have that $D(e^{x^r} \partial_1) = \sum_{j \geq 0} u_{k,j,1} e^{x^r} e^{qy^s} y^j \partial_1$. By $D([\partial_2, e^{x^r} \partial_1]) = 0$, we can prove that $D(e^{x^r} \partial_1) = u_{k,0,1} e^{x^r} \partial_1 = c_1 e^{x^r} \partial_1$ with $c_1 = u_{k,0,1}$. By $D([x\partial_2, e^{x^r} \partial_1]) = -D(e^{x^r} \partial_2)$, we have that $D(e^{x^r} \partial_2) = c_1 e^{x^r} \partial_2$. Similarly, $D(e^{x^r} \partial_2^2) = c_1 e^{x^r} \partial_2^2$. By $D([x\partial_1^2, e^{x^r} \partial_1]) = r^2 D(e^{x^r} x^{2r-1} \partial_1) + r(r-1)D(e^{x^r} x^{2r-1} \partial_1) - D(e^{x^r} \partial_1^2)$, we have that $D(e^{x^r} \partial_1^2) = c_1 e^{x^r} \partial_1^2$. Similarly to the proof of $D(e^{x^r} \partial_u^v) = c_1 e^{x^r} \partial_u^v$, $1 \leq u, v \leq 2$, we can prove that $D(e^{y^s} \partial_u^v) = c_2 e^{y^s} \partial_u^v$. By $D([e^{x^r} \partial_1, e^{x^r} x \partial_1]) = D(e^{2x^r} \partial_1)$, we have that $D(e^{2x^r} \partial_1) = 2c_1 e^{2x^r} \partial_1$. By $D([e^{x^r} \partial_1, e^{x^r} x^{i+1} \partial_1]) = (i+1)D(e^{2x^r} x^i \partial_1)$, we also have that $D(e^{2x^r} x^i \partial_1) = 2c_1 e^{2x^r} x^i \partial_1$. For proof of $p \in N$, we need to consider following two cases:

Case I. Put $p = 2m$. We have that $D([e^{mx^r} \partial_1, e^{mx^r} x^{i+1} \partial_1]) = (i+1)D(e^{2mx^r} x^i \partial_1)$. By induction on p of $e^{px^r} x^i \partial_1$, we are able to prove that

$$D(e^{px^r} x^i \partial_1) = pc_1 e^{px^r} x^i \partial_1.$$

Case II. Put $p = 2m + 1$. Then we have that

$$(9) \quad \begin{aligned} & D([e^{mx^r} \partial_1, e^{(m+1)x^r} x^{i+1} \partial_1]) \\ &= rD(e^{(2m+1)x^r} x^{r+i} \partial_1) + (i+1)D(e^{(2m+1)x^r} x^i \partial_1) \end{aligned}$$

and

$$(10) \quad \begin{aligned} & D([e^{mx^r} x^{i+1} \partial_1, e^{(m+1)x^r} \partial_1]) \\ &= rD(e^{(2m+1)x^r} x^{r+i} \partial_1) - (i+1)D(e^{(2m+1)x^r} x^i \partial_1). \end{aligned}$$

By induction on p of $e^{px^r} x^i \partial_1$ and by subtracting (10) from (9), we are also able to prove that

$$D(e^{px^r} x^i \partial_1) = pc_1 e^{px^r} x^i \partial_1.$$

By Case I and Case II, we have that

$$(11) \quad D(e^{px^r} x^i \partial_1) = pc_1 e^{px^r} x^i \partial_1.$$

By $D([e^{px^r} \partial_1, x \partial_1^2]) = D(e^{px^r} \partial_1^2) - p^2 r^2 D(e^{px^r} x^{2r-2} \partial_1) - pr(r-1)D(e^{px^r} x^{r-2} \partial_1)$, we have that $D(e^{px^r} \partial_1^2) = pc_1 e^{px^r} \partial_1^2$. By $D([x^{i+1} \partial_1^2, e^{px^r} \partial_1]) = p^2 r^2 D(e^{px^r} x^{i+2r-1} \partial_1) + pr(r-1)D(e^{px^r} x^{i+r-1} \partial_1) - (i+1)D(e^{px^r} x^i \partial_1^2)$, we also have that $D(e^{px^r} x^i \partial_1^2) = pc_1 e^{px^r} x^i \partial_1^2$. By $D([x \partial_2, e^{px^r} x^i \partial_1]) = -D(e^{px^r} x^i \partial_2)$ and $D([x \partial_2^2, e^{px^r} x^i \partial_1]) = -D(e^{px^r} x^i \partial_2^2)$, we have that

$$D(e^{px^r} x^i \partial_2^v) = pc_1 e^{px^r} x^i \partial_2^v,$$

where $1 \leq v \leq 2$. Similarly to the proof of the equality $D(e^{px^r} x^i \partial_u^v) = pc_1 e^{px^r} x^i \partial_u^v$, $1 \leq u, v \leq 2$, we can prove that

$$(12) \quad D(e^{qy^s} y^j \partial_u^v) = qc_2 e^{qy^s} y^j \partial_u^v$$

where $1 \leq u, v \leq 2$ and $c_2 \in \mathbb{F}$. Therefore we have proven the lemma. $\square$

Lemma 4. For any derivation $D \in \text{Der}_{\text{anti}}(WN(e^{\pm x^r}, 0, 2)_B^-)$, we have that

$$D(e^{px^r} e^{qy^s} x^i y^j \partial_u^v) = c_1 p e^{px^r} e^{qy^s} x^i y^j \partial_u^v + c_2 q e^{px^r} e^{qy^s} x^i y^j \partial_u^v$$

where $1 \leq u, v \leq 2$.

Proof. Let D be the derivation in the lemma. By $D([y^j \partial_1, e^{px^r} x^{i+1} \partial_1]) = prD(e^{px^r} x^{i+r} y^j \partial_1) + (i+1)D(e^{px^r} x^i y^j \partial_1)$ and $D([x^{i+1} y^j \partial_1, e^{px^r} \partial_1]) = prD(e^{px^r} x^{i+r} y^j \partial_1) - (i+1)D(e^{px^r} x^i y^j \partial_1)$, we have that $D(e^{px^r} x^i y^j \partial_1) = c_1 p e^{px^r} x^i y^j \partial_1$. By $D([e^{px^r} x^i y^{j+1} \partial_1, e^{qy^s} \partial_2]) = -(j+1)D(e^{px^r} e^{qy^s} x^i y^j \partial_1)$, we also have that

$$D(e^{px^r} e^{qy^s} x^i y^j \partial_1) = c_1 p e^{px^r} e^{qy^s} x^i y^j \partial_1 + c_2 q e^{px^r} e^{qy^s} x^i y^j \partial_1.$$

By $D([y^j \partial_1, e^{px^r} x^{i+1} \partial_1^2]) = prD(e^{px^r} x^{i+r} y^j \partial_1^2) + (i+1)D(e^{px^r} x^i y^j \partial_1^2)$ and $D([x^{i+1} y^j \partial_1, e^{px^r} \partial_1^2]) = prD(e^{px^r} x^{i+r} y^j \partial_1^2) - (i+1)iD(e^{px^r} x^{i-1} y^j \partial_1)$, we have that $D(e^{px^r} x^i y^j \partial_1^2) = c_1 p e^{px^r} x^i y^j \partial_1^2$. Since $D([e^{qy^s} \partial_2, e^{px^r} x^i y^{j+1} \partial_1^2]) = (j+1)D(e^{px^r} e^{qy^s} x^i y^j \partial_1^2)$, we are able to prove that

$$D(e^{px^r} e^{qy^s} x^i y^j \partial_1^2) = c_1 p e^{px^r} e^{qy^s} x^i y^j \partial_1^2 + c_2 q e^{px^r} e^{qy^s} x^i y^j \partial_1^2.$$

Similarly, we can prove that

$$\begin{aligned} D(e^{px^r} e^{qy^s} x^i y^j \partial_2) &= c_1 p e^{px^r} e^{qy^s} x^i y^j \partial_2 + c_2 q e^{px^r} e^{qy^s} x^i y^j \partial_2, \\ D(e^{px^r} e^{qy^s} x^i y^j \partial_2^2) &= c_1 p e^{px^r} e^{qy^s} x^i y^j \partial_2^2 + c_2 q e^{px^r} e^{qy^s} x^i y^j \partial_2^2 \end{aligned}$$

hold. Therefore we have proven the lemma. $\square$

Theorem 5. For any $D \in \text{Der}_{\text{anti}}(WN(e^{\pm x^r}, 0, 2)_B^-)$, D is the linear sum of the derivations D_{c_1, c_2} as shown in Note 1 where $c_1, c_2 \in \mathbb{F}$. The additive group $D \in \text{Der}_{\text{anti}}(WN(e^{\pm x^r}, 0, 2)_B^-)$ is isomorphic to the additive group $\mathbb{F}^2$. Every derivation of the algebra $WN(e^{\pm x^r}, 0, 2)_B^-$ is outer.

Proof. The proofs of the theorem are straightforward by Lemma 4, Note 2 and the fact that the derivation of Note 2 cannot be inner. This completes the proof of the theorem. $\square$

Corollary 3. The dimension of $\text{Der}_{\text{anti}}(WN(e^{\pm x^r}, 0, 2)_B^-)$ of the algebra $WN(e^{\pm x^r}, 0, 2)_B^-$ is two. For any derivation D of $\text{Der}_{\text{anti}}(WN(e^{\pm x^r}, 0, 2)_B^-)$, $D(N'_0) = 0$ holds where N'_0 is the zero-homogeneous component of $WN(e^{\pm x^r}, 0, 2)_B^-$ in (4) (see [2] and [8]).

Proof. The proofs of the corollary are straightforward by Theorem 3 and Note 2. $\square$

Remarks. In Proposition 1, even if B_1 and B_2 have the same cardinalities, the algebras $\overline{WN}_{(g_n, \mathfrak{A}), m, s_{B_1}}$ and $\overline{WN}_{(g_n, \mathfrak{A}), m, s_{B_2}}$ cannot be isomorphic. It is interesting problem to find whether two algebras $\overline{WN}_{(g_n, \mathfrak{A}), m, s_{B_1}}$ and $\overline{WN}_{(g_n, \mathfrak{A}), m, s_{B_2}}$ are isomorphic or not. It is also interesting problem to find the automorphism group of an algebra $\overline{WN}_{(g_n, \mathfrak{A}), m, s_{B_1}}$ and its all the derivations. $\square$

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