HOMOGENEOUS AND PRIMARY IDEALS OF $R(+)M$

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ABSTRACT. In this short paper, we prove some new properties on homogeneous ideals and primary ideals of $R(+)M$.

1. Introduction

Let R be a commutative ring with unity and M a unital R -module. M is called *multiplication module* if every submodule N of M has the form IM for some ideal I of R . Equivalently, $N = (N : M)M$.

Let R be a commutative ring with unity and M an R -module. Consider $R(M) = \{(r, m) | r \in R, m \in M\}$ and let (r, m) and (s, n) be two elements of $R(M)$. Define:

1. $(r, m) = (s, n)$ if $r = s$ and $m = n$
2. $(r, m) + (s, n) = (r + s, m + n)$
3. $(r, m)(s, n) = (rs, rn + sm)$

Under these definition, $R(M)$ becomes a commutative ring with unity and $R(M)$ is called the *idealization* of a ring R and an R -module M . Sometimes $R(M)$ is also denoted by $R(+)M$. We can find some basic results about $R(M)$ ([6]). For an ideal I of R , $\sqrt{I} = \{r \in R | r^n \in I \text{ for some positive integer } n\}$ and we know that $\sqrt{I(+)M} = \sqrt{I}(+)M$.

A proper submodule P of an R -module M is said to be *prime* if $rm \in P$ for $r \in R, m \in M$ implies that either $m \in P$ or $r \in (P : M)$. The intersection of all prime submodules containing N is denoted by $M\text{-rad } N$.

In section 2 of this paper we prove some basic properties and Theorem 2.6 about homogeneous ideal of $R(M)$ and in section 3 we prove Theorem 3.2

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, Theorem 3.3 about primary ideals of $R(M)$ and Theorem 3.4 on strongly Laskerian modules

2 . Homogeneous ideals of $R(+)M$

An ideal H of $R(+)M$ is called *homogeneous* if $H = I(+)N$ where I is an ideal of R and N a submodule of M . In this case, $I(+)N = (R(+)M)(I(+)N) = I(+)(IM + N)$ gives $IM \subseteq N$. If $IM \subseteq N$, then M/N is an R/I -module and $g : R(+)M \rightarrow R/I(+)M/N$ defined by $g(r, m) = (r + I, m + N)$ is a ring homomorphism and $\ker(g) = I(+)N$ and hence $I(+)N$ is an ideal of $R(+)M$. So we know that $I(+)N$ is an ideal of $R(+)M$ if and only if $IM \subseteq N$. Every ideal of $R(+)M$ of the form $I(+)N$ is homogeneous. However, ideals of $R(+)M$ need not have the form $I(+)N$, that is, need not be homogeneous. $Z(+)2Z(2, 2)$ is not a homogeneous ideal of $Z(+)2Z([2])$.

PROPOSITION 2.1. *Let R be a ring and N a submodule of an R -module M . Then,*

- (1) *Every ideal of $R(+)M$ containing $0(+)M$ has the form $I(+)M$ for some ideal I of R .*
- (2) *Every ideal of $R(+)M$ is contained in $0(+)N$ has the form $0(+)K$ for some submodule K of N .*
- (3) *For any homogeneous ideal H of $R(+)M$, $H = I(+)N$ where $I = \{r \in R \mid (r, m) \in H \text{ for some } m \in M\}$ and $N = \{m \in M \mid (r, m) \in H \text{ for some } r \in R\}$.*

Proof. (1) Let H be any ideal of $R(+)M$ containing $0(+)M$ and $I = \{r \in R \mid (r, m) \in H \text{ for some } m \in M\}$. For any $r, s \in I$, there exist $m, n \in M$ such that $(r, m), (s, n) \in H$. Since $(r - s, m - n) \in H$, $r - s \in I$. Let $a \in R, r \in I$ and $(r, m) \in H$. Since H is an ideal of $R(+)M$, $(ar, am) = (a, 0)(r, m) \in H$ and $ar \in I$. Thus I is an ideal of R . By the definition of I , if $(r, m) \in H$, then $r \in I$ and $(r, m) \in I(+)M$. If $(a, n) \in H$ then $a \in I$ and there exists $m \in M$ with $(a, m) \in H$. Further since $0(+)M \subseteq H$, $(0, n - m) \in H$ and so $(a, n) = (a, m) + (0, n - m) \in H$. Thus $H = I(+)M$.

(2) Let H be any ideal of $R(+)M$ which is contained in $0(+)N$. Put $K = \{k \in N \mid (0, k) \in H\}$. Then K is a submodule of N and $H = 0(+)K$.

(3) It is Theorem 25.1 of [6]. □

Let N be a submodule of an R -module M and I an ideal of R . The *residual*

of N by I is $(N :_M I) = \{m \in M \mid Im \subseteq N\}$. Then it is a submodule of M containing N ([3]).

The following statement is Lemma 1 in [2]. We give the proof for the first assertion.

PROPOSITION 2.2. *Let R be a ring and M an R -module. Let $I(+)N$ and $J(+)K$ be homogeneous ideals of $R(+)M$. Then,*

$$(I(+)N :_{R(+)M} J(+)K) = (I : J) \cap (N : K)(+)(N :_M J)$$

Furthermore, it is a homogeneous ideal of $R(M)$.

Proof. Let $(r, m) \in (I(+)N : J(+)K)$. Then $(r, m)J(+)K \subseteq I(+)N$ and $rJ \subseteq I, rK + Jm \subseteq N$. Hence $r \in (I : J) \cap (N : K)$. Further $Jm \subseteq rK + Jm \subseteq N$. So $m \in (N :_M J)$. Thus $(I(+)N : J(+)K) \subseteq (I : J) \cap (N : K)(+)(N :_M J)$. If $(r, m) \in (I : J) \cap (N : K)(+)(N :_M J)$, then $rJ \subseteq I, rK \subseteq N$ and $Jm \subseteq N$. So, $(r, m)J(+)K \subseteq rJ(+)rK + Jm \subseteq I(+)N$ and hence $(r, m) \in (I(+)N :_{R(+)M} J(+)K)$. The proof for the second part is in [2]. $\square$

REMARK 1. *From the above Proposition 2.2 we have the following:*

- (1) *If $I(+)N$ is a homogeneous ideal of $R(+)M$, then $\text{ann}(I(+)N) = (\text{ann}I \cap \text{ann}N)(+)(0 :_M I)$.*
- (2) *If M is a faithful R -module and $I(+)N$ is a homogeneous ideal of $R(+)M$, then $\text{ann}(I(+)N) = \text{ann}N(+)(0 :_M I)$.*
- (3) *If M is a faithful multiplication module and $I(+)N$ is a homogeneous ideal of $R(+)M$, then $\text{ann}(I(+)N) = \text{ann}N(+)(\text{ann}I)M$.*

PROPOSITION 2.3. *Let R be a ring and M an R -module. For a principal ideal $R(+)M(a, m)$, the following conditions are equivalent;*

- (1) *$R(+)M(a, m)$ is homogeneous.*
- (2) *$R(+)M(a, m) = Ra(+)(Rm + aM)$.*
- (3) *$(a, 0) \in R(+)M(a, m)$.*
- (4) *there exists $x \in R$ such that $xa = a$ and $xm \in aM$*

Proof. (1) $\Rightarrow$ (2). Since $R(+)M(a, m)$ is homogeneous, $R(+)M(a, m) = I(+)N$ for some ideal I of R and a submodule N of M and $IM \subseteq N$. Since $(1, 0)(a, m) = (a, m) \in I(+)N$, $Ra(+)(Rm + aM) \subseteq I(+)N = R(+)M(a, m)$.

Also $R(+)M(a, m) \subseteq Ra(+)(Rm + aM)$ is clear.

(2) $\Rightarrow$ (1). Since $R(aM) = a(RM) = aM \subseteq (Rm + aM)$, $R(+)M(a, m)$ is homogeneous.

(2) $\Rightarrow$ (3). Clear.

(3) $\Rightarrow$ (4). Since $(a, 0) \in R(+)M(a, m)$, there exists $(x, n) \in R(+)M$ such that $(a, 0) = (x, n)(a, m)$. Hence $xa = a, xm = -an \in aM$.

(4) $\Rightarrow$ (2). Suppose that there exists $x \in R, m' \in M$ such that $xa = a, xm = am'$. Then $(a, 0) = (xa, xm - am') = (x, -m')(a, m) \in R(+)M(a, m)$. Further $(a, m) - (a, 0) = (0, m) \in R(+)M(a, m)$. Therefore, for any $(sa, tm + an) \in Ra(+) (Rm + aM)$, $(sa, tm + an) = (s, n)(a, 0) + (t, 0)(0, m) \in R(+)M(a, m)$. $\square$

A ring R is *presimplifiable* if for $x, y \in R, xy = x$ implies $x = 0$ or y is a unit. Clearly, an integral domain is presimplifiable.

PROPOSITION 2.4. *Quasi local ring is presimplifiable .*

Proof. Let R be a quasi local ring. Suppose that $xy = x$ for $x, y \in R$ and y is not a unit in R . Then $1 - y$ is a unit. If not, $R(1 - y) \neq R$ and $R(1 - y) \subseteq M$, M is a unique maximal ideal of R . So, $1 \in M$, a contradiction. Therefore, there exists $k \in R$ such that $(1 - y)k = k(1 - y) = 1$. Hence $0 = k(x - xy) = kx(1 - y) = x(1 - y)k = x$. $\square$

PROPOSITION 2.5. *Let R be presimplifiable. Then $R(+)M(a, m)$ is homogeneous if and only if $a = 0$ or $m \in aM$.*

Proof. Suppose that $a \neq 0$ and $R(+)M(a, m)$ is homogeneous. By Proposition 2.3, there exists $x \in R$ such that $xa = a, xm \in aM$. Since R is presimplifiable, x is a unit and so $m \in aM$. Conversely, if $a = 0$, take $x = 0 \in R$ in Proposition 2.3-(4). and if $m \in aM$, take $x = 1$ in Proposition 2.3-(4). $\square$

THEOREM 2.6. *Let R be a ring and M an R -module. Every ideal of $R(+)M$ is homogeneous if and only if every principal ideal of $R(+)M$ is homogeneous.*

Proof. Suppose that every principal ideal is homogeneous and let H be an ideal of $R(+)M$. Then $H = \sum_{(a_\alpha, m_\alpha) \in H} R(+)M(a_\alpha, m_\alpha)$
 $= \sum_{\alpha} Ra_\alpha(+) [Rm_\alpha + a_\alpha M]$ (Proposition 2.3)
 $= \sum_{\alpha} Ra_\alpha(+) \sum_{\alpha} [Rm_\alpha + a_\alpha M]$. So, H is homogeneous. The converse is trivial. $\square$

PROPOSITION 2.7. *Let R be presimplifiable. Every ideal of $R(+)M$ is homogeneous if and only if $M = aM$ for every $a \neq 0 \in R$.*

Proof. ($\Rightarrow$) For $a \neq 0 \in R$, and $m \in M$, $R(+)M(a, m)$ is homogeneous. By Proposition 2.3, there exists $x \in R$ such that $xa = a, xm = an, n \in M$. Since R is presimplifiable, x is a unit. Hence $m \in aM$ and so $M \subseteq aM$. Thus $M = aM$.

($\Leftarrow$) It is enough to show that principal ideal of $R(+)M$ is homogeneous (Proposition 2.6.) If $a = 0$, then $R(+)M(a, m)$ is homogeneous by Proposition 2.5 and if $a \neq 0$ then $M = aM$ by assumption. Again by Proposition 2.5, $R(+)M(a, m)$ is homogeneous. $\square$

REMARK 2. Every ideal of $Z(+)Q$ is homogeneous

3. Primary ideals of $R(+)M$

A proper submodule N of an R -module M is called *primary* if whenever $rm \in N$, for some $r \in R$ and $m \in M$, then $m \in N$ or $r^k \in (N : M)$ for some positive integer k .

PROPOSITION 3.1. Let R be a ring and N a submodule of an R -module M . If N is a primary submodule of M , then $(N : M)$ is a primary ideal of R and the converse is true if M is multiplication.

Proof. It comes from Proposition 18 of [8] that if N is a primary submodule then $(N : M)$ is primary.

Let $(N : M)$ be a primary ideal and M a multiplication R -module. Clearly, $N \neq M$. Now let $r \in R, n \in M, rn \in N$, and $n \notin N$. Then $r(Rn : M) \subseteq (Rrn : M) \subseteq (N : M)$. If $(Rn : M) \subseteq (N : M)$ then $Rn = (Rn : M)M \subseteq (N : M)M = N$. So, $n \in N$ and we get a contradiction. Thus $(Rn : M) \not\subseteq (N : M)$. Since $(N : M)$ is primary, there exists a positive integer k such that $r^k \in (N : M)$. $\square$

THEOREM 3.2. Let I be a primary ideal of a ring R and M a finitely generated faithful multiplication module over R . Then IM is a primary submodule of an R -module M .

Proof. Let $IM = M$. Then $IM = RM$. Since M is weak-cancellation ([7]-Theorem 6.6) and faithful, $I = R$, a contradiction. Therefore $IM \neq M$. Now suppose that $r \in R, m \in M, rm \in IM$ and $m \notin IM$. $r(Rm : M) \subseteq (Rrm : M) \subseteq (IM : M) = I + \text{ann}(M) = I$. We know that $(Rm : M) \not\subseteq I$. In fact,

if $(Rm : M) \subseteq I$, then $Rm = (Rm : M)M \subseteq IM$. Hence $m \in IM$ and a contradiction. So, $r(Rm : M) \subseteq I$, $(Rm : M) \not\subseteq I$ and since I is primary, $r^n \in I = (IM : M)$ for some positive integer n . $\square$

REMARK 3. Let M be a finitely generated multiplication R -module and I an ideal of R . Then we always have that $(IM : M) = I + \text{ann}(M)$.

In fact, $IM = (I + \text{ann}(M))M = (IM : M)M$ and since M is weak cancellation, $I + \text{ann}(M) = (IM : M) + \text{ann}(M)$ and so $I + \text{ann}(M) = (IM : M)$.

THEOREM 3.3. Let R be a ring and M an R -module. Let I be an ideal of R and N a submodule of M .

- (1) $I(+)IM$ is a primary ideal of $R(M)$, then I is primary and the converse is true if M is finitely generated, faithful, and multiplication.
- (2) If $(N : M)(+)N$ is a primary ideal of $R(M)$, then N is a primary submodule of M and the converse is true if M is finitely generated multiplication.

Proof of (1). $(\Rightarrow)$ Since $I(+)IM$ is primary, $I \neq R$. Let $ab \in I$, $a \notin I$ for $a, b \in R$. Then $(a, 0)(b, 0) \in I(+)IM$ and $(a, 0) \notin I(+)IM$. Since $I(+)IM$ is primary, $(b, 0)^n \in I(+)IM$ for some positive integer n . Therefore $b^n \in I$ and so I is primary.

$(\Leftarrow)$ Since M is finitely generated multiplication, $I + \text{ann}(M) = (IM : M)$ by Remark 3. However since M is faithful, $I = (IM : M)$. Now let $r \in R$, $n \in M$, $rn \in IM$ and $n \notin IM$. Then $r(Rn : M) \subseteq (Rrn : M) \subseteq (IM : M) = I$

However $(Rn : M) \not\subseteq I$. Since M is multiplication, $Rn = (Rn : M)M$ and if $(Rn : M) \subseteq I$ then $(Rn : M)M \subseteq IM$ and $n \in IM$, contradiction. Since I is primary, there exists a positive integer n such that $r^n \in I = (IM : M)$ and so IM is primary. By Proposition 18 of [1], $I(+)IM$ is primary. $\square$

Proof of (2). $(\Rightarrow)$ Since $(N : M)(+)N$ is primary, $N \neq M$. Let $rm \in N$ and $m \notin N$ for $r \in R$ and $m \in M$. Then $(r, 0)(0, m) \in (N : M)(+)N$ and $(0, m) \notin (N : M)(+)N$. Since $(N : M)(+)N$ is primary, $(r, 0)^n \in (N : M)(+)N$ for some positive integer n . Hence $r^n \in (N : M)$ and N is primary.

$(\Leftarrow)$ Let $I = (N : M)$. Then $N = (N : M)M = IM$ and

$I(+)N = (N : M)(+)N$ is a homogeneous ideal of $R(M)$. Since $M\text{-rad}N = M\text{-rad}IM$ and N is primary, $(I + \text{ann}(M))(+)N = [(N : M) + (0 : M)](+)N = (N : M)(+)N = I(+)N$ is primary by Proposition 18 of [1]. $\square$

A primary submodule Q of an R -module M is said to be *strongly primary* if $(\sqrt{(Q : M)})^n M \subseteq Q$ for some positive integer n . An R -module M is called *strongly Laskerian* if every submodule of M is a finite intersection of primary submodules of M and every primary submodule is strongly primary. A ring R is *strongly Laskerian ring* if every ideal of R is a finite intersection of primary ideals of R and each primary ideal of R contains a power of its radical ([4],[5]).

THEOREM 3.4. *Let R be a ring and M an R -module. If $R(M)$ is strongly Laskerian, then R is also strongly Laskerian. Assuming further that M is finitely generated, faithful and multiplication then M is strongly Laskerian.*

Proof. Suppose that $R(M)$ is strongly Laskerian and $\mathfrak{q}$ a primary ideal of R . Then $\mathfrak{q}(+)M$ is a primary ideal of $R(M)$. There exists a positive integer k such that $(\sqrt{\mathfrak{q}(+)M})^k \subseteq \mathfrak{q}(+)M$ and further $\sqrt{\mathfrak{q}(+)M} = \sqrt{\mathfrak{q}}(+)M$. Therefore $(\sqrt{\mathfrak{q}(+)M})^k = (\sqrt{\mathfrak{q}})^k(+)((\sqrt{\mathfrak{q}})^{k-1}M) \subseteq \mathfrak{q}(+)M$. So, $(\sqrt{\mathfrak{q}})^k \subseteq \mathfrak{q}$. Since every ideal is a finite intersection of primary ideals of R ([1]-Corollary 21), R is a strongly Laskerian.

Assume that M is a finitely generated, faithful and multiplication. Then by Corollary 21 of [1] we know that every submodule of M is a finite intersection of primary submodules of M . Hence it is enough to prove that every primary submodule is strongly primary. Let Q be a primary submodule of M . Then $(Q : M)$ is primary and $(Q : M)(+)M$ is also primary in $R(M)$. Since $R(M)$ is strongly Laskerian, $(\sqrt{(Q : M)(+)M})^s \subseteq (Q : M)(+)M$ for some positive integers. On the other hand, $(\sqrt{(Q : M)(+)M})^s = (\sqrt{(Q : M)})^s(+)((\sqrt{(Q : M)})^{s-1}M) \subseteq (Q : M)(+)M$. Thus $(\sqrt{(Q : M)})^s \subseteq (Q : M)$ and since M is multiplication,

$(\sqrt{(Q : M)})^s M \subseteq (Q : M)M = Q$ and hence M is strongly Laskerian. $\square$

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