ROUGH STRONG IDEALS IN BH -ALGEBRAS

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Abstract. The notion of strong ideal in BH -algebra is introduced and investigated some properties of it. Rough sets and rough strong ideals in BH -algebras are established and some related their properties are discussed.

1. Introduction

Y. Imai and K. Iséki introduced two classes of abstract algebras: BCK -algebras and BCI -algebras ([2,3]). It is known that the class of BCK -algebras is a proper subclass of the class of BCI -algebras. BCK -algebras have some connections with other areas: D. Mundici [8] proved MV -algebras are categorically equivalent to bounded commutative algebra, and J. Meng [9] proved that implicative commutative semigroups are equivalent to a class of BCK -algebras. W. A. Dudek and X. Zhang [1] defined the new concept of ideals in BCC -algebras and described connections between such ideals and congruences. Y. B. Jun, E. H. Roh, and H. S. Kim [4] introduced the notion of a BH -algebra, which is a generalization of BCK/BCI -algebras. They defined the notions of ideal, maximal ideal and translation ideal and investigated some properties. E. H. Roh and S. Y. Kim [11] estimated the number of BH^* -subalgebras of order i in a transitive BH^* -algebras by using Hao's method.

In this paper, we introduce the notion of strong ideals in BH -algebras and investigate some properties of it. We also define the notion of rough sets in BH -algebras. Using a strong ideal in BH -algebras, we obtain some relations between strong ideals and upper(lower) rough strong ideals in BH -algebras.

2. Preliminaries

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By a *BH-algebra* ([4]), we mean an algebra $(X; *, 0)$ of type $(2, 0)$ satisfying the following conditions:

- (I) $x * x = 0$,
- (II) $x * 0 = x$,
- (III) $x * y = 0$ and $y * x = 0$ imply $x = y$, for all $x, y \in X$.

For brevity, we also call X a *BH-algebra*. In X we can define a binary operation “ $\leq$ ” by $x \leq y$ if and only if $x * y = 0$. A non-empty subset S of a *BH-algebra* X is called a *subalgebra* of X if, for any $x, y \in S$, $x * y \in S$, i.e., S is a closed under binary operation.

Definition 2.1. ([4]) A non-empty subset A of a *BH-algebra* X is called an *ideal* of X if it satisfies:

- (I1) $0 \in A$,
- (I2) $x * y \in A$ and $y \in A$ imply $x \in A$, $\forall x, y \in X$.

An ideal A of a *BH-algebra* X is said to be a *translation ideal* of X if it satisfies:

- (I3) $x * y \in A$ and $y * x \in A$ imply $(x * z) * (y * z) \in A$ and $(z * x) * (z * y) \in A$, $\forall x, y, z \in X$.

Obviously, $\{0\}$ and X are ideals of X . We will call $\{0\}$ and X a *zero ideal* and a *trivial ideal*, respectively. An ideal A is said to be *proper* if $A \neq X$. A mapping $f : X \rightarrow Y$ of *BH-algebras* is called a *homomorphism* if $f(x * y) = f(x) * f(y)$ for all $x, y \in X$. For a homomorphism $f : X \rightarrow Y$ of *BH-algebras*, the *kernel* of f , denoted by $\text{Ker}(f)$, defined to be the set

$$\text{Ker}(f) = \{x \in X \mid f(x) = 0\}.$$

Definition 2.2. ([11]) A *BH-algebra* X is called a *BH*-algebra* if it satisfies the identity $(x * y) * x = 0$ for all $x, y \in X$.

Example 2.3. ([4]) Let $X := \{0, a, b, c\}$ be a *BH-algebra* which is not a *BCK-algebra* with the following Cayley table:

$*$	0	1	2	3
0	0	1	0	0
1	1	0	0	0
2	2	2	0	3
3	3	3	3	0

Then $A := \{0, 1\}$ is a translation ideal of X .

3. Strong ideal of a BH -algebra

Definition 3.1. A non-empty subset A of a BH -algebra X is called a *strong ideal* of X if it satisfies (I1) and

(I4) $(x * y) * z, y \in A$ imply $x * z \in A$.

Lemma 3.2. In a BH -algebra, any strong ideal is an ideal.

Proof. Putting $z := 0$ in (I4), we obtain (I2). $\square$

Lemma 3.3. In a BH^* -algebra X , any ideal is a subalgebra.

Proof. Let A be an ideal of X . Then $0 \in A$ and $(x * y) * x = 0$ for any $x, y \in A$. Then for $x, y \in A$, we have $(x * y) * x \in A$, which implies $x * y \in A$. $\square$

Corollary 3.4. Any strong ideal of BH^* -algebra is a subalgebra.

Example 3.5. (1) Let $X := \{0, 1, 2, 3, 4, 5\}$ be a set with the following Cayley table:

$*$	0	1	2	3	4	5
0	0	0	0	0	0	5
1	1	0	0	0	0	1
2	2	2	0	0	1	1
3	3	2	1	0	1	1
4	4	4	4	4	0	1
5	5	5	5	5	5	0

Then X is a BH -algebra which is not a BCK/BCI -algebra, since $(4 * (4 * 5)) * 5 = (4 * 1) * 5 = 4 * 5 = 1 \neq 0$. Let $S := \{0, 1, 2, 3, 4\}$. It is easy to see that S is a subalgebra and a strong ideal of X .

(2) Let $X := \{0, 1, 2, 3, 4\}$ be a set with the following Cayley table:

$*$	0	1	2	3	4
0	0	0	0	0	4
1	1	0	1	0	0
2	2	2	0	0	0
3	3	3	1	0	0
4	4	3	4	3	0

Then X is a BH -algebra, which is not a BCK/BCI -algebra, since $((4 * 2) * (4 * 3)) * (3 * 2) = (4 * 3) * 1 = 3 * 1 = 3 \neq 0$. Let $A := \{0, 1, 2, 3\}$. Then A is a subalgebra, not an ideal, and not a strong ideal of X , since $4 * 3 = 3 \in A$, but $4 \notin A$. Let $B := \{0, 1\}$. It is easy to show that

B is a subalgebra, an ideal of X , and not a strong ideal of X , since $(4 * 1) * 2 = 3 * 2 = 1 \in B$, but $4 * 2 = 4 \notin B$.

For any element a in a BH -algebra, we define a set $\downarrow a := \{x \in X \mid x \leq a\}$, called an *initial section* of a .

Proposition 3.6. *For any element a of a BH^* -algebra X , the initial section $\downarrow a := \{x \in X \mid x \leq a\}$ is an ideal of X if and only if for any $x, y, z \in X$, $x * y \leq z$ and $y \leq z$ imply $x \leq z$.*

Proof. Assume that for each $a \in X$, $\downarrow a$ is an ideal of X . Let $x, y, z \in X$ be such that $x * y \leq z$ and $y \leq z$. Then $x * y \in \downarrow z$ and $y \in \downarrow z$. Since $\downarrow z$ is an ideal of X , it follows from (I2) that $x \in \downarrow z$, i.e., $x \leq z$.

Conversely, consider $\downarrow z$ for any $z \in X$. Obviously $0 \in \downarrow z$ for any $z \in X$, since X is a BH^* -algebra. For $y \in \downarrow z$ and any $a \in X$, let $a * y \leq z$. Since $y \in \downarrow z$, it follows the hypothesis that $a \leq z$, i.e., $a \in \downarrow z$. Thus $\downarrow z$ is an ideal of X . $\square$

Proposition 3.7. *Let X be a BH -algebra. Then a subalgebra A of X is a strong ideal of X if and only if $\forall x, y, z \in X, x \in A, y * z \in X - A$ imply $(y * x) * z \in X - A$.*

Proof. Assume that a subalgebra A of X is a strong ideal of X and let $x, y, z \in X$ be such that $x \in A$ and $y * z \in X - A$. If $(y * x) * z \notin X - A$, then $(y * x) * z \in A$. Since A is a strong ideal of X and $x \in A$, we have $y * z \in A$. This is a contradiction.

Conversely, assume that $\forall x, y, z \in X, x \in A, y * z \in X - A$ imply $(y * x) * z \in X - A$. Since A is a subalgebra of X , we have $0 \in A$. For every $x \in A$, let $(y * x) * z \in A$. If $y * z \notin A$, then $(y * x) * z \in X - A$ by assumption. This is a contradiction. Hence $y * z \in A$. Thus A is a strong ideal of X . $\square$

Putting $z := 0$ in the above Proposition we have Corollary 3.8.

Corollary 3.8. *Let A be a subalgebra of a BH -algebra X . Then A is an ideal of X if and only if $\forall x, y \in X, x \in A, y \in X - A$ imply $y * x \in X - A$.*

Proposition 3.9. *Let A be an ideal of a BH -algebra X . If B is an ideal of A , then it is an ideal of X .*

Proof. Since B is an ideal of A , we have $0 \in B$. Let $y, x * y \in B$ for some $x \in X$. If $x \in A$, then $x \in B$, since B is an ideal of A . If $x \in X - A$, then $y, x * y \in B \subseteq A$ and so $x \in A$ because A is an ideal of

X . Thus we have $x \in B$, since B is an ideal of A . This completes the proof. $\square$.

A proper ideal of a BH -algebra X is called *maximal* if it is not properly contained in any proper ideal of X . In our convention, any BCK/BCI -algebra is a BH -algebra, but there are BH -algebras which is not BCK/BCI -algebra. Such BH -algebras are called *proper*.

Theorem 3.10. *Any BH -algebra may be viewed as a maximal strong ideal of some BH -algebra.*

Proof. If S is a (proper) BH -algebra and $e \notin S$, then $X := S \cup \{e\}$ with the multiplication defined by

$$x * y := \begin{cases} x * y & \text{for } x, y \in S \\ 0 & \text{for } y = e, x \in X \\ e & \text{for } x = e, y \neq e \end{cases}$$

is a (proper) BH -algebra and e is the greatest element of X . It is easy to see that S is a maximal strong ideal of X . $\square$

A non-zero element $a \in X$ is called an *atom* of a BH -algebra X if $x \leq a$ implies $x = 0$ or $x = a$.

Lemma 3.11 *A non-zero element $a \in X$ is an atom of X if $\{0, a\}$ is an ideal of X .*

Proof. Straightforward. $\square$

In general, the converse is not true as seen in the following example.

Example 3.12. Let $X := \{0, a, b, c\}$ be a set with the following Cayley table:

$*$	0	a	b	c
0	0	0	0	c
a	a	0	0	0
b	b	a	0	0
c	c	b	b	0

Then X is a proper BH -algebra. An element a is an atom, but $\{0, a\}$ is not an ideal.

Lemma 3.13. *If every non-zero element of a BH^* -algebra X is an atom, then any subalgebra of X is an ideal of X .*

Proof. Let S be a subalgebra of X and let $x, y * x \in S$. Since $y * x \leq y$ for all $x, y \in X$ and y is an atom of X , we have $y * x = 0$ or $y = y * x \in S$.

If $y * x = 0$, then $y \leq x$ which gives $y = 0$ or $y = x$. Thus $y \in S$, which completes the proof. $\square$

From the above Lemmas we obtain the following Theorem.

Theorem 3.14. *A BH^* -algebra contains only atoms if and only if every its subalgebra is an ideal.*

For any element x and y of a BH -algebra, $x * y^n$ denotes $(\cdots((x * y) * y) * \cdots) * y$ in which y occurs n times.

Definition 3.15. Let n be a positive integer. A non-zero element a of a BH -algebra X is called an n -atom of X if $x * a^n = 0$ implies $x = 0$ or $x = a$.

The 1-atom is precisely an atom.

Example 3.16. Let X be a BH -algebra as in Example 3.5(2), Then 1 and 2 are n -atoms of X for every positive integer n .

Theorem 3.17. *Let X be a BH -algebra and $0 \neq a \in X$. If $\{0, a\}$ is an ideal of X , then a is an n -atom of X for positive integer n .*

Proof. Let x be a non-zero element of X such that $x * a^2 = 0$. Then $(x * a) * a = x * a^2 = 0 \in \{0, a\}$. Since $\{0, a\}$ is an ideal of X , it follows that $x * a \in \{0, a\}$ so that $x * a = 0$ or $x * a = a$. In each case, we know that $x \in \{0, a\}$ and so $x = a$ because $x \neq 0$. Hence a is a 2-atom of X . Assume that a is a k -atom of X , where $k \in \mathbb{N}$. Let x be a non-zero element of X such that $x * a^{k+1} = 0$. Then

$$(x * a^k) * a = x * a^{k+1} = 0 \in \{0, a\}.$$

It follows from (I2) that $x * a^k \in \{0, a\}$ so that $x * a^k = 0$ or $x * a^k = a$. If $x * a^k = 0$, then $x = a$ by induction hypothesis. If $x * a^k = a$, then $(x * a^{k-1}) * a = x * a^k \in \{0, a\}$. By using (I2) repeatedly, we get $x \in \{0, a\}$ and thus $x = a$ because $x \neq 0$. Therefore a is an n -atom. $\square$

Theorem 3.18. *Let $f : X \rightarrow Y$ be a homomorphism of BH -algebras. If B is a strong ideal of Y , then $f^{-1}(B)$ is a strong ideal of X .* $\square$

Proof. Since $f(0) = 0$, we have $0 \in f^{-1}(B)$. Let $x, y, z \in X$ be such that $(x * y) * z \in f^{-1}(B)$ and $y \in f^{-1}(B)$. Then $(f(x) * f(y)) * f(z) = f((x * y) * z) \in B$ and $f(y) \in B$. Since B is a strong ideal of Y , it follows from (I4) that $f(x * z) = f(x) * f(z) \in B$ so that $x * z \in f^{-1}(B)$. Hence $f^{-1}(B)$ is a strong ideal of X . $\square$

Corollary 3.19. *Let $f : X \rightarrow Y$ be a homomorphism of BH -algebras. Then $\text{Ker } f := \{x \in X \mid f(x) = 0\}$ is a strong ideal of X .*

4. Rough sets in BH -algebras

In what follows let X denote a BH -algebra unless otherwise specified.

Let I be a translation ideal of X . Define a relation ρ on X by $(x, y) \in \rho$ if and only if $x * y \in I$ and $y * x \in I$. Then ρ is an equivalence relation on X related to a translation ideal I of X . Moreover ρ satisfies $(x, y) \in \rho$ and $(u, v) \in \rho$ imply $(x * u, y * v) \in \rho$. Hence ρ is a congruence relation on X . We denote by $[a]_\rho$ the ρ -congruence class containing the element $a \in X$. Let X/ρ be the set of all ρ -equivalence classes on X , i.e., $X/\rho := \{[a]_\rho | a \in X\}$. For any $[x]_\rho, [y]_\rho \in X/\rho$, if we define

$$[x]_\rho * [y]_\rho := [x * y]_\rho = \{z \in X | (z, x * y) \in \rho\},$$

then it is well defined, since ρ is a congruence relation. A congruence relation ρ on a BH -algebra X is said to be *regular* if $[x]_\rho * [y]_\rho = [0]_\rho = [y]_\rho * [x]_\rho$ implies $[x]_\rho = [y]_\rho$ for any $[x]_\rho, [y]_\rho \in X/\rho$.

Theorem 4.1. *Let X be a BH -algebra and let ρ be a congruence relation on X . Then ρ is regular if and only if X/ρ is a BH -algebra.*

Proof. Straightforward. $\square$

Let X be a BH -algebra and ρ be an congruence relation on X and let $\mathcal{P}(X)$ denote the power set of X . For all $x \in X$, let $[x]_\rho$ denote the ρ -congruence class of x . Define the functions $\rho_-, \rho^- : \mathcal{P}(X) \rightarrow \mathcal{P}(X)$ as follows: for any $\emptyset \neq A \in \mathcal{P}(X)$,

$$\rho_-(A) := \{x \in X | [x]_\rho \subseteq A\}$$

and

$$\rho^-(A) := \{x \in X | [x]_\rho \cap A \neq \emptyset\}.$$

$\rho_-(A)$ is called the ρ -lower approximation of A while $\rho^-(A)$ is called the ρ -upper approximation of A . For a non-empty subset A of X ,

$$\rho(A) = (\rho_-(A), \rho^-(A))$$

is called a *rough set* with respect to ρ of $\mathcal{P}(X) \times \mathcal{P}(X)$ if $\rho_-(A) \neq \rho^-(A)$. A subset A of X is said to be *definable* if $\rho_-(A) = \rho^-(A)$. The pair (X, ρ) is called an *approximation space*.

Proposition 4.2. *Let X be a BH -algebra and $\emptyset \neq A, B \subseteq X$. Let ρ and λ be congruence relations on X . Then the following hold:*

- (1) $\rho_-(A) \subseteq A \subseteq \rho^-(A)$,
- (2) $\rho^-(A \cup B) = \rho^-(A) \cup \rho^-(B)$,
- (3) $\rho_-(A \cap B) = \rho_-(A) \cap \rho_-(B)$,
- (4) $A \subseteq B$ implies $\rho_-(A) \subseteq \rho_-(B)$,

- (5) $A \subseteq B$ implies $\rho^-(A) \subseteq \rho^-(B)$,
- (6) $\rho_-(A) \cup \rho_-(B) \subseteq \rho_-(A \cup B)$,
- (7) $\rho^-(A \cap B) \subseteq \rho^-(A) \cap \rho^-(B)$,
- (8) $\rho \subseteq \lambda$ implies $\rho_-(A) \supseteq \lambda_-(A)$,
- (9) $\rho \subseteq \lambda$ implies $\rho^-(A) \subseteq \lambda^-(A)$.

Proof. Straightforward. $\square$

Corollary 4.3. Let ρ and λ be congruence relations on a BH -algebra X and let $\emptyset \neq A \subseteq X$. Then $\rho_-(A) \cap \lambda_-(A) \subseteq (\rho \cap \lambda)_-(A)$.

Proof. It follows immediately from Proposition 4.2 (8). $\square$

Let X be a BH -algebra and let $\emptyset \neq A, B \subseteq X$. Define $A * B := \{a * b \mid a \in A, b \in B\}$.

Theorem 4.4. Let ρ be a congruence relation on a BH -algebra X and let $\emptyset \neq A, B \subseteq X$. Then $\rho^-(A) * \rho^-(B) \subseteq \rho^-(A * B)$.

Proof. Let c be any element of $\rho^-(A) * \rho^-(B)$. Then $c = a * b$ for some $a \in \rho^-(A)$ and $b \in \rho^-(B)$. Thus there exist elements $x, y \in X$ such that $x \in [a]_\rho \cap A$ and $y \in [b]_\rho \cap B$. Hence $x \in [a]_\rho, y \in [y]_\rho, x \in A$ and $y \in B$. Since ρ is a congruence relation on X , $x * y \in [a]_\rho * [b]_\rho = [a * b]_\rho$. Since $x * y \in A * B$, we have $c = a * b \in \rho^-(A * B)$. $\square$

Theorem 4.5. Let ρ be a congruence relation on a BH -algebra X and let $\emptyset \neq A, B \subseteq X$. If $\rho_-(A * B)$ is non-empty, then $\rho_-(A) * \rho_-(B) \subseteq \rho_-(A * B)$.

Proof. Let $c \in \rho_-(A) * \rho_-(B)$. Then $c = a * b$ for some $a \in \rho_-(A)$ and $b \in \rho_-(B)$. Thus we have $[a]_\rho \subseteq A$ and $[b]_\rho \subseteq B$. Since ρ is a congruence relation on X , $[a * b]_\rho = [a]_\rho * [b]_\rho \subseteq A * B$ and so $c = a * b \in \rho_-(A * B)$, completing the proof. $\square$

Remark. The condition that $\rho_-(A * B)$ is non-empty in Theorem 4.5 is necessary.

Example 4.6. Let $X := \{0, a, b, c, d\}$ be a BH -algebra which is not a BCK/BCI -algebra with the following Cayley table:

$*$	0	a	b	c	d
0	0	0	0	0	0
a	a	0	0	a	0
b	b	b	0	0	b
c	c	c	c	0	c
d	d	d	d	d	0

Then $I := \{0, a\}$ is a translation ideal of X . Let ρ be a congruence relation on X such that $\{0, a\}$, $\{b\}$, $\{c\}$, and $\{d\}$ are all ρ -congruences of X . If we take $A := \{b, c\}$, $B := \{c\}$, then $A * B = \{0\}$ and $\rho_-(A * B) = \emptyset$, $\rho_-(A) = \{b, c\}$, $\rho_-(B) = \{c\}$ and hence $\rho_-(A) * \rho_-(B) = \{0\}$.

A non-empty subset A of a BH -algebra X is called an *upper* (*lower*, respectively) *rough subalgebras* of X if $\rho^-(A)$ ($\rho_-(A)$, respectively) is a subalgebra of X .

Proposition 4.7. *Let ρ be a congruence relation on a BH -algebra X and let $\emptyset \neq A \subseteq X$. If A is a subalgebra of X , then A is an upper rough subalgebra of X .*

Proof. Since $0 \in A$ and $A \subseteq \rho^-(A)$, we have $0 \in \rho^-(A)$. If we put $B := A$ in Theorem 4.4, then the proposition follows Proposition 4.2(5). $\square$

Proposition 4.8. *Let ρ be a congruence relation on a BH -algebra X and let A be a subalgebra of X . If $\rho_-(A)$ is non-empty, then it is a subalgebra of X , i.e., A is a lower rough subalgebra of X .*

Proof. If we take $B := A$ in Theorem 4.5, then $\rho_-(A) * \rho_-(A) \subseteq \rho_-(A * A) \subseteq \rho_-(A)$ by applying Proposition 4.2(4). $\square$

Let X be a BH -algebra and let $\emptyset \neq A \subseteq X$. Let ρ be an congruence relation on X . Then A is called an *upper* (*lower*, respectively) *rough (strong) ideal* of X if $\rho^-(A)$ ($\rho_-(A)$, respectively) is an (strong) ideal of X .

Theorem 4.9. *Let ρ be a congruence relation on a BH -algebra X . If A is an ideal of a BH -algebra X , then it is an upper rough ideal of X .*

Proof. Since A is an ideal of X , $0 \in A$ and hence $A \cap [0]_\rho \neq \emptyset$, i.e., $0 \in \rho^-(A)$. Let $x, y \in X$ with $x * y, y \in \rho^-(A)$. Then $([x]_\rho * [y]_\rho) \cap A = [x * y]_\rho \cap A \neq \emptyset$ and $[y]_\rho \cap A \neq \emptyset$. This means that there exist $\alpha \in A$ and $q \in A$ such that $\alpha \in [x]_\rho * [y]_\rho$ and $q \in [y]_\rho$. Thus $\alpha = p * q$ for some $p \in [x]_\rho$ and $q \in [y]_\rho$. Since A is an ideal of X , $p \in A$. Hence $p \in [x]_\rho \cap A$, i.e., $x \in \rho^-(A)$, completing the proof. $\square$

Theorem 4.10. *Let ρ be a congruence relation on a BH -algebra X . If A is a strong ideal of a BH -algebra X , then it is an upper rough strong ideal of X .*

Proof. Since A is a strong ideal of X , $0 \in A$ and hence $A \cap [0]_\rho \neq \emptyset$, i.e., $0 \in \rho^-(A)$. Let $x, y, z \in X$ with $(x * y) * z, y \in \rho^-(A)$. Then $(([x]_\rho * [y]_\rho) * [z]_\rho) \cap A = [(x * y) * z]_\rho \cap A \neq \emptyset$ and $[y]_\rho \cap A \neq \emptyset$. This

means that there exist $\alpha \in A$ and $q \in A$ such that $\alpha \in ([x]_\rho * [y]_\rho) * [z]_\rho$ and $q \in [y]_\rho$. Thus $\alpha = (p * q) * r$ for some $p \in [x]_\rho, q \in [y]_\rho$ and $r \in [z]_\rho$. Since A is a strong ideal of X , $p * r \in A$. Hence $p * r \in ([x]_\rho * [z]_\rho) \cap A$, i.e., $x * z \in \rho^-(A)$, completing the proof. $\square$

Proposition 4.11. *Let X be a BH^* -algebra and let $\emptyset \neq A, B \subseteq X$. If D is an ideal of X and $A \cap D \neq \emptyset \neq B \cap D$, then $(A * B) \cap D \neq \emptyset$.*

Proof. Assume that $a \in A \cap D$ and $b \in B \cap D$. Then $a * b \in A * B$. By Lemma 3.3, we have $a * b \in D$, proving $(A * B) \cap D \neq \emptyset$. $\square$

Proposition 4.11 does not hold for a BH -algebra in general, since not every ideal of a BH -algebra is a subalgebra of X .

Theorem 4.12. *Let ρ be a congruence relation on a BH -algebra X and let A be an ideal of X . If $\rho_-(A)$ is non-empty, then A is a lower rough ideal of X , i.e., $\rho_-(A)$ is an ideal of X .*

Proof. By Proposition 4.8, $\rho_-(A)$, if it is non-empty, is a subalgebra of X , and hence $0 \in \rho_-(A)$. Assume that $x * y, y \in \rho_-(A)$ for any $x \in X$. Then $[x]_\rho * [y]_\rho = [x * y]_\rho \subseteq A$ and $[y]_\rho \subseteq A$. If $\alpha \in [x]_\rho$, then $(\alpha, x) \in \rho$. Since ρ is a congruence relation on X , we have $(\alpha * y, x * y) \in \rho$. Hence $\alpha * y \in [x * y]_\rho \subseteq A$. Since A is an ideal of X and $y \in A$, we obtain $\alpha \in A$. Thus $[x]_\rho \subseteq A$, proving that so $x \in \rho_-(A)$. $\square$

Theorem 4.13. *Let ρ be a congruence relation on a BH -algebra X let A be a strong ideal of X . If $\rho_-(A)$ is non-empty, then A is a lower rough strong ideal of X , i.e., $\rho_-(A)$ is a strong ideal of X .*

Proof. By Proposition 4.8, $\rho_-(A)$, if it is non-empty, is a subalgebra of X , and hence $0 \in \rho_-(A)$. Assume that $(x * y) * z, y \in \rho_-(A)$ for any $x \in X$. Then $([x]_\rho * [y]_\rho) * [z]_\rho = [(x * y) * z]_\rho \subseteq A$ and $[y]_\rho \subseteq A$. If $\alpha \in [x]_\rho$, then $(\alpha, x) \in \rho$. Since ρ is a congruence relation on X , we have $(\alpha * y, x * y) \in \rho$ and so $((\alpha * y) * z, (x * y) * z) \in \rho$. Hence $(\alpha * y) * z \in [(x * y) * z]_\rho \subseteq A$. Since A is a strong ideal of X and $y \in A$, we obtain $\alpha * z \in A$. Thus $[x]_\rho * [z]_\rho \subseteq A$, proving that $x * z \in \rho_-(A)$. $\square$

Let ρ be a regular congruence relation on a BH -algebra X . The lower and upper approximations can be presented in an equivalent form as shown below:

$$\begin{aligned}\rho_-(A)/\rho &= \{[x]_\rho \in X/\rho \mid [x]_\rho \subseteq A\} \\ \rho^-(A)/\rho &= \{[x]_\rho \in X/\rho \mid [x]_\rho \cap A \neq \emptyset\}.\end{aligned}$$

Proposition 4.14. *Let ρ be a regular congruence relation on a BH-algebra X . If A is a subalgebra of X , then $\rho^-(A)/\rho$ is a subalgebra of the quotient BH-algebra X/ρ .*

Proof. Since A is a subalgebra of X , there exists an element $x \in A$ and hence $[x]_\rho \cap A \neq \emptyset$, i.e., $\rho^-(A)/\rho \neq \emptyset$. Let $[x]_\rho$ and $[y]_\rho$ be any elements of $\rho^-(A)/\rho$. Then $[x]_\rho \cap A \neq \emptyset$ and $[y]_\rho \cap A \neq \emptyset$. This means that there exist $a, b \in X$ such that $a \in [x]_\rho \cap A$ and $b \in [y]_\rho \cap A$. Then $a * b \in [x]_\rho * [y]_\rho$. Since A is a subalgebra of X , $a * b \in A$. Hence $([x]_\rho * [y]_\rho) \cap A \neq \emptyset$. This means that $[x]_\rho * [y]_\rho \in \rho^-(A)/\rho$, completing the proof. $\square$

Proposition 4.15. *Let ρ be a regular congruence relation on a BH-algebra X . If A is a subalgebra of X , then $\rho_-(A)/\rho$ is, if it is non-empty, a subalgebra of the quotient BH-algebra X/ρ .*

Proof. Straightforward. $\square$

Theorem 4.16. *Let ρ be a regular congruence relation on a BH-algebra X . If A is a strong ideal of X , then $\rho^-(A)/\rho$ is a strong ideal of the quotient BH-algebra X/ρ .*

Proof. Since $0 \in A$, $A \cap [0]_\rho \neq \emptyset$ and hence $[0]_\rho \in \rho^-(A)/\rho$. Let $[x]_\rho, [y]_\rho, [z]_\rho \in X/\rho$ with $([x]_\rho * [y]_\rho) * [z]_\rho, [y]_\rho \in \rho^-(A)/\rho$. Then $(([x]_\rho * [y]_\rho) * [z]_\rho) \cap A \neq \emptyset$ and $[y]_\rho \cap A \neq \emptyset$. Hence there exists $\alpha \in [(x * y) * z]_\rho \cap A$ and $q \in [y]_\rho \cap A$. Since $[(x * y) * z]_\rho \neq [0]_\rho$ and $[y]_\rho \neq [0]_\rho$, we have $\alpha \neq 0$ and $q \neq 0$. Thus $\alpha = (p * q) * r$ for some $p \in [x]_\rho$, $q \in [y]_\rho$ and $r \in [z]_\rho$. Since A is a strong ideal of X , we have $p * r \in A$. Hence $p * r \in ([x]_\rho * [y]_\rho) \cap A$, proving $[x]_\rho * [z]_\rho \in \rho^-(A)/\rho$. $\square$

Theorem 4.17. *Let ρ be a regular congruence relation on a BH-algebra X . If A is a strong ideal of X , then $\rho_-(A)/\rho$ is, if it is non-empty, a strong ideal of the quotient BH-algebra X/ρ .*

Proof. By Proposition 4.15, $\rho_-(A)/\rho$, if it is non-empty, is a subalgebra of X/ρ , and hence $[0]_\rho \in \rho_-(A)/\rho$. Let $[x]_\rho, [y]_\rho, [z]_\rho \in X/\rho$ with $([x]_\rho * [y]_\rho) * [z]_\rho, [y]_\rho \in \rho_-(A)/\rho$. Then $[(x * y) * z]_\rho \subseteq A$ and $[y]_\rho \subseteq A$. Let $\alpha \in [(x * y) * z]_\rho \subseteq A$ and $q \in [y]_\rho \subseteq A$. Then $\alpha = (p * q) * r$ for some $p \in [x]_\rho$, $q \in [y]_\rho$ and $r \in [z]_\rho$ such that $(p * q) * r \in A$ and $q \in A$. Since A is a strong ideal of X , we have $p * r \in A$. Hence $p * r \in [x]_\rho * [z]_\rho \subseteq A$, proving $[x]_\rho * [z]_\rho \subseteq \rho_-(A)/\rho$. $\square$

Theorem 4.18. *Let ρ be a regular congruence relation on a BH-algebra X . If A is an upper rough ideal of X , then $\rho^-(A)/\rho$ is an ideal of X/ρ .*

Proof. Since $0 \in \rho^-(A)$, we have $[0]_\rho \cap A \neq \emptyset$ and hence $[0]_\rho \in \rho^-(A)/\rho$. Let $[x]_\rho * [y]_\rho = [x * y]_\rho, [y]_\rho \in \rho^-(A)/\rho$ for some $[x]_\rho \in X/\rho$. Then $([x]_\rho * [y]_\rho) \cap A = [x * y]_\rho \cap A \neq \emptyset$ and $[y]_\rho \cap A \neq \emptyset$. Hence $x * y, y \in \rho^-(A)$. Since $\rho^-(A)$ is an ideal of X , we have $x \in A$. Thus $x \in [x]_\rho \cap A \neq \emptyset$, proving $[x]_\rho \in \rho^-(A)/\rho$. $\square$

Theorem 4.19. *Let ρ be a regular congruence relation on a BH-algebra X . If A is a lower rough ideal of X , then $\rho_-(A)/\rho$ is, if it is non-empty, an ideal of the quotient BH-algebra X/ρ .*

Proof. Since $\rho_-(A)/\rho \neq \emptyset$, $\rho_-(A)/\rho$ is a subalgebra of X/ρ and hence $[0]_\rho \in \rho_-(A)/\rho$. Let $[x]_\rho * [y]_\rho, [y]_\rho \in \rho_-(A)/\rho$ for some $[x]_\rho \in X/\rho$. Hence $[x * y]_\rho \subseteq A$ and $[y]_\rho \subseteq A$. Therefore $x * y \in \rho_-(A), y \in \rho_-(A)$. Since $\rho_-(A)$ is an ideal of X , we have $x \in \rho_-(A)$. Therefore $[x]_\rho \subseteq A$. Thus $[x]_\rho \in \rho_-(A)/\rho$. $\square$

References

- [1] W. A. Dudek and X. Zhang, *On ideals and congruences in BCC-algebras*, Czechoslov. Math. J., **48**(1998), 21-29.
- [2] K. Iséki, *On BCI-algebras*, Mathematics Seminar Notes, **8**(1980), 125-130.
- [3] K. Iséki and S. Tanaka, *An introduction to the theory of BCK-algebras*, Math. Japonica, **23**(1978), 1-26.
- [4] Y. B. Jun, E. H. Roh and H. S. Kim, *On BH-algebras*, Scientiae Math. **1**(1998), 347-354.
- [5] N. Kuroki, *Rough Ideals in semigroups*, Inform. Sci., **100**(1995), 139-163.
- [6] N. Kuroki and J. N. Mordeson, *Structure of rough sets and rough groups*, J. Fuzzy Math., **5**(1997), 183-191.
- [7] C. R. Lim and H. S. Kim, *Rough Ideals in BCK/BCI-algebras*, Bull. Polish Acad. Sci. Math., **51**(2003), 59-67.
- [8] D. Mundici, *MV-algebras are categorically equivalent to bounded commutative BCK-algebras*, Math. Japon., **31**(1986), 889-894.
- [9] J. Meng, *Implicative commutative semigroups are equivalent to a class of BCK-algebras*, Semigroup Forum, **50**(1995), 89-96.
- [10] J. Meng and Y. B. Jun, *BCK-algebras*, Kyung Moon Sa, Seoul, 1994.
- [11] E. H. Roh and S. Y. Kim, *On BH*-subalgebras of transitive BH*-algebras*, Far East J. Math. Sci., **1**(1999), 255-263.

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