A BIJECTIVE PROOF OF $r = 1$ REDUCTION FORMULA FOR LITTLEWOOD-RICHARDSON COEFFICIENTS

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Abstract. Inspired by the reduction formulae between intersection numbers on Grassmannians obtained by Griffiths-Harris and the factorization theorem of Littlewood-Richardson coefficients by King, Tollu and Toumazet, eight reduction formulae has been discovered by the author and others. In this paper, we prove $r = 1$ reduction formula by constructing a bijective map between suitable sets of Littlewood-Richardson tableaux.

0. Introduction

We introduce basic notations to define Littlewood-Richardson coefficient in this section. More details can be found, for example, in [6] or [14]. By a partition, we mean a weakly decreasing sequence of nonnegative integers with finite number of positive integers. Let $\lambda = (\lambda_1, \lambda_2, \dots)$ be a partition. The *size* of λ is defined as $|\lambda| = \sum_i \lambda_i$, the total sum of the parts, and the *length* of λ , denoted by $\ell(\lambda)$, is defined as the number of positive integers in λ .

For each partition λ , we associate its *Young diagram*, a left-justified array of boxes with λ_p boxes in its p th row. We will let (p, q) -position of λ denote the box at the p th row and the q th column in the Young diagram of λ . For a given partition λ , $\tilde{\lambda}$ indicates the *conjugate* of λ , whose diagram has λ_p boxes in its p th column. A *Young tableau*, or simply a *tableau*, of shape λ is a filling of a Young diagram of shape

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λ with integers that is weakly increasing across each row and strictly increasing down each column.

For two partitions $\lambda = (\lambda_1, \lambda_2, \dots)$ and $\nu = (\nu_1, \nu_2, \dots)$, we write $\lambda \subseteq \nu$, if $\lambda_i \leq \nu_i$ for all i . For partitions λ and ν with $\lambda \subseteq \nu$, the *skew diagram*, or *skew shape*, ν/λ is the diagram consisting boxes of ν which are not the boxes of λ . We will write $(p, q) \in \nu/\lambda$ if $\lambda_p < q \leq \nu_p$, i.e there is a box at the p th row and q th column in the skew diagram of ν/λ .

A *skew tableau* of shape ν/λ is a filling of boxes of a skew diagram ν/λ with integers, where entries are weakly increasing in rows and strictly increasing in columns. For a given (skew) tableau T of shape ν/λ and $(p, q) \in \nu/\lambda$, we will let $T(p, q)$ denote the *entry* in the box at (p, q) -position of the tableau T . The *content* of a (skew) tableau T is a sequence of integer $\mu = (\mu_1, \mu_2, \dots)$ such that there are μ_i i 's in T . The *reverse row word* of a skew tableau T , denoted by $w(T)$, is the word obtained by reading the entries of T from right to left and top to bottom starting from the first row. A word $w = x_1 \cdots x_r$ is called a *lattice word* if, for any $s \leq r$ and i , $x_1 \cdots x_s$ contains at least as many i 's as it contains $(i+1)$'s. A skew tableau T is a *Littlewood-Richardson skew tableau* (LR-tableau) if its reverse row word $w(T)$ is a lattice word. If T is an LR-tableau of shape ν/λ , the lattice word condition enforce that the content μ of T is a partition and $\mu \subseteq \nu$.

Definition 0.1. For given partitions λ, μ and ν with $\lambda, \mu \subseteq \nu$, the *Littlewood-Richardson coefficient* (LR-coefficient) $c_{\lambda, \mu}^\nu$ is the number of LR-tableaux on the shape ν/λ of content μ .

The following is a well-known identity between LR-coefficients under conjugation.

Lemma 0.2 ([9, 6]). *For partitions λ, μ, ν , the following equality holds:*

$$c_{\lambda, \mu}^\nu = c_{\tilde{\lambda}, \tilde{\mu}}^{\tilde{\nu}}.$$

The Littlewood-Richardson coefficients occur in many mathematical subjects. They explain multiplication rule of the product of Schur functions in ring of symmetric functions. They also count the multiplicity in the decomposition of tensor products of irreducible polynomial representations of general linear groups, or in the decomposition of induced representations in the representation theory of the symmetric group. Moreover, the Littlewood-Richardson coefficients appear as intersection numbers in the Schubert calculus of Grassmannians.

For a given partition λ , by $\lambda \setminus \lambda_I$ where I is a (finite) subset of the indices, we mean the partition obtained by deleting all the i th part

for $i \in I$ from λ . By a reduction formula for Littlewood-Richardson coefficient $c_{\lambda, \mu}^{\nu}$, is meant a formula of the type $c_{\lambda, \mu}^{\nu} = c_{\lambda', \mu'}^{\nu'}$, where either $\lambda' = \lambda \setminus \lambda_I$ or λ' is the partition obtained by successively decreasing the first i parts by 1 for $i \in I$, with similar formulae applying to the parts of μ and ν for J and K in which J and K necessary have the same cardinality as I . Note that the second type reduction formulae may be obtained from the first type reduction formulae applying the identity under conjugation, Lemma 0.2.

Inspired by the reduction formulae between intersection numbers on Grassmannians obtained by Griffiths-Harris [7] and the factorization theorem on Littlewood-Richardson coefficients due to King, Tollu and Toumazet [10, 11], the author, in collaboration with S. Cho and E.-K. Jung, has established eight such reduction formulae, $r = 1$, $r = n - 1$, $r = 2$ and $r = n - 2$ and their conjugates, where n is an integer bigger than the length of ν , in a sequence of previous papers [1, 3, 2, 4, 5]. A bijective proof, using LR-tableaux, of reduction formulae for $r = n - 1$ and $r = n - 2$ was announced in [1] and [4]. But detailed proofs for $r = 1$ and $r = 2$ reduction formulae have not been appeared in printed form. The purpose of this paper is to obtain a bijective proof of $r = 1$ reduction formulae. In [3] and [5], only the statement and algorithm of bijective map for $r = 1$ reduction formula appeared and no proof was given.

1. Statements and Algorithm

In this section, we give a precise statement of our $r = 1$ reduction formula and describe a bijection which will be used to prove the reduction formula. Recall, for a given partition λ , by $\lambda \setminus \lambda_{\{i\}}$ where i is an index of the partition λ , we mean the partition obtained by deleting the i th part from λ .

Theorem 1.1 ($r = 1$ reduction formula). *We let λ, μ and ν be partitions with $\ell(\lambda), \ell(\mu), \ell(\nu) \leq n$ and $c_{\lambda, \mu}^{\nu} > 0$. Suppose that there are $1 \leq i, j, k \leq n$ such that $i + j = k + 1$ and $\nu_k = \lambda_i + \mu_j$. Then*

$$c_{\lambda, \mu}^{\nu} = c_{\lambda \setminus \lambda_{\{i\}}, \mu \setminus \mu_{\{j\}}}^{\nu \setminus \nu_{\{k\}}}.$$

For a skew tableau T with n rows and integers $1 \leq h, \ell \leq n$, we let $\mathbf{n}_T^h(\ell)$ be the number of ℓ 's in the h th row of T . We will need the following lemmas adopted from [1].

Lemma 1.2. *A skew tableau T with n rows is an LR-tableau, i.e. the reverse row word $w(T)$ is a lattice word, if and only if, for all $1 < p, s \leq n$,*

$$\sum_{h=1}^p \mathbf{n}_T^h(s) \leq \sum_{h=1}^{p-1} \mathbf{n}_T^h(s-1).$$

Lemma 1.3. *Suppose that s appears in the q th row of an LR-tableau T , then $q \geq s$.*

In the rest of this paper, we assume that $i + j = k + 1$, $\lambda_i + \mu_j = \nu_k$. The following is a key lemma that allows Definition 1.8 is valid.

Lemma 1.4. *Let T be an LR-tableau of shape ν/λ with content μ . For $j + 1 \leq \ell \leq k$, we have*

$$(1.5) \quad \sum_{h=\ell}^k \mathbf{n}_T^h(\ell) = \sum_{h=\ell-1}^{k-1} \mathbf{n}_T^h(\ell-1),$$

$$(1.6) \quad \sum_{h=j}^k \mathbf{n}_T^k(h) = \mu_j,$$

and

$$(1.7) \quad \sum_{h=j}^k \mathbf{n}_T^h(j) = \mu_j.$$

Proof. First note that the column (strict) increasingness of T and Lemma 1.3 imply the entry in the $(k, \lambda_i + 1)$ -box is at least $k - i + 1 = j$. Then,

$$\begin{aligned} \mu_j &= \nu_k - \lambda_i \\ &\leq \mathbf{n}_T^k(j) + \mathbf{n}_T^k(j+1) + \cdots + \mathbf{n}_T^k(k-2) + \mathbf{n}_T^k(k-1) + \mathbf{n}_T^k(k) \\ &\leq \mathbf{n}_T^k(j) + \mathbf{n}_T^k(j+1) + \cdots + \mathbf{n}_T^k(k-2) + \left(\mathbf{n}_T^k(k-1) + \mathbf{n}_T^{k-1}(k-1) \right) \\ &\leq \mathbf{n}_T^k(j) + \mathbf{n}_T^k(j+1) + \cdots + \left(\mathbf{n}_T^k(k-2) + \left(\mathbf{n}_T^{k-1}(k-2) + \mathbf{n}_T^{k-2}(k-2) \right) \right) \\ &\quad \vdots \\ &\leq \mathbf{n}_T^k(j) + \mathbf{n}_T^{k-1}(j) + \cdots + \mathbf{n}_T^j(j) \leq \mu_j. \end{aligned}$$

Therefore all inequalities are actually equalities and we obtain desired relations. $\square$

Let T be an LR-tableau of shape ν/λ with content μ . The following algorithm is to construct the corresponding LR-tableau $\Phi(T)$ of shape $(\nu \setminus \nu_{\{k\}})/(\lambda \setminus \lambda_{\{i\}})$, with content $\mu \setminus \mu_{\{j\}}$ in a bijective way.

Definition 1.8. The reduced LR-tableau $\Phi(T)$ is obtained by applying the following algorithm:

- Step 1: **for** $\ell = k$ **downto** $j + 1$ **do**
 Empty all boxes containing ℓ in k th row.
 Replace all $(\ell - 1)$'s in $(k - \ell + 1)$ consecutive rows from the $(\ell - 1)$ st row to the $(k - 1)$ st row with ℓ 's.
end for
- Step 2: Empty all boxes containing j in the k th row.
- Step 3: **for** $\ell = j + 1$ **to** n **do**
 Replace all ℓ 's with $(\ell - 1)$'s.
end for
- Step 4: Slide each box at (p, q) -position, $1 \leq p \leq k - 1$, $\lambda_i + 1 \leq q$, one step down to remove all empty boxes.

Note (1.7) guarantees that we have removed all j 's in T after Step 2.

Example 1.9. Let $n = 6$ and $\lambda = (5, 5, 3, 2, 2, 0)$, $\mu = (8, 7, 5, 5, 4, 1)$, $\nu = (11, 9, 8, 8, 8, 3)$. We have $i = 3$, $j = 3$, $k = 5$, and $\lambda_3 + \mu_3 = \nu_5 = 8$.

2. Bijjective Proof

In this section, we will show that Φ is a bijective map between the set of LR-tableaux of shape ν/λ and content μ and the set of LR-tableaux of shape $(\nu \setminus \nu_{\{k\}})/(\lambda \setminus \lambda_{\{i\}})$ and content $\mu \setminus \mu_{\{j\}}$. For convenience, we denote the tableau obtained from T by applying the Step 1 to Step 3

by T° , hence T° is a filling of boxes of a skew diagram ν/λ with empty boxes in the k th row. The following two lemmas follow directly from Definition 1.8.

Lemma 2.1. *For $(p, q) \in \nu/\lambda$, we have*

$$T^\circ(p, q) = \begin{cases} \text{empty box} & \text{if } p = k \text{ and } q \geq \lambda_i + 1, \\ T(p, q) - 1, & \text{if } p > k \text{ and } T(p, q) \geq j + 1, \\ T(p, q), & \text{otherwise.} \end{cases}$$

Lemma 2.2. *For $(p, q) \in (\nu \setminus \nu_{\{k\}})/(\lambda \setminus \lambda_{\{i\}})$, we have*

$$\Phi(T)(p, q) = \begin{cases} T^\circ(p + 1, q), & \text{if } i \leq p \leq k - 1 \text{ and } q \leq \lambda_i, \\ T^\circ(p, q), & \text{if } 1 \leq p \leq k - 1 \text{ and } q \geq \lambda_i + 1, \\ T^\circ(p + 1, q), & \text{if } p \geq k. \end{cases}$$

Lemma 2.3. *For p with $i \leq p \leq k$, we have $T(p, \lambda_i + 1) \geq p - i + 1$. Also, if $(p, \lambda_i) \in \nu/\lambda$, then $T(p, \lambda_i) \leq p - i$ for $i < p \leq k$.*

Proof. Note $(\ell, \lambda_i + 1) \in \nu/\lambda$ for each $\ell = i, i + 1, \dots, p$. Thus, $T(p, \lambda_i + 1) \geq p - i + 1$ because of the column strictness of T .

Now note if $(p, \lambda_i) \in \nu/\lambda$ for $i < p \leq k$, then $(k, \lambda_i) \in \nu/\lambda$. Equation (1.6) and the condition $\lambda_i + \mu_j = \nu_k$ imply that $T(k, \lambda_i) \leq j - 1$. Then, because of the column strictness of T again, we have $T(p, \lambda_i) \leq j - 1 - (k - p) = p - i$. $\square$

Proposition 2.4. $\Phi(T)$ is column increasing.

Proof. It is enough to show that T° is column increasing if we ignore empty boxes. And, from Lemma 2.1 and Lemma 2.3, it is clear that T° is column increasing. $\square$

Proposition 2.5. $\Phi(T)$ is weakly increasing in rows.

Proof. It is clear that rows of T° are weakly increasing if we ignore empty boxes from Lemma 2.1. From Lemma 2.1 and Lemma 2.2, we only need to compare $T^\circ(p, \lambda_i)$ and $T^\circ(p - 1, \lambda_i + 1)$ for $i < p \leq k$. Now Lemma 2.3 says that

$$T^\circ(p, \lambda_i) = T(p, \lambda_i) \leq p - i \leq T(p - 1, \lambda_i + 1) = T^\circ(p - 1, \lambda_i + 1),$$

and hence we have the row weakly increasingness of $\Phi(T)$. $\square$

Finally, we show that $\Phi(T)$ is an LR-tableau, i.e., the row word $w(\Phi(T))$ is a lattice word. We need the following lemma which immediately follows from Lemma 2.1 and Lemma 2.2.

Lemma 2.6. For $1 \leq p, s \leq n - 1$,

$$\mathbf{n}_{\Phi(T)}^p(s) = \begin{cases} \mathbf{n}_T^p(s) + \mathbf{n}_T^{p+1}(s) & \text{if } i \leq p \leq k - 1 \text{ and } s = p - i + 1, \\ \mathbf{n}_T^{p+1}(s) & \text{if } i \leq p \leq k - 1 \text{ and } s \leq p - i, \\ \mathbf{n}_T^{p+1}(s) & \text{if } k \leq p \leq n - 1 \text{ and } s \leq j - 1, \\ \mathbf{n}_T^{p+1}(s + 1) & \text{if } k \leq p \leq n - 1 \text{ and } s \geq j, \\ \mathbf{n}_T^p(s) & \text{otherwise.} \end{cases}$$

Proposition 2.7. The reverse row word $w(\Phi(T))$ is a lattice word.

Proof. From Lemma 1.2, it is enough to show

$$\sum_{h=1}^p \mathbf{n}_{\Phi(T)}^h(s) \leq \sum_{h=1}^{p-1} \mathbf{n}_{\Phi(T)}^h(s - 1)$$

for any $1 < p, s \leq n - 1$.

If $1 < p \leq k - 1$ and $s > p - i$, then

$$\sum_{h=1}^p \mathbf{n}_{\Phi(T)}^h(s) = \sum_{h=1}^p \mathbf{n}_T^h(s) \leq \sum_{h=1}^{p-1} \mathbf{n}_T^h(s - 1) = \sum_{h=1}^{p-1} \mathbf{n}_{\Phi(T)}^h(s - 1).$$

If $i < p \leq k - 1$ and $s \leq p - i$, then

$$\begin{aligned} & \sum_{h=1}^p \mathbf{n}_{\Phi(T)}^h(s) \\ &= \sum_{h=s}^{s+i-2} \mathbf{n}_{\Phi(T)}^h(s) + \mathbf{n}_{\Phi(T)}^{s+i-1}(s) + \sum_{h=s+i}^p \mathbf{n}_{\Phi(T)}^h(s) \\ &= \sum_{h=s}^{s+i-2} \mathbf{n}_T^h(s) + \mathbf{n}_T^{s+i-1}(s) + \mathbf{n}_T^{s+i}(s) + \sum_{h=s+i}^p \mathbf{n}_T^{h+1}(s) \quad \text{by Lemma 2.6} \\ &= \sum_{h=1}^{p+1} \mathbf{n}_T^h(s). \end{aligned}$$

Similarly, we also have

$$\sum_{h=1}^{p-1} \mathbf{n}_{\Phi(T)}^h(s - 1) = \sum_{h=1}^p \mathbf{n}_T^h(s - 1).$$

Hence, because $w(T)$ is a lattice word, we have

$$\sum_{h=1}^p \mathbf{n}_{\Phi(T)}^h(s) = \sum_{h=1}^{p+1} \mathbf{n}_T^h(s) \leq \sum_{h=1}^p \mathbf{n}_T^h(s - 1) = \sum_{h=1}^{p-1} \mathbf{n}_{\Phi(T)}^h(s - 1).$$

If $p \geq k$ and $s \leq j - 1$, then

$$\sum_{h=1}^p \mathbf{n}_{\Phi(T)}^h(s) = \sum_{h=1}^{p+1} \mathbf{n}_T^h(s) \leq \sum_{h=1}^p \mathbf{n}_T^h(s-1) = \sum_{h=1}^{p-1} \mathbf{n}_{\Phi(T)}^h(s-1).$$

Now we note that, for $p \geq k$ and $s \geq j$,

$$\begin{aligned} \sum_{h=1}^p \mathbf{n}_{\Phi(T)}^h(s) &= \sum_{h=1}^{k-1} \mathbf{n}_T^h(s) + \sum_{h=k+1}^{p+1} \mathbf{n}_T^h(s+1) \\ (2.8) \quad &= \sum_{h=1}^k \mathbf{n}_T^h(s+1) + \sum_{h=k+1}^{p+1} \mathbf{n}_T^h(s+1) \quad \text{by (1.5)} \\ &= \sum_{h=1}^{p+1} \mathbf{n}_T^h(s+1). \end{aligned}$$

If $p = k$ and $s = j$, then we have

$$\begin{aligned} \sum_{h=1}^k \mathbf{n}_{\Phi(T)}^h(j) &= \sum_{h=1}^{k+1} \mathbf{n}_T^h(j+1) \leq \sum_{h=1}^k \mathbf{n}_T^h(j) \\ &\leq \sum_{h=1}^{k-1} \mathbf{n}_T^h(j-1) \leq \sum_{h=1}^k \mathbf{n}_T^h(j-1) = \sum_{h=1}^{k-1} \mathbf{n}_{\Phi(T)}^h(j-1). \end{aligned}$$

If $p = k$ and $s > j$, then

$$\sum_{h=1}^k \mathbf{n}_{\Phi(T)}^h(s) = \sum_{h=1}^{k+1} \mathbf{n}_T^h(s+1) \leq \sum_{h=1}^k \mathbf{n}_T^h(s) \leq \sum_{h=1}^{k-1} \mathbf{n}_T^h(s-1) = \sum_{h=1}^{k-1} \mathbf{n}_{\Phi(T)}^h(s-1).$$

If $p > k$ and $s > j$, then we have

$$\sum_{h=1}^p \mathbf{n}_{\Phi(T)}^h(s) = \sum_{h=1}^{p+1} \mathbf{n}_T^h(s+1) \leq \sum_{h=1}^p \mathbf{n}_T^h(s) = \sum_{h=1}^{p-1} \mathbf{n}_{\Phi(T)}^h(s-1).$$

□

Now we show Φ is in fact a bijective map between the set of LR-tableaux of shape ν/λ and content μ and the set of LR-tableaux of shape $(\nu \setminus \nu_{\{k\}})/(\lambda \setminus \lambda_{\{i\}})$ and content $\mu \setminus \mu_{\{j\}}$. Assume T and S are given LR-tableaux of shape ν/λ and content μ such that $\Phi(T) = \Phi(S)$. Lemma 2.1 and Lemma 2.2 imply that $T(p, q) = S(p, q)$ unless $p = k$ and $q \geq \lambda_i + 1$. Then, because both T and S have content μ and rows are weakly increasing, there is a unique way to fill boxes at positions

(k, q) for $q = \lambda_i + 1, \dots, \nu_k$ for T and S . Therefore, we obtain $T = S$, and hence Φ is an injective map.

Moreover, in general, $c_{\lambda, \mu}^\nu \geq c_{\lambda \setminus \lambda_{\{i\}}, \mu \setminus \mu_{\{j\}}}^{\nu \setminus \nu_{\{k\}}}$ for any $1 \leq i, j, k \leq n$ with $\lambda_i + \mu_j = \nu_k$ (see [8], and also [15]). Hence, we have the following theorem.

Theorem 2.9. *The map Φ is a bijective map between the set of LR-tableaux of shape ν/λ and content μ and the set of LR-tableaux of shape $(\nu \setminus \nu_{\{k\}})/(\lambda \setminus \lambda_{\{i\}})$ and content $\mu \setminus \mu_{\{j\}}$.*

3. Application

Theorem 1.1 is rather restrictive to be applied because we have to check the condition that $c_{\lambda, \mu}^\nu > 0$ first. However, using the Horn's inequalities in [12, 13], we can enhance Theorem 1.1, and remove the condition $c_{\lambda, \mu}^\nu > 0$. More detailed technique to obtain the extension from Theorem 1.1 can be found in [5].

Theorem 3.1 (Extension of Theorem 1.1). *Let λ, μ and ν be partitions with $\ell(\lambda), \ell(\mu), \ell(\nu) \leq n$ and $|\nu| = |\lambda| + |\mu|$. Suppose that there are $1 \leq i, j, k \leq n$ satisfying the following condition;*

$$i + j \leq k + 1.$$

- (a) *If $\lambda_i + \mu_j < \nu_k$, then $c_{\lambda, \mu}^\nu = 0$.*
- (b) *If $\lambda_i + \mu_j = \nu_k$, then $c_{\lambda, \mu}^\nu = c_{\lambda \setminus \lambda_{\{i\}}, \mu \setminus \mu_{\{j\}}}^{\nu \setminus \nu_{\{k\}}}$.*

For a partition λ and an integer $i \leq \ell(\lambda)$, we denote the integer sequence $(\lambda_1 - 1, \lambda_2 - 1, \dots, \lambda_i - 1, \lambda_{i+1}, \dots)$ by $\lambda - (1^i)$. Note if $\lambda_i > \lambda_{i+1}$, then $\lambda - (1^i)$ is a partition obtained from λ by removing a column of length i . A conjugated form of Theorem 3.1 can be obtained by applying Lemma 0.2. Again, one may find detailed proof to obtain the following theorem from [5]

Theorem 3.2 (Conjugation of Theorem 3.1). *Let λ, μ and ν be partitions with $\ell(\lambda), \ell(\mu), \ell(\nu) \leq n$ and $|\nu| = |\lambda| + |\mu|$. Suppose that there are λ_i, μ_j, ν_k such that $\lambda_i > \lambda_{i+1}$, $\mu_j > \mu_{j+1}$ and $\nu_k > \nu_{k+1}$ with $\lambda_{i+1} + \mu_{j+1} + 1 \leq \nu_k$.*

- (a) *If $i + j < k$, then $c_{\lambda, \mu}^\nu = 0$.*
- (b) *If $i + j = k$, then $c_{\lambda, \mu}^\nu = c_{\lambda - (1^i), \mu - (1^j)}^{\nu - (1^k)}$.*

Moreover, the formula can be applied by formally setting $\lambda_0 = \mu_0 = \nu_1$ and $\nu_{n+1} = 0$.

Example 3.3. The following gives an example of applying Theorem 3.1 and Theorem 3.2 repeatedly to compute a Littlewood-Richardson coefficient. We use notations $\stackrel{1}{=}$ and $\stackrel{\tilde{1}}{=}$ to indicate that Theorem 3.1 or Theorem 3.2 is used respectively.

$$\begin{aligned}
& c_{(13,13,12,11,11,11,9,9,7,5,4,3,1),(9,8,7,6,6,5,5,5,4,2,2)}^{(15,14,14,13,13,13,13,13,11,11,10,10,9,9)} \\
& \stackrel{1}{=} c_{(13,13,12,11,11,11,9,9,7,5,4,3,1),(8,7,6,6,5,5,5,4,2,2)}^{(15,14,14,13,13,13,13,13,11,11,10,10,9)} \\
& \stackrel{1}{=} c_{(13,13,12,11,11,11,9,9,7,5,4,3),(7,6,6,5,5,5,4,2,2)}^{(15,14,14,13,13,13,13,13,11,11,10,10)} \\
& \stackrel{1}{=} c_{(13,13,12,11,11,11,9,9,7,5,4),(6,6,5,5,5,4,2,2)}^{(15,14,14,13,13,13,13,13,11,11,10)} \\
& \stackrel{1}{=} c_{(13,13,12,11,11,11,9,9,7,5),(6,5,5,5,4,2,2)}^{(15,14,14,13,13,13,13,13,11,11)} \\
& \stackrel{1}{=} c_{(13,13,12,11,11,11,9,9,7),(5,5,5,4,2,2)}^{(15,14,14,13,13,13,13,11)} \stackrel{1}{=} c_{(13,13,12,11,11,9,9,7),(5,5,5,4,2)}^{(15,14,14,13,13,13,13,11)} \\
& \stackrel{1}{=} c_{(13,12,11,11,9,9,7),(5,5,5,4,2)}^{(15,14,14,13,13,13,11)} \\
& \stackrel{1}{=} c_{(12,11,11,9,9,7),(5,5,5,4,2)}^{(15,14,14,13,13,11)} \stackrel{\tilde{1}}{=} c_{(12,11,11,9,9,7),(4,4,4,3,1)}^{(14,13,13,12,12,11)} \\
& \stackrel{1}{=} c_{(12,11,11,9,9),(4,4,3,1)}^{(14,13,13,12,12)} \stackrel{1}{=} c_{(12,11,9,9),(4,4,3)}^{(14,13,13,12)} \stackrel{1}{=} c_{(12,11,9),(4,3)}^{(14,13,12)} \stackrel{1}{=} c_{(11,9),(4,3)}^{(14,13)} \\
& \stackrel{1}{=} c_{(11),(3)}^{(14)} \stackrel{1}{=} c_{\emptyset,\emptyset}^{\emptyset} = 1
\end{aligned}$$

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