SCALARIZATION METHODS FOR VECTOR VARIATIONAL INEQUALITIES

B.-S. LEE[†], M. FIRDOSH KHAN[‡] AND SALAHUDDIN[§]

1. Introduction and preliminaries

In studying vector problems including vector optimization problems, vector variational inequality problems and vector equilibrium problems, there have been usually traditional concentrations on scalarization approaches which enable us to replace the vector problems under consideration with equivalent scalar problems [3, 4, 6, 8-10].

In particular, in [10], Slavov discussed some scalarization techniques and one application of the multi-objective optimization problem into a mathematical economics.

In [6], Jimenez et al. developed a scalarization method in order to obtain scalar versions of their results on the necessary and sufficient conditions for strict minimizers of a general vector optimization problem, through a variational approach.

In [8], Konnov considered a scalarization approach to connect vector variational inequalities into an equivalent scalar variational inequalities with a set-valued cost mapping. He gave an equivalence between weak and strong solutions of set-valued vector variational inequalities and suggested a new gap function for vector variational inequalities. He applied his results to vector optimization, vector network equilibrium and vector migration equilibrium problems.

In [4], Guu et al. extended the scalarization approaches of Giannessi et. al [3] to set-valued vector optimization problems and set-valued weak vector optimization problems. The scalar variational inequalities given by them is different from those given by Konnov [8].

In [9], Liu et al. introduced four kinds of scalar variational inequalities for studying vector variational inequalities for set-valued mappings by using the scalarization approach considered by Konnov [8].

In this paper, we obtain more generalized results than the corresponding results of Liu et al. [9].

An ordered Banach space (Y, P) is a real Banach space ordered by a nonempty closed, convex and pointed cone $P \subseteq Y$, that is, $\lambda P \subset P$ for $\lambda > 0$, $P + P = P$ and $P \cap \{-P\} = \{0\}$ with the apex at the origin in the following form of

$$x \geq y \Leftrightarrow x - y \in P \text{ for } x, y \in Y$$

and

$$x \not\geq y \Leftrightarrow x - y \notin P \text{ for } x, y \in Y.$$

If the interior $\text{int}P$ of P is nonempty, then a weak order in Y is defined by

$$y < x \Leftrightarrow x - y \in \text{int}P \text{ for } x, y \in Y$$

and

$$y \not< x \Leftrightarrow x - y \notin \text{int}P \text{ for } x, y \in Y.$$

Throughout this paper, X is a real Banach space with its topological dual X^* , $(Y, P) \equiv (\mathbb{R}^n, P)$, K a nonempty convex subset of X , N_0 be a nonempty subset of X^* , N a nonempty subset of the vector space $L(X, Y)$ of all the continuous linear mappings from X to Y and $\langle \ell, x \rangle$ the value of $\ell \in L(X, Y)$ at $x \in X$, where $P = \mathbb{R}_+^n = \{\lambda = (\lambda_1, \lambda_2, \dots, \lambda_n) \in \mathbb{R}^n; \lambda_i > 0, i = 1, 2, \dots, n\}$. Let $T_i : K \rightarrow 2^{X^*}$ ($i = 1, 2, \dots, n$) be a set-valued mapping, $T_0 : K \rightarrow 2^{X^*}$ and $T : K \rightarrow 2^{L(X, Y)}$ be set-valued mappings defined by $T_0(x) = \text{conv}\left(\bigcup_{i=1}^n T_i(x)\right)$, the convex hull of $\bigcup_{i=1}^n T_i(x)$ and $T(x) = \prod_{i=1}^n T_i(x)$. Let $M_i : K \times N_0 \rightarrow 2^{X^*}$ ($i = 1, 2, \dots, n$) be a set-valued mapping, $M : K \times N \rightarrow 2^{L(X, Y)}$ a set-valued mapping defined by

$$M(x, s) = \prod_{i=1}^n M_i(x, s_i) \text{ for } s = (s_1, s_2, \dots, s_n) \in N \subset L(X, Y)$$

and $M_0 : K \times N_0 \rightarrow 2^{X^*}$ be a set-valued mapping defined by

$$M_0(x, s_i) = \text{conv} \left(\bigcup_{i=1}^n M_i(x, s_i) \right) \text{ for } x \in K \text{ and } s = (s_1, s_2, \dots, s_n) \in N.$$

Let f_i ($i = 1, 2, \dots, n$) : $K \rightarrow Y$ be a mapping and $f(x) = (f_1(x), \dots, f_n(x))$ for $x \in K$ and $B = \{(\lambda_1, \lambda_2, \dots, \lambda_n) \in \mathbb{R}_+^n : \sum_{i=1}^n \lambda_i = 1\}$, which is a nonempty compact and convex subset of $\mathbb{R}^n$.

If $f(x) \equiv u$ for all $x \in K$ and $M_0(x^*, s^*) = s^*$, then we obtain some corresponding results which have been investigated by many authors [1, 2, 5, 8, 9].

By using a scalarization system, Konnov [8] and Liu et al. [9] converted vector variational inequalities into equivalent scalar variational inequalities. Inspired by works of Konnov [8] and Liu et al. [9], in this paper we introduce four kinds of generalized scalar variational inequalities for studying VVI-S and VVI-W and then we study some relationships between those variational inequalities under suitable conditions.

The following well-known Kneser Minimax Theorem is essential in the proving of our main results.

Proposition 1.1 [7]. Let A be a nonempty convex set in a vector space and B a nonempty compact convex set in a Hausdorff topological vector space. Suppose that g is a real-valued function on $A \times B$ such that for each fixed $a \in A$, $g(a, \cdot)$ is lower semicontinuous and convex on B and for each fixed $b \in B$, $g(\cdot, b)$ is concave on A . Then

$$\min_{b \in B} \sup_{a \in A} g(a, b) = \sup_{a \in A} \min_{b \in B} g(a, b).$$

2. Main results

We consider the following Stampacchia-type variational inequalities:

(VVI-S) Vector variational inequality with the strong solution set S_{VS} ;

Find $x^* \in K$ such that there exists $s^* \in T(x^*)$ satisfying

$$\langle M(x^*, s^*), y - x^* \rangle + f(y) - f(x^*) \not\leq 0, \text{ for } y \in K.$$

(VVI-W) Vector variational inequality with the weak solution set S_{VW} ;

Find $x^* \in K$ such that for each $y \in K$ there exists $s^* \in T(x^*)$ satisfying

$$\langle M(x^*, s^*), y - x^* \rangle + f(y) - f(x^*) \not\leq 0.$$

(SVI-S) Scalar variational inequality with the strong solution set S_{SS} ;
Find $x^* \in K$ such that there exists $e^* \in T_0(x^*)$ satisfying

$$\langle M_0(x^*, e^*), y - x^* \rangle + \langle \lambda, f(y) - f(x^*) \rangle \geq 0, \text{ for } y \in K, \lambda \in B.$$

(SVI-1) Scalar variational inequality with the solution set S_{S1} ;
Find $x^* \in K$ such that for each $y \in K$, there exists $e^* \in T_0(x^*)$ satisfying

$$\langle M_0(x^*, e^*), y - x^* \rangle + \langle \lambda, f(y) - f(x^*) \rangle \geq 0, \text{ for } \lambda \in B.$$

(SVI-2) Scalar variational inequality with the solution set S_{S2} ;
Find $x^* \in K$ such that there exist $e^* \in T_0(x^*)$, $\lambda \in B$ satisfying

$$\langle M_0(x^*, e^*), y - x^* \rangle + \langle \lambda, f(y) - f(x^*) \rangle \geq 0, \text{ for } y \in K.$$

(SVI-W) Scalar variational inequality with the weak solution set S_{SW} ;
Find $x^* \in K$ such that for each $y \in K$, there exist $e^* \in T_0(x^*)$, $\lambda \in B$ satisfying

$$\langle M_0(x^*, e^*), y - x^* \rangle + \langle \lambda, f(y) - f(x^*) \rangle \geq 0.$$

Remark 2.1. Obviously we obtain the following relations

$$S_{VS} \subset S_{VW}, \quad S_{SS} \subset S_{S1} \subset S_{SW} \quad \text{and} \quad S_{SS} \subset S_{S2} \subset S_{SW}.$$

By using the scalarization approaches, we obtain the other relations between the solution sets under suitable conditions.

Theorem 2.1. Let $M_0 : K \times N_0 \rightarrow X^*$ be a mapping such that $x \rightarrow M_0(\cdot, x)$ is continuous. Assume that for each $x \in K$, $T_i(x)$ is nonempty convex and weakly* compact and f_i is convex ($i = 1, 2, \dots, n$). Then $S_{S2} = S_{SW}$ holds.

Proof. Since $T_i(x)$ ($i = 1, 2, \dots, n$) is nonempty convex and weakly* compact and B is nonempty compact and convex, so are $T_0(x)$ and $T_0(x) \times B$. Let $x^* \in S_{SW}$, then for each $y \in K$, there exists $e^* \in T_0(x^*)$

and $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_n) \in B$ such that

$$\begin{aligned} & \langle M_0(x^*, e^*), y - x^* \rangle + \sum_{i=1}^n \lambda_i (f_i(y) - f_i(x^*)) \\ &= \langle M_0(x^*, e^*), y - x^* \rangle + \langle \lambda, f(y) - f(x^*) \rangle \geq 0. \end{aligned}$$

Define a function $g : K \times T_0(x^*) \times B \rightarrow \mathbb{R}$ by

$$\begin{aligned} g(a, b, \alpha) &= \langle M_0(x^*, b), x^* - a \rangle + \langle \alpha, f(x^*) - f(a) \rangle \\ &= \langle M_0(x^*, b), x^* - a \rangle + \sum_{i=1}^n \alpha_i (f_i(x^*) - f_i(a)) \end{aligned}$$

for $a \in K$, $b \in T_0(x^*)$ and $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n) \in B$. Then it follows that

$$\begin{aligned} & \sup_{a \in K} \min_{(b, \alpha) \in T_0(x^*) \times B} g(a, b, \alpha) \\ &= \sup_{a \in K} \min_{(b, \alpha) \in T_0(x^*) \times B} \{ \langle M_0(x^*, b), x^* - a \rangle + \sum_{i=1}^n \alpha_i (f_i(x^*) - f_i(a)) \} \\ &\leq 0. \end{aligned}$$

On the other hand, for each fixed $a \in A$, $g(a, \cdot, \cdot)$ is continuous (with respect to the second argument in the weak* topology of X^* and with respect to the third one in the norm topology of $\mathbb{R}^n$) and convex on B and for each fixed $(b, \alpha) \in T_0(x^*) \times B$, $g(\cdot, b, \alpha)$ is concave on A from the convexity of f_i ($i = 1, 2, \dots, n$). By Proposition 1.1 we have

$$\begin{aligned} & \min_{(b, \alpha) \in T_0(x^*) \times B} \sup_{a \in K} \{ \langle M_0(x^*, b), x^* - a \rangle + \langle \alpha, f(x^*) - f(a) \rangle \} \\ &= \min_{(b, \alpha) \in T_0(x^*) \times B} \sup_{a \in K} g(a, b, \alpha) \\ &= \sup_{a \in K} \min_{(b, \alpha) \in T_0(x^*) \times B} g(a, b, \alpha) \leq 0. \end{aligned}$$

Thus there are $h^* \in T_0(x^*)$ and $\delta \in B$ such that

$$\langle M_0(x^*, h^*), x^* - y \rangle + \langle \delta, f(x^*) - f(y) \rangle \leq 0 \text{ for all } y \in K,$$

which implies that $x^* \in S_{S2}$. $\square$

Corollary 2.1. Let $M_0 : K \times N_0 \rightarrow X^*$ be a mapping such that $x \rightarrow M_0(x, \cdot)$ is continuous. Assume that for each $x \in K$, $T_i(x)$ is nonempty convex and weakly* compact and f_i is convex ($i = 1, 2, \dots, n$). Then

$$S_{SS} \subseteq S_{S1} \subseteq S_{S2} = S_{SW}.$$

Now we characterize the solution sets S_{VS} and S_{VW} via S_{SS} , S_{S1} , S_{S2} and S_{VW} .

Theorem 2.2. Let $M_0 : K \times N_0 \rightarrow X^*$ be a mapping such that $x \rightarrow M_0(x, \cdot)$ is continuous. Assume that for each $x \in K$, $T_i(x)$ is a nonempty convex and weakly* compact and f_i is convex ($i = 1, 2, \dots, n$), then the following assertions hold:

- (i) $S_{VS} \subseteq S_{VW} \subseteq S_{S2} = S_{SW}$.
- (ii) If $T_0(x) = \bigcup_{i=1}^n T_i(x)$ for $x \in K$, then

$$S_{SS} \subseteq S_{VS} \text{ and } S_{S1} \subseteq S_{VW}.$$

Proof. (i) Let $x^* \in S_{VW}$, then for each $y \in K$,

$$\langle M(x^*, s^*), y - x^* \rangle + f(y) - f(x^*) \not\geq 0 \text{ for some } s^* \in T(x^*).$$

Hence for some $j \in \{1, 2, 3, \dots, n\}$, there is $s_j^* \in T_j(x^*)$ such that

$$\langle M_j(x^*, s_j^*), y - x^* \rangle + f_j(y) - f_j(x^*) \geq 0.$$

Set $e^* = s_j^*$ and $\gamma = (\gamma_1, \gamma_2, \dots, \gamma_n)$ with $\gamma_j = 1$ and $\gamma_i = 0$ for $i \neq j$, then $e^* \in T_0(x^*)$, $\gamma \in B$ and consequently

$$\langle M_0(x^*, e^*), y - x^* \rangle + \langle \gamma, f(y) - f(x^*) \rangle \geq 0,$$

which implies that $x^* \in S_{SW}$.

(ii) Assume that for each $x \in K$, $T_0(x) = \bigcup_{i=1}^n T_i(x)$. Let $x^* \in S_{SS}$, then there exists

$$e^* \in T_0(x^*) \text{ such that } \langle M_0(x^*, e^*), y - x^* \rangle + \langle \lambda, f(y) - f(x^*) \rangle \geq 0$$

for $y \in K$, $\lambda \in B$.

Since for each $x \in K$, $T_0(x) = \bigcup_{i=1}^n T_i(x)$, there is $j \in \{1, 2, \dots, n\}$ such that $e^* \in T_j(x^*)$, set $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_n)$ with $\lambda_j = 1$ and $\lambda_i = 0$ for $i \neq j$. From the above inequality, we have

$$\langle M_0(x^*, e^*), y - x^* \rangle + f_j(y) - f_j(x^*) \geq 0, \text{ for } y \in K.$$

Choose arbitrary elements $s_i^* \in T_i(x^*)$ for $i \neq j$, and set $s_j^* = e^*$ and $s^* = (s_1^*, s_2^*, \dots, s_n^*) \in T(x^*)$. It follows that

$$\langle M(x^*, s^*), y - x^* \rangle + f(y) - f(x^*) \not\geq 0, \text{ for } y \in K,$$

which implies that $x^* \in S_{VS}$. Similarly, we can show that $S_{S1} \subseteq S_{VW}$. $\square$

Remark 2.2. We obtain some results in [9] corresponding to our results as corollaries.

Open problem. We can consider the similar result for Minty-type vector variational inequalities.

References

- [1] G.Y. Chen, X.X. Huang and X.Q. Yang, *Vector optimization: Set-valued Variational Analysis*, Springer-Verlag, Berlin, Heidelberg, 2005.
- [2] F. Giannessi, *Vector Variational Inequalities and Vector Theory Variational Equilibrium*, Kluwer Academic Publishers, Dordrecht, Boston, London, 2000.
- [3] F. Giannessi, G. Mastroeni and L. Pellegrini, *On the theory of vector optimization and variational inequalities. Image space analysis and separation*, in : F. Giannessi(Ed.) *Vector variational Inequalities and Vector Equilibria*, Kluwer Academic Publishers, Dordrecht, Boston, London, 2000, pp. 153–215.
- [4] S.-M. Guu, N.-J. Huang and J. Li, *Scalarization approaches for set-valued vector optimization problems and vector variational inequalities*, J. Math. Anal. Appl. **356** (2009), 564–576.
- [5] N. Hadjesavvas and S. Schaible, *Quasimonotonicity and pseudomonotonicity in variational inequalities and equilibrium problem*: In Generalized Convexity Generalized Monotonicity (Edited by J.P. Crouzeix, J.E. Martinez-Legaz and M. Volle) Academic Publishers, Dordrecht, 257–275, 1998.
- [6] B. Jimenez, V. Novo and M. Sama, *Scalarization and optimality conditions for strict minimizers in multiobjective optimization via contingent epiderivatives*, J. Math. Anal. Appl. **352** (2009), 788–798.
- [7] H. Kneser, *Sur un theoreme fundamental dela theorie des Jeux*, C.R. Acad. Sci. Paris **234** (1952), 2418–2420.
- [8] I.V. Konnov, *A scalarization approach for variational inequalities with applications*, J. Global Optim. **32** (2005), 517–527.
- [9] Z.B. Liu, N.J. Huang and B.S. Lee, *Scalarization approaches for generalized vector variational inequalities*, Nonlinear Anal. Forum **12**(1) (2007), 119–124.
- [10] Z.D. Slavov, *Scalarization techniques or relationship between a social welfare function and a Pareto optimality concept*, Appl. Math. & Comp. **172** (2006), 464–471.

[†] Department of Mathematics,
Kyungsung University,
Busan 608-736, Korea
E-mail: bslee@ks.ac.kr

[‡] S.S. School (Boys) Aligarh Muslim University,
Aligarh-202002, India
E-mail: khan.mfk@yahoo.com

[§] Department of Mathematics,
Aligarh Muslim University,
Aligarh-202002, India
E-mail: salahuddin12@mailcity.com