A NOTE ON QUASI-TOPOLOGICAL SPACES

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Abstract. We study some properties of quasi-topological spaces. Also we introduce the concept of Q -continuity and investigate several types of quasi-topology on an SGNS.

1. Introduction and preliminaries

In [2, 3], Császár introduced the notion of quasi-topology which is a generalized topology [1]. In [6], the author introduced the strong generalized neighborhood systems which are generalizations of neighborhood systems. The strong generalized neighborhood system induces a strong generalized neighborhood space (briefly SGNS). We introduced the two concepts of interior and closure on an SGNS and investigate some properties in [6, 7]. In this paper, we introduce and study the notions of Q -continuity in a q -space. In particular, we investigate several types of quasi-topology on an SGNS.

Definition 1.1 ([6]). Let X be a nonempty set with the power set $\exp(X)$ and $\psi : X \rightarrow \exp(\exp(X))$. Then ψ is called a *strong generalized neighborhood system* on X if it satisfies the following:

- (1) $x \in V$ for $V \in \psi(x)$;
- (2) for $U, V \in \psi(x)$, $V \cap U \in \psi(x)$.

Then the pair (X, ψ) is called a *strong generalized neighborhood space* (briefly SGNS) on X . Then $V \in \psi(x)$ is called a *strong generalized neighborhood* of $x \in X$.

$\psi : X \rightarrow \exp(\exp(X))$ is called a *generalized neighborhood system* [1] on X if it satisfies $x \in V$ for $V \in \psi(x)$. It is obvious that every strong generalized neighborhood system is a generalized neighborhood system.

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Definition 1.2 ([6]). Let (X, ψ) be an SGNS on X and $A \subseteq X$. Then the interior and closure of A on ψ (denoted by $\iota(A)$, $\gamma(A)$, respectively) are defined as following:

$$\begin{aligned}\iota(A) &= \{x \in A : \text{there exists } V \in \psi(x) \text{ such that } V \subseteq A\}; \\ \gamma(A) &= \{x \in X : V \cap A \neq \emptyset \text{ for all } V \in \psi(x)\}.\end{aligned}$$

Theorem 1.3 ([6]). Let (X, ψ) be an SGNS on X . We obtain the following.

- (1) $\iota(A) \subseteq A$ and $A \subseteq \gamma(A)$, for all $A \subseteq X$;
- (2) $\iota(A \cap B) = \iota(A) \cap \iota(B)$ and $\gamma(A \cup B) = \gamma(A) \cup \gamma(B)$, for all $A, B \in 2^X$;
- (3) $\gamma(A) = X - \iota(X - A)$, $\iota(A) = X - \gamma(X - A)$.

Definition 1.4 ([7]). Let (X, ψ) be an SGNS on X and $A \subseteq X$. Then the weak- interior and weak-closure of A on X (denoted by $I(A)$, $C(A)$, respectively) are defined as following:

- (1) $I(A) = \{x \in A : A \in \psi(x)\}$.
- (2) $C(A) = \{x \in X : X - A \notin \psi(x)\}$.

Theorem 1.5 ([7]). Let (X, ψ) be an SGNS and $A, B \subseteq X$. Then the following hold.

- (1) $I(A) \subseteq A$ and $A \subseteq C(A)$.
- (2) $I(A) \cap I(B) \subseteq I(A \cap B)$ and $C(A \cup B) \subseteq C(A) \cup C(B)$.
- (3) $I(A) = X - C(X - A)$ and $C(A) = X - I(X - A)$.
- (4) $I(A) \subseteq \iota(A)$ and $\gamma(A) \subseteq C(A)$.

A collection $\mathbf{H}$ of subsets of X is called an *m-family* [5] on X if $\cap \mathbf{H} \neq \emptyset$. Let ψ be an SGNS in X and let $\mathbf{H}$ be an *m-family* on X . Then we say that an *m-family* $\mathbf{H}$ converges to $x \in X$ if $\mathbf{H}$ is finer than $\psi(x)$ i.e., $\psi(x) \subseteq \mathbf{H}$ [6]. Let (X, ψ) be a SGNS on X and $A \subset X$.

2. Quasi-topological spaces

Definition 2.1. Let X be a nonempty set. A subfamily q_X of the power set $\exp(X)$ is called a *quasi-topology* [3] on X if it satisfies the following:

- (1) $\emptyset \in q_X$.
- (2) For $U_1, U_2 \in q_X$, $U_1 \cap U_2 \in q_X$.
- (3) If $U_i \in q_X$ for $i \in J$, then $\cup_{i \in J} U_i \in q_X$.

Then the pair (X, q_X) is called a *q-space* on X . Then $V \in q_X$ is called a *q-open* set and the complement of a *q-open* set is a *q-closed* set.

The collection of all q -open sets (resp. q -closed sets) in a q -space X will be denoted by $QO(X)$ (resp. $QC(X)$). We set $Q(x) = \{U \in QO(X) : x \in U\}$.

Remark 2.2. Every topological space is a q -space. Clearly the converse is not always true.

Definition 2.3. Let (X, q_X) be a q -space. For a subset A of X , the q -closure of A and the q -interior of A are defined as follows:

- (1) $qCl(A) = \cap\{F : A \subseteq F, X - F \in q_X\}$.
- (2) $qInt(A) = \cup\{U : U \subseteq A, U \in q_X\}$.

Theorem 2.4. Let (X, q_X) be a q -space and $A \subseteq X$.

- (1) $x \in qInt(A)$ if and only if there exists an element $U \in Q(x)$ such that $U \subseteq A$.
- (2) $x \in qCl(A)$ if and only if $A \cap V \neq \emptyset$ for all $V \in Q(x)$.

Proof. It is obvious. □

Theorem 2.5. Let (X, q_X) be a q -space and $A \subseteq X$.

- (1) A is q -open if and only if $qInt(A) = A$.
- (2) A is q -closed if and only if $qCl(A) = A$.

Proof. From Definition 2.3, it is obvious. □

Theorem 2.6. Let (X, q_X) be a q -space and $A, B \subseteq X$.

- (1) If $A \subseteq B$, then $qInt(A) \subseteq qInt(B)$ and $qCl(A) \subseteq qCl(B)$.
- (2) $qCl(X - A) = X - qInt(A)$ and $qInt(X - A) = X - qCl(A)$.

Proof. (1) and (2) are obvious. □

Theorem 2.7. Let (X, q_X) be a q -space and $A, B \subseteq X$.

- (1) $qInt(A) \subseteq A$.
- (2) $qInt(A \cap B) = qInt(A) \cap qInt(B)$.
- (3) $qInt(qInt(A)) = qInt(A)$.

Proof. Obvious. □

Theorem 2.8. Let (X, q_X) be a q -space and $A, B \subseteq X$.

- (1) $A \subseteq qCl(A)$.
- (2) $qCl(A \cup B) = qCl(A) \cup qCl(B)$.
- (3) $qCl(qCl(A)) = qCl(A)$.

Proof. From Theorem 2.6(2), it is obvious. □

Remark 2.9. Let (X, q_X) be a q -space. Then in general, $qInt(X) = X$ and $qCl(\emptyset) = \emptyset$ are not true. For the example, consider $X = \{a, b, c\}$ and a quasi-topology $q_X = \{\emptyset, \{a\}, \{b\}, \{a, b\}\}$. Then $qInt(X) = \{a, b\} \neq X$ and $qCl(\emptyset) = \{c\} \neq \emptyset$.

3. Q -continuity and quasi-topologies on an SNGS

Definition 3.1. Let $f : X \rightarrow Y$ be a function between q -spaces (X, q_X) and (Y, q_Y) . Then f is said to be Q -continuous if for every $A \in QO(Y)$, $f^{-1}(A)$ is in $QO(X)$.

Theorem 3.2. Let $f : X \rightarrow Y$ be a function on two q -spaces (X, q_X) and (Y, q_Y) . Then the following statements are equivalent:

- (1) f is Q -continuous.
- (2) For every $A \in QC(Y)$, $f^{-1}(A)$ is in $QC(X)$.
- (3) $f(qCl(A)) \subseteq qCl(f(A))$ for $A \subseteq X$.
- (4) $qCl(f^{-1}(V)) \subseteq f^{-1}(qCl(V))$ for $B \subseteq Y$.
- (5) $f^{-1}(qInt(V)) \subseteq qInt(f^{-1}(V))$ for $B \subseteq Y$.

Proof. (1) $\Leftrightarrow$ (2) Obvious.

(1) $\Rightarrow$ (3) Let $x \in qCl(A)$. If $f(x)$ is not in $qCl(f(A))$, then there is a q -open subset V containing $f(x)$ such that $V \cap f(A) = \emptyset$. By Q -continuity, $f^{-1}(V)$ is q -open and since $f^{-1}(V) \cap A = \emptyset$, $x \notin qCl(A)$; a contradiction.

(3) $\Rightarrow$ (4) Let $A = f^{-1}(B)$ for $B \subseteq Y$; then by (3), $f(qCl(A)) \subseteq qCl(f(A)) = qCl(f(f^{-1}(B))) \subseteq qCl(B)$. Thus we have (4).

(4) $\Rightarrow$ (5) By Theorem 2.6, it is obvious.

(5) $\Rightarrow$ (1) Let $V \in QO(Y)$. By (5), $f^{-1}(V) \subseteq qInt(f^{-1}(V))$. Thus $f^{-1}(V) \in QO(X)$. □

Let (X, q_X) be a q -space and $\mathbf{H}$ an m -family on X . Then we say that an m -family $\mathbf{H}$ q -converges to $x \in X$ if $\mathbf{H}$ is finer than $Q(x)$ i.e., $Q(x) \subseteq \mathbf{H}$. Let $f : X \rightarrow Y$ be a function; then it is obvious $f(\mathbf{H}) = \{f(F) : F \in \mathbf{H}\}$ is an m -family on Y .

Theorem 3.3. Let $f : X \rightarrow Y$ be a bijective function between q -spaces (X, q_X) and (Y, q_Y) . Then f is Q -continuous iff for an m -family $\mathbf{H}$ q -converging to $x \in X$, $f(\mathbf{H})$ q -converges to $f(x)$.

Proof. Suppose f is Q -continuous and $\mathbf{H}$ is an m -family q -converging to $x \in X$. By hypothesis and surjectivity, we get $Q(f(x)) \subseteq f(Q(x)) \subseteq f(\mathbf{H})$, so that $f(\mathbf{H})$ q -converges to $f(x)$.

For the converse, let $U \in Q(f(x))$ for $x \in X$. Since $Q(x)$ q -converges to x , by hypothesis, we get $Q(f(x)) \subseteq f(Q(x))$. From f is injectivity, it follows $f^{-1}(U) \in Q(x)$. □

Definition 3.4. Let $f : X \rightarrow Y$ be a function between q -spaces (X, q_X) and (Y, q_Y) . Then f is said to be q -open if for every $A \in QO(X)$, $f(A) \in QO(Y)$.

Theorem 3.5. Let $f : (X, q_X) \rightarrow (Y, q_Y)$ be a function between two q -spaces. Then the following are equivalent:

- (1) f is q -open.
- (2) $f(qInt(A)) \subseteq qInt(f(A))$ for $A \subseteq X$.
- (3) $qInt(f^{-1}(B)) \subseteq f^{-1}(qInt(B))$ for $B \subseteq Y$.

Proof. (1) $\Rightarrow$ (2) Suppose that f is q -open and $y \in f(qInt(A))$. Then there exists $x \in qInt(A)$ such that $f(x) = y$, and so there is $U \in Q(x)$ such that $U \subseteq A$. Since f is q -open, $y \in f(U) \in Q(f(x))$ and $f(U) \subseteq f(A)$, so that we have $y \in qInt(f(A))$.

(2) $\Rightarrow$ (1) For the converse, let $A \in QO(X)$; then by hypothesis $f(A) = f(qInt(A)) \subseteq qInt(f(A))$. Hence we have $f(A) \in QO(Y)$.

(2) $\Leftrightarrow$ (3) Obvious. $\square$

Lemma 3.6. Let (X, ψ) be a $SGNS$ and $A, B \subseteq X$. If $A \subseteq B$, then $\iota(A) \subseteq \iota(B)$ and $\gamma(A) \subseteq \gamma(B)$.

Proof. Obvious. $\square$

Theorem 3.7. Let (X, ψ) be an $SGNS$ and $A \subseteq X$. Then $q_\iota = \{A \subseteq X : \iota(A) = A\}$ is a quasi-topology on X .

Proof. It is obvious that $\emptyset \in q_\iota$. For $U, V \in q_\iota$, by Theorem 1.3(2), $U \cap V \subseteq q_\iota$. If $U_i \in q_\iota$ for $i \in \Lambda$, then $\cup U_i \subseteq \cup \iota(U_i) \subseteq \iota(\cup U_i)$. Hence $\cup U_i \in q_\iota$. $\square$

Let X be a nonempty set. We recall that a family $\tau \subseteq exp(X)$ is called a supratopology on X [4] if $X \in \tau$ and τ is closed under arbitrary union.

Theorem 3.8. Let (X, ψ) be an $SGNS$. Then $Sq_X(X) = \{A \subseteq X : A \subseteq \gamma(\iota(A))\}$ is a generalized topology on X and it indeed is a supratopology.

Proof. It is obvious that $\emptyset \in Sq_X$. Suppose that there exists a q -closed subset $F \neq X$ such that $\iota(X) \subseteq F \subseteq X$. Then $X - F$ is a nonempty q -open set such that $X - F \cap \iota(X) \neq \emptyset$ and so $\iota(X) \subseteq X - F \cup \iota(X)$. It is a contradiction.

For each $i \in \Lambda$, if $U_i \in Sq_X(X)$, then $U_i \subseteq \gamma(\iota(U_i)) \subseteq \gamma(\iota(\cup U_i))$. Hence $\cup U_i \in Sq_X(X)$. $\square$

Theorem 3.9. Let (X, ψ) be an *SGNS*. Then $\alpha q(X) = \{A \subseteq X : S \cap A \in Sq_X(X) \text{ for all } S \in Sq_X(X)\}$ is a quasi-topology on X and it indeed is a topology.

Proof. It is obvious that $X, \emptyset \in \alpha q(X)$. Let $A, B \in \alpha q(X)$. Then for all $S \in Sq_X(X)$, since $(S \cap A) \in Sq_X(X)$, $S \cap (A \cap B) = ((S \cap A) \cap B) \in Sq_X(X)$. Hence $A \cap B \in \alpha q(X)$.

For each $i \in \Lambda$, if $U_i \in \alpha q(X)$, then for all $S \in Sq_X(X)$ since $(S \cap U_i) \in Sq_X(X)$, by Theorem 3.8, $S \cap \cup U_i = \cup(S \cap U_i) \in \alpha q(X)$. Hence $\cup U_i \in \alpha q(X)$. $\square$

Theorem 3.10. Let (X, ψ) be an *SGNS*. If $S \in Sq_X(X)$ and $U \in q_\iota$, then $S \cap U \in Sq_X(X)$.

Proof. From Theorem 2.4, It is obvious. $\square$

Theorem 3.11 ([7]). Let (X, ψ) be an *SGNS*, $A \subseteq X$ and let $\mathbf{B} = \{A \subseteq X : I(A) = A\}$. Then $q_I = \{\cup \sigma : \sigma \subseteq \mathbf{B}\}$ is a quasi-topology on X .

Theorem 3.12. Let (X, ψ) be an *SGNS*. Then q_ι is finer than q_I .

Proof. From Definition 1.2 and Definition 1.5, it is obvious. $\square$

Remark 3.13. Let (X, ψ) be an *SGNS*. Then the inclusions are satisfied:

$$q_I \subseteq q_\iota \subseteq \alpha q(X) \subseteq Sq_X(X)$$

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