CANONICAL LEFT CELLS AND THE SHORTEST LENGTH ELEMENTS IN THE DOUBLE COSETS OF WEYL GROUPS

NAMHEE KWON

Abstract. Let G be the general linear group $GL(n, \mathbb{C})$, W_0 the Weyl group of G and W the extended affine Weyl group of G . Then, it is well-known that W is a union of the double cosets W_0xW_0 as x moves over the set of dominant weights of W . It is also known that each double coset W_0xW_0 contains a unique element m_x of the shortest length. These shortest length elements belong to what are called the canonical left cells. However, it is still an open problem to find the canonical left cell containing a given m_x . One of the main purposes of this paper is to introduce a new approach to attack this question. In particular, we will present a conjecture which explicitly describes the canonical left cells containing an element m_x . We will show that our conjecture is true for some specific types of m_x .

1. Introduction

Let G be the general linear group $G = GL(n, \mathbb{C})$, and let T be the maximal torus of G consisting of diagonal matrices. Then the Weyl group W_0 of G is isomorphic to the symmetric group S_n of n letters, and the extended affine Weyl group is given $W = X(T) \rtimes S_n$ for the character group $X(T)$.

According to [2], we can define the equivalence relations $\sim_L$, $\sim_R$ and $\sim_{LR}$ on the extended affine Weyl group W . We call the corresponding equivalence classes the *left*, *right* and *two-sided cells*, respectively. Concerning the two-sided cells of W , a couple of interesting facts are known. First, we can associate each two-sided cell of W with a partition of n . Second, if a two-sided cell $\underline{c}$ corresponds to a partition λ , then we can consider two specific left cells Γ_λ and $\Gamma_{\underline{c}}$ in the two-sided cell $\underline{c}$

Received October 28, 2010. Accepted February 8, 2011.

2000 Mathematics Subject Classification. 20G99, 20G05

Key words and phrases. Weyl group, two-sided cell, canonical left cell

(see Section 2 for the details). These left cells play important roles in representation theory. In particular, we call the left cell $\Gamma_{\underline{c}}$ the *canonical left cell* of the two-sided cell $\underline{c}$. Let us now take the right cell $\Gamma_{\underline{c}}^{-1}$ and consider intersection $\Gamma_{\underline{c}} \cap \Gamma_{\underline{c}}^{-1}$. Then, by the definition of the canonical left cell, the elements of $\Gamma_{\underline{c}} \cap \Gamma_{\underline{c}}^{-1}$ can be characterized as the minimal length elements in the various double cosets of W_0 in W . In more details, let us first choose a unique element m_x of minimal length in each double coset $W_0 x W_0$ as x moves over the set of dominant weights in $X(T)$. Then it is known that

$$\coprod_{\underline{c} \in \mathcal{C}} (\Gamma_{\underline{c}} \cap \Gamma_{\underline{c}}^{-1}) = \{m_x \mid x \in \Lambda^+\},$$

where $\mathcal{C}$ denotes the set of two-sided cells in W .

A natural question arises:

(1.1)

Determine an intersection $\Gamma_{\underline{c}} \cap \Gamma_{\underline{c}}^{-1}$ containing the given element m_x .

Question (1.1) is still an open problem: a partial answer for the square-type elements m_x can be found in [6, Section 3]. More explicitly, Xi proved that the square-type elements $m_{x_{\mu_1}^2 x_{\mu_1+\mu_2}^2 \cdots x_{\mu_1+\cdots+\mu_s}^2}$ are contained in the two-sided cell $\underline{c}$ corresponding to the dual partition of $(\mu_1, \dots, \mu_s)$. Nonetheless, we have one serious defect in Xi's approach. Namely, there is no way to determine the two-sided cell $\underline{c}$ containing the general element m_x . (We even can not guess the two-sided cell $\underline{c}$ in Xi's approach.) For this reason, in this paper we propose a new approach which is different from that of [6]. In more details, we first consider two disjoint unions $\coprod_{\underline{c} \in \mathcal{C}} (\Gamma_{\underline{c}} \cap \Gamma_{\underline{c}}^{-1})$ and $\coprod_{\lambda \in P(n)} (\Gamma_{\lambda} \cap \Gamma_{\lambda}^{-1})$, where $P(n)$ is the set of partitions of n . Then, we construct a bijection between $\coprod_{\underline{c} \in \mathcal{C}} (\Gamma_{\underline{c}} \cap \Gamma_{\underline{c}}^{-1})$ and $\coprod_{\lambda \in P(n)} (\Gamma_{\lambda} \cap \Gamma_{\lambda}^{-1})$ by using Xi's bijection appearing in [6]. We conjecture that under our bijection the given intersection $\Gamma_{\underline{c}} \cap \Gamma_{\underline{c}}^{-1}$ will be assigned to the intersection $\Gamma_{\lambda} \cap \Gamma_{\lambda}^{-1}$ if λ is the partition corresponding to the two-sided cell $\underline{c}$. We prove that our conjecture is correct for some specific types of m_x , and this reflects Xi's partial answer to Question (1.1). We remark that our conjecture can yield a description of the canonical left cell containing an arbitrary element m_x regardless to the type of a dominant weight x if it turns out to be true.

2. Preliminaries

In the remaining part of this paper, we will fix the general linear group $G = GL(n, \mathbb{C})$ and the maximal torus T consisting of diagonal matrices.

According to [1] and [3], we can view the extended affine Weyl group W of G as the set of permutations of $\mathbb{Z}$. Namely, it is known that the group $\widetilde{W} = \{\sigma \mid \sigma : \mathbb{Z} \rightarrow \mathbb{Z} \text{ is a permutation such that } \sigma(i+n) = \sigma(i)+n \text{ and } \sum_{i=1}^n \sigma(i) \equiv \sum_{i=1}^n i \pmod{n}\}$ is isomorphic to the extended affine Weyl group W of G . Let $s_1, \dots, s_{n-1}$ be the generators of the symmetric group S_n , and $t_1, \dots, t_n$ the generators of the character group $X(T)$. Then each s_i and t_i are understood in the group $\widetilde{W}$ as follows:

$$(2.1) \quad s_i(j) = \begin{cases} j+1, & \text{if } j \equiv i \pmod{n}, \\ j-1, & \text{if } j \equiv i+1 \pmod{n}, \\ j, & \text{otherwise;} \end{cases}$$

$$(2.2) \quad t_i(j) = \begin{cases} j+n, & \text{if } j \equiv i \pmod{n}, \\ j, & \text{otherwise.} \end{cases}$$

Henceforth, we will identify the extended affine Weyl group W with the group $\widetilde{W}$. Under this identification, we recall that the set Λ^+ of dominant weights in W is described

$$\Lambda^+ = \{x_1^{a_1} \cdots x_n^{a_n} \mid a_1, \dots, a_{n-1} \in \mathbb{Z}_{\geq 0} \text{ and } a_n \in \mathbb{Z}\},$$

where $x_i = t_1 \cdots t_i$. Thus we can identify Λ^+ with the set

$$(2.3) \quad \{(a_1 + \cdots + a_n, \dots, a_{n-1} + a_n, a_n)\}$$

of all non-increasing n -tuples of integers.

Let $\underline{c}$ be a given two-sided cell of W . Then, the two-sided cell $\underline{c}$ yields the corresponding partition $\lambda = (\lambda_1, \dots, \lambda_r)$ of n . More explicitly, let w be an element of W . Then we can choose the maximal cardinality d_i among all possible cardinalities of d -chain family sets of w of index i (see [5] for the definition of d -chain family sets). So, we have a non-decreasing n numbers $d_1 \leq d_2 \leq \cdots \leq d_n = n$. It is known that $d_1 \geq d_2 - d_1 \geq \cdots \geq d_n - d_{n-1}$, and we have a partition $(d_1, d_2 - d_1, \dots, d_n - d_{n-1})$ of n which is called the *partition associated with w* , and denoted by $\lambda(w)$. It is also known that $w_1 \sim_{LR} w_2$ if and only if $\lambda(w_1) = \lambda(w_2)$ [3, 5]. Thus we obtain the corresponding partition $\lambda(\underline{c})$ for the two-sided cell $\underline{c}$ of W .

Let us now write w_λ for the longest element of the subgroup of W generated by

$$s_1, \dots, s_{\lambda_1-1}, s_{\lambda_1+1}, \dots, s_{\lambda_1+\lambda_2-1}, s_{\lambda_1+\lambda_2+1}, \dots, s_{\lambda_1+\dots+\lambda_r-1}.$$

Then, it is known that $w_\lambda \in \underline{c}$ if the two-sided cell $\underline{c}$ corresponds to the partition λ [5]. Thus, we can find the unique left cell Γ_λ of $\underline{c}$ containing w_λ . On the other hand, we have another important left cell in $\underline{c}$ called *canonical left cell*. In more details, set $R(w) = \{s_i \mid ws_i \leq w\}$ for $w \in W$. Then, the canonical left cell $\Gamma_{\underline{c}}$ of $\underline{c}$ is defined as the unique left cell satisfying $R(\Gamma_{\underline{c}}) \subset \{s_0\}$, where s_0 is an element in W defined by $s_0(0) = 1, s_0(1) = 0$ and $s_0(i) = i$ if $i \neq 0, 1 \pmod{n}$. We remark that $R(\Gamma_{\underline{c}})$ is well-defined because $R(u_1) = R(u_2)$ for $u_1 \sim_L u_2$ [2].

It is also well-known that $W = \bigcup_{x \in \Lambda^+} W_0 x W_0$ for the Weyl group W_0 of G , and there exists a unique element m_x of minimal length in each double coset $W_0 x W_0$ [4].

Throughout the paper, the following notations will be in force:

- $\mathcal{C}$ denotes the set of two-sided cells in the extended affine Weyl group W .
- $P(n)$ denotes the set of partitions of n .

For the elements m_x , we have the following interesting properties [6]:

$$(2.4) \quad \begin{cases} (1) & \coprod_{\underline{c} \in \mathcal{C}} (\Gamma_{\underline{c}} \cap \Gamma_{\underline{c}}^{-1}) = \{m_x \mid x \in \Lambda^+\}, \\ (2) & m_x \neq m_y \text{ if } x \neq y \text{ and } x, y \in \Lambda^+, \\ (3) & m_{xx_n^a} = x_n^a m_x = \omega^{an} m_x \text{ if } x \in \Lambda^+, \end{cases}$$

where $\omega : \mathbb{Z} \rightarrow \mathbb{Z}$ is the permutation in W defined by $\omega(i) = i + 1$.

3. Main results

We begin by a combinatorial description of $\coprod_{\lambda \in P(n)} (\Gamma_\lambda \cap \Gamma_\lambda^{-1})$. The key ingredient of this combinatorial model is the bijection appeared in [6]. More explicitly, let $\lambda = (\lambda_1, \dots, \lambda_r) \in P(n)$ and let $\lambda^* = (\mu_1^{n_1}, \dots, \mu_p^{n_p})$ be the dual partition of λ . Here each index n_k indicates that the components μ_k are repeated n_k times. Then, in [6] the author construct explicitly the following bijection: $\epsilon_\lambda : \Gamma_\lambda \cap \Gamma_\lambda^{-1} \rightarrow \{(\beta_1, \dots, \beta_m) \mid \beta_i \text{ is a non-increasing } n_i\text{-tuple of integers}\}$.

Let us now consider the set $\Omega = \{(\alpha, (\mu_1, \dots, \mu_l)) \mid \alpha = (k_1^{a_1}, \dots, k_l^{a_l}) \in P(n) \text{ and each } \mu_i \text{ is a non-increasing } a_i\text{-tuple of integers}\}$. Then the set Ω is decomposed into $\coprod_{\lambda \in P(n)} \Omega_\lambda$,

where Ω_λ is $\{(\alpha, (\mu_1, \dots, \mu_l)) \in \Omega \mid \alpha = \lambda\}$. Since each ϵ_λ is bijective, the maps $\Phi_\lambda : \Gamma_{\lambda^*} \cap \Gamma_{\lambda^*}^{-1} \longrightarrow \Omega_\lambda$ defined as $\Phi_\lambda(x) = (\lambda, \epsilon_{\lambda^*}(x))$ yield bijections for each $\lambda \in P(n)$. By combining the bijections Φ_λ , we obtain the following bijection:

$$(3.1) \quad \Phi : \coprod_{\lambda \in P(n)} (\Gamma_\lambda \cap \Gamma_\lambda^{-1}) \longrightarrow \Omega,$$

where $\Phi|_{\Gamma_\lambda \cap \Gamma_\lambda^{-1}} = \Phi_{\lambda^*}$ for each $\lambda \in P(n)$.

Next, let $x = (a_1^{t_1}, \dots, a_l^{t_l})$ be a non-increasing n -tuple of integers. Then, by Equation (2.3) we have $x \in \Lambda^+$ and $t_1 + \dots + t_l = n$. Thus, each dominant weight $x \in \Lambda^+$ yields a partition of n , and we denote this partition by $\rho(x)$.

For $\lambda \in P(n)$, set $\Lambda_\lambda^+ = \{x \in \Lambda^+ \mid \rho(x) = \lambda\}$. Then, there exists a bijection between Λ_λ^+ and Ω_λ because both Λ_λ^+ and Ω_λ are countable sets. Hence, we can construct a bijection $\Theta : \Lambda^+ \longrightarrow \Omega$ which sends Λ_λ^+ to Ω_λ for each $\lambda \in P(n)$. (In our construction below, it is not necessary to know explicitly the bijections $\Lambda_\lambda^+ \longrightarrow \Omega_\lambda$ ($\lambda \in P(n)$).)

It is clear from Equation (2.4) that there exists a natural bijection $j : \coprod_{\underline{c} \in \mathcal{C}} (\Gamma_{\underline{c}} \cap \Gamma_{\underline{c}}^{-1}) \longrightarrow \Lambda^+$ defined by $j(m_x) = x$ ($x \in \Lambda^+$). So, we have the following diagram:

$$\begin{array}{ccc} \coprod_{\underline{c} \in \mathcal{C}} (\Gamma_{\underline{c}} \cap \Gamma_{\underline{c}}^{-1}) & \xrightarrow{j} & \Lambda^+ \\ \gamma \downarrow & & \downarrow \Theta \\ \coprod_{\lambda \in P(n)} (\Gamma_\lambda \cap \Gamma_\lambda^{-1}) & \xleftarrow{\Phi^{-1}} & \Omega, \end{array}$$

where we set $\gamma = \Phi^{-1} \circ \Theta \circ j$.

The following theorem shows that the bijection γ preserves certain elements of $\coprod_{\underline{c} \in \mathcal{C}} (\Gamma_{\underline{c}} \cap \Gamma_{\underline{c}}^{-1})$ under the viewpoint of the bijective correspondence between $\mathcal{C}$ and $P(n)$.

Theorem 3.1. *Let $\lambda = (\lambda_1, \dots, \lambda_r)$ be a partition of n , and let $\lambda^* = (\mu_1, \dots, \mu_s)$ be the dual partition of λ . Suppose that $m_{x_{\mu_1}^{a_1} x_{\mu_1 + \mu_2}^{a_2} \dots x_{\mu_1 + \dots + \mu_s}^{a_s}} \in \Gamma_{\underline{c}} \cap \Gamma_{\underline{c}}^{-1}$ for some two-sided cell $\underline{c}$, where $a_1, \dots, a_{s-1} \in \mathbb{Z}_{\geq 2}$ and $a_s \in \mathbb{Z}$. Then, $\underline{c}$ is the two-sided cell corresponding to the partition λ and $\gamma(m_{x_{\mu_1}^{a_1} x_{\mu_1 + \mu_2}^{a_2} \dots x_{\mu_1 + \dots + \mu_s}^{a_s}}) \in \Gamma_\lambda \cap \Gamma_\lambda^{-1}$.*

Proof. We first notice that $W = \langle \omega \rangle \rtimes W_{aff}$, where W_{aff} is the affine Weyl group of G and ω is the permutation defined in Equation

(2.4). Then it is easy to check that for $u, v \in W_{aff}$

$$(3.2) \quad \omega^i u \sim_R \omega^j v \quad \text{if } u \sim_R v.$$

Thus, we obtain

$$\begin{aligned} m_{x_{\mu_1}^{a_1} x_{\mu_1+\mu_2}^{a_2} \cdots x_{\mu_1+\cdots+\mu_s}^{a_s}} &= \omega^{a_s n} m_{x_{\mu_1}^{a_1} x_{\mu_1+\mu_2}^{a_2} \cdots x_{\mu_1+\cdots+\mu_{s-1}}^{a_{s-1}}} \quad (\text{by Equation (2.4)}) \\ &\sim_R m_{x_{\mu_1}^{a_1} x_{\mu_1+\mu_2}^{a_2} \cdots x_{\mu_1+\cdots+\mu_{s-1}}^{a_{s-1}}} \quad (\text{by Equation (3.2)}) \\ &\sim_R x_{\mu_1} x_{\mu_1+\mu_2} \cdots x_{\mu_1+\cdots+\mu_{s-1}} \quad (\text{by [6, Lemma 3.3.3]}) \\ &\sim_R x_n x_{\mu_1} x_{\mu_1+\mu_2} \cdots x_{\mu_1+\cdots+\mu_{s-1}} \quad (\text{by [6, Lemma 3.2.2]}) \\ &\sim_R x_{\mu_1} x_{\mu_1+\mu_2} \cdots x_{\mu_1+\cdots+\mu_s} \\ &\quad (\text{since } \mu_1 + \cdots + \mu_s = n \text{ and } x_n = \omega^n \text{ is in the center of } W). \end{aligned}$$

Hence, $x_{\mu_1} x_{\mu_1+\mu_2} \cdots x_{\mu_1+\cdots+\mu_s}$ and $m_{x_{\mu_1}^{a_1} x_{\mu_1+\mu_2}^{a_2} \cdots x_{\mu_1+\cdots+\mu_s}^{a_s}}$ are contained in the same right cell. This implies that they are in the same two-sided cell. However, by [6, Lemma 3.2.1] $x_{\mu_1} x_{\mu_1+\mu_2} \cdots x_{\mu_1+\cdots+\mu_s}$ is an element of the two-sided cell corresponding to the partition λ . So the element $m_{x_{\mu_1}^{a_1} x_{\mu_1+\mu_2}^{a_2} \cdots x_{\mu_1+\cdots+\mu_s}^{a_s}}$ is also contained in the two-sided cell corresponding to λ . Hence, by assumption the canonical left cell $\Gamma_{\underline{c}}$ actually belongs to the two-sided cell corresponding to λ . This means that $\underline{c}$ is the two-sided cell corresponding to λ .

On the other hand, we note that

$$\begin{aligned} x_{\mu_1}^{a_1} x_{\mu_1+\mu_2}^{a_2} \cdots x_{\mu_1+\cdots+\mu_s}^{a_s} &= x_1^0 \cdots x_{\mu_1-1}^0 x_{\mu_1}^{a_1} x_{\mu_1+1}^0 \cdots x_{\mu_1+\mu_2-1}^0 x_{\mu_1+\mu_2}^{a_2} x_{\mu_1+\mu_2+1}^0 \\ &\quad \cdots x_{\mu_1+\cdots+\mu_s}^{a_s}. \end{aligned}$$

Then, by Equation (2.3) the element $m_{x_{\mu_1}^{a_1} x_{\mu_1+\mu_2}^{a_2} \cdots x_{\mu_1+\cdots+\mu_s}^{a_s}}$ corresponds to a non-increasing n -tuple

$$(3.3) \quad ((a_1 + \cdots + a_s)^{\mu_1}, \dots, (a_{s-1} + a_s)^{\mu_{s-1}}, a_s^{\mu_s})$$

under the map j , where $(a_r + \cdots + a_s)^{\mu_r}$ denotes that the component $(a_r + \cdots + a_s)$ is repeated μ_r times. From our construction of Θ , the bijection Θ sends the n -tuple (3.3) to an element of Ω whose first coordinate is equal to the partition λ^* . Finally, the elements in Ω with first coordinate λ^* are assigned to the elements in $\Gamma_{\lambda} \cap \Gamma_{\lambda}^{-1}$ under the map Φ^{-1} . The theorem now follows. $\square$

Example 3.2. Let us consider the case of $G = GL(10, \mathbb{C})$ and take a partition $\mu = (4, 2, 2, 1, 1)$ of 10. Then the element $m_{x_4^2 x_6^2 x_8^2 x_9^2 x_{10}^2}$ is contained in the piece $\Gamma_{\underline{c}} \cap \Gamma_{\underline{c}}^{-1}$ if the two-sided cell $\underline{c}$ corresponds to the dual partition $\mu^* = (5, 3, 1, 1)$ of μ .

Let us now state our conjecture concerned with a description of the canonical left cell containing a given element m_x . Based on Theorem (3.1), we conjecture the following.

Conjecture 3.3. *The bijection γ in Theorem (3.1) sends an intersection $\Gamma_{\underline{c}} \cap \Gamma_{\underline{c}}^{-1}$ to the intersection $\Gamma_{\lambda} \cap \Gamma_{\lambda}^{-1}$, where λ is the partition corresponding to $\underline{c}$.*

If we assume Conjecture (3.3), then we can describe the canonical left cell $\Gamma_{\underline{c}}$ which contains an arbitrary element m_x .

For example, for a given $m_{x_1x_5x_{10}}$ that corresponds to a non-increasing 10-tuple of integers $(3, 2, 2, 2, 2, 1, 1, 1, 1, 1)$, Conjecture (3.3) says that the element $m_{x_1x_5x_{10}}$ will be contained in an intersection piece $\Gamma_{\underline{c}} \cap \Gamma_{\underline{c}}^{-1}$ if the two-sided cell $\underline{c}$ corresponds to the partition $(3, 2, 2, 2, 1)$.

References

- [1] H. Eriksson and K. Eriksson, *Affine Weyl groups as infinite permutations*, The electronic J. of Combinatorics **5** (1998), 1-31.
- [2] D. Kazhdan and G. Lusztig, *Representation of Coxeter groups and Hecke algebras*, Invent. Math. **53** (1979), 165-184.
- [3] G. Lusztig, *Some examples of square integrable representations of semisimple p -adic groups*, Trans. Amer. Math. Soc. **277** (1983), 623-653.
- [4] G. Lusztig and N. Xi, *Canonical left cells in affine Weyl group*, Adv. in Math **72** (1988), 284 - 288.
- [5] J. Shi, *Kazhdan-Lusztig cells of certain affine Weyl group*, LNM 1179, Springer-Verlag, Berlin, 1986.
- [6] N. Xi, *The based ring of two-sided cells of affine Weyl groups of type $\widetilde{A_{n-1}}$* , Mem. Amer. Math Soc. **157** (2002), no. 749.

Namhee Kwon
 Department of Mathematics, Daegu University,
 Daegu, 712-714, Korea.
 E-mail: nkwon@daegu.ac.kr