IDENTIFICATION METHOD FOR DIGITAL SPACES

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Abstract. The aim of the paper is to develop an identification method for digital spaces and to study its digital homotopic properties related to a strong k -deformation retract.

1. Introduction

To study a compression of digital spaces, the strong k -deformation retract in [9] has been often used in digital topology. Further, an *adjunction space* plays an important role in algebraic topology [16]. More precisely, consider two topological spaces X and Y . For a subspace $A \subset X$, let $f : A \rightarrow Y$ be a continuous map. Then the identification space (or an adjunction space) of X to Y induced by the map f , denoted by $X \cup_f Y$, is the disjoint union of X and Y by identifying $x \in A$ into $f(x) \in Y$ for all $x \in A$.

Since the digital version of an adjunction space plays an important in digital topology, *e.g.* digital image synthesis, digital image weaving and so forth. Since it has an intrinsic feature (see Remark 3.1 and Example 3.2), the present paper studies an identification method for digital spaces (see Theorem 3.4). More precisely, with some hypothesis we establish the digital version of an adjunction space and investigate its digital homotopic properties related to a strong k -deformation retract.

2. Preliminaries

Let $\mathbf{Z}^n$ represents the set of points in the Euclidean n D space with integer coordinates. Since a *digital picture* is commonly represented as a quadruple $(\mathbf{Z}^n, k, \bar{k}, X)$, where $n \in \mathbf{N}$, the finite space $X \subset \mathbf{Z}^n$ is the set

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of points of which we regard as belonging to the space with k -adjacency depicted, k represents an adjacency relation for X , $\bar{k}$ represents an adjacency relation for $\mathbf{Z}^n - X$ [14, 15], and $\mathbf{N}$ is the set of natural numbers. But in this paper we are not concerned with $\bar{k}$ -adjacency between two points in $\mathbf{Z}^n - X$. We say that the pair (X, k) is a *digital space with k -adjacency* (briefly, a *digital space*). For $\{a, b\} \subset \mathbf{Z}$ with $a \leq b$, $[a, b]_{\mathbf{Z}} = \{a \leq n \leq b | n \in \mathbf{Z}\}$ is called a digital interval with 2-adjacency [2]. In relation to the study of a multidimensional digital space $X \subset \mathbf{Z}^n$ with one of the k -adjacency relation of $\mathbf{Z}^n$, as a generalization of the k -adjacency relations of $\mathbf{Z}^2$ and $\mathbf{Z}^3$, we have often used the following *k -adjacency of relations* of $\mathbf{Z}^n$, $n \in \mathbf{N}$:

For a natural number m with $1 \leq m \leq n$, two distinct points

$$p = (p_1, p_2, \dots, p_n), q = (q_1, q_2, \dots, q_n) \in \mathbf{Z}^n$$

are called $k(m, n)$ - (briefly, k -) adjacent if

- there are at most m indices i such that $|p_i - q_i| = 1$ and
- for all other indices i such that $|p_i - q_i| \neq 1, p_i = q_i$.

By using the above operator, according to the two numbers $m, n \in \mathbf{N}$, we can represent the $k(m, n)$ (or k)-adjacency relations of $\mathbf{Z}^n$ [12], as follows.

$$k := k(m, n) = \sum_{i=n-m}^{n-1} 2^{n-i} C_i^m, \text{ where } C_i^m = \frac{n!}{(n-i)! i!}. \quad (2.1)$$

For an adjacency relation k of $\mathbf{Z}^n$, a simple k -path with $l+1$ elements in $\mathbf{Z}^n$ is assumed to be an injective sequence $(x_i)_{i \in [0, l]_{\mathbf{Z}}} \subset \mathbf{Z}^n$ such that x_i and x_j are k -adjacent if and only if either $j = i + 1$ or $i = j + 1$ [14]. If $x_0 = x$ and $x_l = y$, then we say that the length of the simple k -path, denoted by $l_k(x, y)$, is the number l . A simple closed k -curve with l elements in $\mathbf{Z}^n$, denoted by $SC_k^{n, l}$ [5], is the simple k -path $(x_i)_{i \in [0, l-1]_{\mathbf{Z}}}$, where x_i and x_j are k -adjacent if and only if $j = i + 1(\text{mod } l)$ or $i = j + 1(\text{mod } l)$ [14].

In the study of a digital isomorphism, a digital homotopy and various properties of a digital space, we have often used the following digital k -neighborhood of a point $x \in X$ with radius $\varepsilon \in \mathbf{N}$ [4] (see also [5]): For a digital space (X, k) in $\mathbf{Z}^n$, the digital k -neighborhood of $x_0 \in X$ with radius ε is defined in X to be the following subset of X :

$$N_k(x_0, \varepsilon) = \{x \in X | l_k(x_0, x) \leq \varepsilon\} \cup \{x_0\}, \quad (2.2)$$

where $l_k(x_0, x)$ is the length of a shortest simple k -path from x_0 to x and $\varepsilon \in \mathbf{N}$.

Motivated by both the digital continuity in [15] and the (k_0, k_1) -continuity in [2], by using (2.1), we can present digital continuity which can be substantially used for studying digital spaces in $\mathbf{Z}^n, n \in \mathbf{N}$, as follows.

Proposition 2.1. [11] *Let (X, k_0) and (Y, k_1) be digital spaces in $\mathbf{Z}^{n_0}$ and $\mathbf{Z}^{n_1}$, respectively. A function $f : X \rightarrow Y$ is (k_0, k_1) -continuous if and only if for every $x \in X$ $f(N_{k_0}(x, 1)) \subset N_{k_1}(f(x), 1)$.*

In Proposition 2.1 if $k_0 = k_1$, then we use the terminology a k_0 -continuous map instead of a (k_0, k_1) -continuous map.

Since a digital space (X, k) can be considered to be a digital k -graph, we may use the term a (k_0, k_1) -isomorphism as in [6] rather than a (k_0, k_1) -homeomorphism as in [2], as follows.

Definition 1. [2] (see also [6, 11]) *For two digital spaces (X, k_0) in $\mathbf{Z}^{n_0}$ and (Y, k_1) in $\mathbf{Z}^{n_1}$, a map $h : X \rightarrow Y$ is called a (k_0, k_1) -isomorphism if h is a (k_0, k_1) -continuous bijection and further, $h^{-1} : Y \rightarrow X$ is (k_1, k_0) -continuous. Then we use the notation $X \approx_{(k_0, k_1)} Y$. If $n_0 = n_1$ and $k_0 = k_1$, then we speak out a k_0 -isomorphism and use the notation $X \approx_{k_0} Y$.*

For a digital space (X, k) and $A \subset X$, (X, A) is called a digital space pair with k -adjacency [7]. Furthermore, if A is a singleton set $\{x_0\}$, then (X, x_0) is called a pointed digital space [14]. Based on the pointed digital homotopy in [3], the following notion of k -homotopy relative to a subset $A \subset X$ has been often used in studying a k -homotopic thinning [12] and a strong k -deformation retract of a digital space (X, k) in $\mathbf{Z}^n$ [8] (see also [9]).

Definition 2. [8] (see also [9, 12]) *Let $((X, A), k_0)$ and (Y, k_1) be a digital space pair and a digital space, respectively. Let $f, g : X \rightarrow Y$ be (k_0, k_1) -continuous functions. Suppose there exist $m \in \mathbf{N}$ and a function $F : X \times [0, m]_{\mathbf{Z}} \rightarrow Y$ such that*

- for all $x \in X$, $F(x, 0) = f(x)$ and $F(x, m) = g(x)$;
- for all $x \in X$, the induced function $F_x : [0, m]_{\mathbf{Z}} \rightarrow Y$ given by $F_x(t) = F(x, t)$ for all $t \in [0, m]_{\mathbf{Z}}$ is $(2, k_1)$ -continuous;
- for all $t \in [0, m]_{\mathbf{Z}}$, the induced function $F_t : X \rightarrow Y$ given by $F_t(x) = F(x, t)$ for all $x \in X$ is (k_0, k_1) -continuous.

Then we say that F is a (k_0, k_1) -homotopy between f and g [3].

- Furthermore, for all $t \in [0, m]_{\mathbf{Z}}$, then suppose the induced map F_t on A is a constant which is the prescribed function from A to Y . In other words, $F_t(x) = f(x) = g(x)$ for all $x \in A$ and for all $t \in [0, m]_{\mathbf{Z}}$.

Then we call F a (k_0, k_1) -homotopy relative to A between f and g , and we say that f and g are (k_0, k_1) -homotopic relative to A in Y , $f \simeq_{(k_0, k_1)rel A} g$ in symbols.

In Definition 2, if $A = \{x_0\} \subset X$, then we say that F is a pointed (k_0, k_1) -homotopy at $\{x_0\}$ [3]. When f and g are pointed (k_0, k_1) -homotopic in Y , we denote by $f \simeq_{(k_0, k_1)} g$. In addition, if $k_0 = k_1$ and $n_0 = n_1$, then we say that f and g are pointed k_0 -homotopic in Y and use the notation $f \simeq_{k_0} g$ and $f \in [g]$ which means the k_0 -homotopy class of g . If, for some $x_0 \in X$, 1_X is k -homotopic to the constant map with the space x_0 relative to $\{x_0\}$, then we say that (X, x_0) is *pointed k -contractible* [3]. Indeed, the notion of k -contractibility is slightly different from the contractibility in Euclidean topology [3] (see also [5, 10]).

We say that a k -continuous map $r : (X', k) \rightarrow (X, k)$ is a k -retraction [2] if $X \subset X'$ and $r(x) = x$ for all $x \in X$. Then, we say that X is a k -retract of X' and further, a point $x \in X' - X$ is said to be k -retractable [9]. In terms of the digital k -retract, the notion of *strong k -deformation retract* was developed in [8] (see also [9]), as follows.

Definition 3. [8] (see also [9, 12]) For a digital space pair $((X, A), k)$, A is said to be a *strong k -deformation retract* of X if there is a k -retraction r of X onto A such that $F : i \circ r \simeq_{k-rel.A} 1_X$. Then, a point $x \in X - A$ is called *strong k -deformation retractable*. For a space (X, k) , we can delete all strong k -deformation retractable points from X . Then, this processing is called a *k -homotopic thinning*.

In Definition 3 a strong k -deformation retractable point is called a *k -homotopic thinning point* [8] (see also [9, 12]). For a space (X, k) , a *simple k -point* is one whose removal does not change the digital topological property of (X, k) up to k -connectivity, k -cavity, and k -hole [1]. Deleting all simple k -point of (X, k) , we obtain a digital space $(X', k) \subset (X, k)$ called a *k -thinning set* of (X, k) [1]. Indeed, there is a difference between a k -thinning [1] and a k -homotopic thinning [9] (see also [12]).

By using the *trivial extension* in [3] and the *Khalimsky operation* in [13], the paper [3] establishes the k -fundamental group: For a digital space (X, k) , consider a k -loop f with a base point x_0 , we denote $[f]_X$ (briefly, $[f]$) the k -homotopy class of f in X . Then for a k -loop f_1 with the same base point $x_0 \in X$, $f_0 \in [f]$ means that the two k -loops f and f_0 have trivial extensions that can be joined by a k -homotopy keeping the end point fixed [5] (see also [3]). Furthermore, if $f_1, f_2, g_1, g_2 \in F^k(X, x_0)$, $f_1 \in [f_2]$, and $g_1 \in [g_2]$, then $f_1 * g_1 \in [f_2 * g_2]$, i.e., $[f_1 * g_1] = [f_2 * g_2]$ [3, 13]. Then we use the notation $\pi^k(X, x_0) = \{[f] | f \in$

$F^k(X, x_0)\}$ which is a group [3] with the operation $[f] \cdot [g] = [f * g]$ and called the (digital) k -fundamental group of (X, x_0) [3], where the base point is assumed to be a point which cannot be deleted by a strong deformation retract [9]. If X is pointed k -contractible, then $\pi^k(X, x_0)$ is trivial [3].

Let $((X, A), k)$ be a digital space pair with k -adjacency. A map $f : ((X, A), k_0) \rightarrow ((Y, B), k_1)$ is called (k_0, k_1) -continuous if f is (k_0, k_1) -continuous and $f(A) \subset B$ [7]. If $A = \{a\}$, $B = \{b\}$, we write $(X, A) = (X, a)$, $(Y, B) = (Y, b)$, and we say that f is a pointed (k_0, k_1) -continuous map [14]. A (k_0, k_1) -continuous map $f : ((X, x_0), k_0) \rightarrow ((Y, y_0), k_1)$ induces a group homomorphism $f_* : \pi^{k_0}(X, x_0) \rightarrow \pi^{k_1}(Y, y_0)$ given by $f_*([\alpha]) = [f \circ \alpha]$, where $[\alpha] \in \pi^{k_0}(X, x_0)$ [3].

Motivated by both 8-contractibility of $SC_8^{2,4}$ [3] and non-8-contractibility of $SC_8^{2,6}$ [5], the papers [5] (see also [8, 12]) proved that $\pi^k(SC_k^{n,l})$ is an infinite cyclic group, where $SC_k^{n,l}$ is not k -contractible. Precisely, $\pi^k(SC_k^{n,l}) \simeq (l\mathbf{Z}, +)$, where $SC_k^{n,l}$ is not k -contractible and “ $\simeq$ ” means a group isomorphism. In addition, $\pi^k(SC_k^{n,4})$ is trivial if $k = 3^n - 1, n \in \mathbf{N} - \{1\}$.

3. Identification Method for Digital Spaces and its Digital Homotopic Properties

This section proposes the digital version of an adjunction space in algebraic topology and further, studies it with a strong k -deformation retract.

Definition 4. For $(A, k) \subset (X, k)$, let $f : (A, k) \rightarrow (Y, k)$ be a k -continuous map. Then a digital identification space of X to Y induced by the map f , denoted by $X \cup_f Y$ with k -adjacency, is the disjoint union of X and Y by identifying $x \in A$ into $f(x) \in Y$ for all $x \in A$.

In relation to the study of an adjunction space, in algebraic topology we have the following result: For two topological spaces X and Y , consider a subset $A \subset X$. Then for a continuous map $f : A \rightarrow Y$, where $A \subset D \subset X$, assume that D is a strong deformation retract of X . Then $D \cup_f Y$ is also a strong deformation retract of $X \cup_f Y$. However, this kind of approach cannot be acceptable in digital topology, as follows.

Remark 3.1. For two digital spaces (X, k) and (Y, k) , consider a subset $A \subset X$. Then for a k -continuous map $f : A \rightarrow Y$, where $A \subset D \subset X$, assume that D is a strong k -deformation retract of X . Then

$D \cup_f Y$ need not be a strong k -deformation retract of $X \cup_f Y$ (see Example 3.2).

Example 3.2. Consider the digital spaces X , $A, D \subset X$, and Y in Figure 1. To be specific, consider the subsets $D := \{x_i | i \in [0, 5]_{\mathbf{Z}}\}$ and $A := \{x_1, x_2\} \subset X := \{x_i | i \in [0, 8]_{\mathbf{Z}}\}$. Then the digital space D is a strong 8-deformation retract of X . Then consider the 8-continuous map $f : A \rightarrow Y$ such that $f(x_i) = y_i, i \in \{1, 2\}$ (see Figure 1). By using the identification method for digital spaces of Definition 4, we obtain the digital space $(Z, 8)$ in Figure 1. Then, owing to the establishment of a simple closed 8-curve represented by C_3 in the space $(Z, 8)$, the adjunction space $D \cup_f Y$ is not a strong 8-deformation retract of $X \cup_f Y := Z$.

FIGURE 1. Feature of the digital version of an adjunction space

Lemma 3.3. [8] (see also [9, 12]) If (A, x_0) is a strong k -deformation retract of (X, x_0) , then $\pi^k(X, x_0)$ is isomorphic to $\pi^k(A, x_0)$.

In view of Remark 3.1, we strongly need to study the digital version of an adjunction space in algebraic topology and geometric topology, as follows.

Theorem 3.4. For two digital spaces (X, k) and (Y, k) , consider a subset $A \subset X$ and a k -continuous map $f : A \rightarrow Y$, where $A \subset D \subset X$. Assume that D is a strong k -deformation retract of X and the digital adjunction space $X \cup_f Y$ has the following property: $f(X \setminus D)$ is not k -adjacent to Y and $f(X \setminus D) \cap Y = \emptyset$. Then $\pi^k(D \cup_f Y, z_0)$ is isomorphic to $\pi^k(X \cup_f Y, z_0)$, where z_0 is not a strongly k -deformation retractable point in $X \cup_f Y$.

Proof. Before proving the assertion, if we omit the hypothesis that the digital adjunction space $X \cup_f Y$ has the following property: $f(X \setminus D)$ is not k -adjacent to Y and $f(X \setminus D) \cap Y = \emptyset$, then this assertion cannot be successful (see Example 3.2). Suppose that $f(X \setminus D)$ is k -adjacent to Y (see Cases 1-3) or $f(X \setminus D) \cap Y \neq \emptyset$ (see Case 4). Then we can show the assertion is not valid with the following cases.

(Case 1) In case the digital adjunction space $X \cup_f Y$ makes a new simple closed k -curve which is not k -contractible. For instance, consider the case of Figure 1 (see Example 3.2). As discussed in Example 3.2, the assertion cannot be valid.

(Case 2) In case the digital adjunction space $X \cup_f Y$ makes a new simple closed k -curve which is k -contractible. For instance, consider the case of Figure 2. More precisely, consider the digital spaces X , $A, D \subset X$, and Y in Figure 2. To be specific, consider the subsets $D := \{x_i | i \in [0, 5]_{\mathbf{Z}}\}$ and $A := \{x_1, x_2\} \subset X := \{x_i | i \in [0, 8]_{\mathbf{Z}}\}$. Then, due to the 8-contractibility of $SC_8^{2,4} := \{x_3, x_6, x_7, x_8\} \subset X$, the digital space D is a strong 8-deformation retract of X .

Let us now consider the 8-continuous map $f : A \rightarrow Y$ such that $f(x_i) = y_i, i \in \{1, 2\}$ (see Figure 2). By using the identification method of Definition 4, we obtain the digital space $(Z, 8)$ in Figure 2. Owing to the simple closed 8-curve which is equal to the set $\{z_i | i \in [0, 3]_{\mathbf{Z}}\}$ in the space $(Z, 8)$, the adjunction space $D \cup_f Y$ is not a strong 8-deformation retract of $X \cup_f Y := Z$.

(Case 3) Even if the digital adjunction space $X \cup_f Y$ does not make a new simple closed k -curve of Cases 1 and 2 above, in case $f(X \setminus D)$ is k -adjacent to Y , we may have some difficulty in processing the given strong k -deformation retract of X onto D via f in $X \cup_f Y$.

(Case 4) Suppose that $f(X \setminus D) \cap Y \neq \emptyset$. Then the condition may invoke some difficulty in processing the given strong k -deformation retract of X onto D via f in $X \cup_f Y$.

Concretely, if we omit the hypothesis of this theorem, then the assertion cannot be valid.

With the hypothesis of the theorem (see Figure 3), let us prove the assertion. Let $\bar{i} : X \rightarrow X \cup_f Y, \bar{j} : Y \rightarrow X \cup_f Y$ be the induced maps by the inclusions $\bar{i} : X \rightarrow X \cup Y, \bar{j} : Y \rightarrow X \cup Y$, respectively. Then $\bar{i}$ and $\bar{j}$ are k -continuous maps. Let $R_t : X \rightarrow X$ be a strong k -deformation retract of X onto D . Put

$$F_t := \bar{i} \circ R_t : X \rightarrow X \cup_f Y \quad \text{and} \quad G_t := \bar{j} : Y \rightarrow X \cup_f Y.$$

By using the maps F_t and G_t , we can define the k -homotopy

FIGURE 2. Explanation of a digital adjunction space

$$H_t : X \cup Y \rightarrow X \cup_f Y$$

such that

$$H_t(z) = \begin{cases} F_t(z) & \text{if } z \in X, \text{ and} \\ G_t(z) & \text{if } z \in Y. \end{cases}$$

Since $H_t(a) = H_t(f(a))$ for all $a \in A$, the map H_t induces the new k -homotopy

$$H_t^* : X \cup_f Y \rightarrow X \cup_f Y$$

such that

$$H_t^* \circ \bar{i} = F_t = \bar{i} \circ R_t \text{ and } H_t^* \circ \bar{j} = G_t = \bar{j}.$$

Thus H_t is a k -homotopy for a strong k -deformation of $X \cup_f Y$ onto $D \cup_f Y$.

In addition, suppose z_0 is a strongly k -deformation retractable point in $X \cup_f Y$. Then, since the process of the strong k -deformation retract of the theorem cannot be valid (see Definition 2).

By Lemma 3.3, the assertion is proved. $\square$

Example 3.5. Consider the digital spaces X , $A, D \subset X$, and Y in Figure 3. To be specific, consider the subsets $D := \{x_i | i \in [0, 5]_{\mathbf{Z}}\}$ and $A := \{x_1, x_2\} \subset X := \{x_i | i \in [0, 8]_{\mathbf{Z}}\}$. Then the digital space D is a strong 8-deformation retract of X . Then consider the 8-continuous map $f : A \rightarrow Y$ such that $f(x_i) = y_i, i \in \{1, 2\}$ (see Figure 3). By using the identification method for digital spaces of Definition 4, we obtain the digital space $(Z, 8)$ in Figure 3 which satisfies the hypothesis of Theorem

3.4. Then the adjunction space $D \cup_f Y$ is a strong 8-deformation retract of $X \cup_f Y := Z$.

FIGURE 3. Configuration of Theorem 3.4.

4. Summary

We have studied a method of establishing an adjunction space in the discrete case which can be used in computer science. Indeed, the results in this paper can be used in computer science in relation to image processing, discrete geometry, image synthesis and so forth.

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