STABILITY FOR JORDAN LEFT DERIVATIONS MAPPING INTO THE RADICAL OF BANACH ALGEBRAS

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Abstract. In this article, we take account of stability for ring Jordan left derivations and ring left derivations and we also deal with problems for the radical ranges of linear Jordan left derivations and linear left derivations.

1. Introduction and preliminaries

Throughout this article, $\mathcal{A}$ denotes an algebra over the real or complex field $\mathbb{F}$. An additive mapping $d : \mathcal{A} \rightarrow \mathcal{A}$ is said to be a *ring left derivation* (resp., *ring derivation*) if $d(xy) = xd(y) + yd(x)$ (resp., $d(xy) = xd(y) + d(x)y$) holds for all $x, y \in \mathcal{A}$. Furthermore, if $d(\lambda x) = \lambda d(x)$ is valid for all $\lambda \in \mathbb{F}$ and all $x \in \mathcal{A}$, then d is a *linear left derivation* (resp., *linear derivation*). An additive mapping $d : \mathcal{A} \rightarrow \mathcal{A}$ is called a *ring Jordan left derivation* (resp., *ring Jordan derivation*) if $d(x^2) = 2xd(x)$ (resp., $d(x^2) = xd(x) + d(x)x$) is fulfilled for all $x \in \mathcal{A}$. In addition, if $d(\lambda x) = \lambda d(x)$ holds for all $\lambda \in \mathbb{F}$ and all $x \in \mathcal{A}$, then d is a *linear Jordan left derivation* (resp., *linear Jordan derivation*).

Let us introduce the historical background of our investigation. The stability problem of functional equations has originally been formulated by Ulam [21]: *Under what condition does there exists a homomorphism near an approximate homomorphism?* Hyers [9] answered the problem of Ulam under the assumption that the groups are Banach spaces. A generalized version of the theorem of Hyers for approximately additive mappings was given by Aoki [1] and for approximately linear mappings

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was presented by Rassias [17] by considering an unbounded Cauchy difference. The paper work of Rassias [17] has had a lot of influence in the development of what call the *generalized Hyers-Ulam stability* of functional equations.

Since then, a great deal of work has been done by a number of authors and the problems concerned with the generalizations and the applications of the stability to a number of functional equations have been developed as well (for instance, [10, 15, 16]). In particular, the stability result concerning derivations between operator algebras was first obtained by Šemrl [18]. Badora [2] gave a generalization of the Bourgin's result [5] and he also dealt with the Hyers-Ulam stability and the Bourgin-type superstability of derivations in [3].

Singer and Wermer [19] obtained a fundamental result which started investigation into the ranges of linear derivations on Banach algebras. The result, which is called the Singer-Wermer theorem, states that any continuous linear derivation on a commutative Banach algebra maps into the radical. They also made a very insightful conjecture, namely that the assumption of continuity is unnecessary. This was known as the Singer-Wermer conjecture and was proved by Thomas [20]. The Singer-Wermer conjecture implies that any linear derivation on a commutative semisimple Banach algebra is identically zero which is the result of Johnson [12]. On the other hand, Hatori and Wada [8] showed that a zero operator is the only derivation on a commutative semisimple Banach algebra with the maximal ideal space without isolated points. Note that this differs from the above result of Johnson. Based on these facts and a private communication with Watanabe [13], Miura *et al.* proved the generalized Hyers-Ulam stability and Bourgin-type superstability of derivations on Banach algebras in [13].

The main purpose of this article is offer the generalized Hyers-Ulam stability of ring Jordan left derivations and ring left derivations. In addition, based on this stability, we investigate the problems for the radical ranges of linear Jordan left derivations and linear left derivations.

2. Main results

Throughout this article, let a be a fixed rational number with $a > 1$. If a is not integer, there exist unique nonnegative integers b, p and q such that $a = b + \frac{q}{p}$, $0 < \frac{q}{p} < 1$ and $(p, q) = 1$. If a is an integer, we let $a = b$. We now establish the stability of ring Jordan left derivations.

Theorem 2.1. *Let $\mathcal{A}$ be a Banach algebra. If $\varphi, \phi : \mathcal{A} \times \mathcal{A} \rightarrow [0, \infty)$ are mappings such that*

$$(2.1) \quad \tilde{\varphi}(x, y) = \sum_{k=0}^{\infty} \frac{\varphi(a^k x, a^k y)}{a^k} < \infty, \quad \lim_{n \rightarrow \infty} \frac{\phi(a^n x, y)}{a^n} = 0$$

for all $x, y \in \mathcal{A}$, and we also assume that $\sum_{k=2}^n (\cdot) = 0$ if $n < 2$. Suppose that a mapping $f : \mathcal{A} \rightarrow \mathcal{A}$ satisfies the following functional inequalities

$$(2.2) \quad \|f(x + y) - f(x) - f(y)\| \leq \varphi(x, y)$$

for all $x, y \in \mathcal{A}$, and

$$(2.3) \quad \|f(x \circ y) - 2xf(y) - 2yf(x)\| \leq \phi(x, y)$$

for all $x, y \in \mathcal{A}$, where $x \circ y = xy + yx$ is the product x and y . Then there exists a unique ring Jordan left derivation $d : \mathcal{A} \rightarrow \mathcal{A}$ such that

$$(2.4) \quad \|f(x) - d(x)\| \leq \frac{1}{a} \left[\tilde{\varphi}\left(\frac{q}{p}x, bx\right) + \frac{q}{p} \sum_{i=2}^p \tilde{\varphi}\left(\frac{1}{p}x, \frac{i-1}{p}x\right) \right. \\ \left. + \sum_{i=2}^q \tilde{\varphi}\left(\frac{1}{p}x, \frac{i-1}{p}x\right) + \sum_{i=2}^b \tilde{\varphi}(x, (i-1)x) \right]$$

for all $x \in \mathcal{A}$. Moreover, the relation

$$(2.5) \quad x\{f(y) - d(y)\} = 0$$

holds for all $x \in \mathcal{A}$.

Proof. It follows from Theorem 2.1 [14] that there exists a unique additive mapping $d : \mathcal{A} \rightarrow \mathcal{A}$ satisfying (2.4), where

$$(2.6) \quad d(x) = \lim_{n \rightarrow \infty} \frac{f(a^n x)}{a^n}$$

for all $x \in \mathcal{A}$.

Now, we are going to prove that d is a ring left derivation: The condition (2.3) implies that a function $\Delta : \mathcal{A} \times \mathcal{A} \rightarrow \mathcal{A}$ defined by

$$(2.7) \quad \Delta(x, y) = f(x \circ y) - 2xf(y) - 2yf(x)$$

satisfies the following property

$$(2.8) \quad \lim_{n \rightarrow \infty} \frac{\Delta(a^n x, y)}{a^n} = 0.$$

Using (2.6), (2.7) and (2.8), we have that

$$\begin{aligned}
 (2.9) \quad d(x \circ y) &= \lim_{n \rightarrow \infty} \frac{f(a^n x \circ y)}{a^n} \\
 &= \lim_{n \rightarrow \infty} \left\{ 2xf(y) + 2y \frac{f(a^n y)}{a^n} + \frac{\Delta(a^n x, y)}{a^n} \right\} \\
 &= 2xf(y) + 2yd(x)
 \end{aligned}$$

for all $x, y \in \mathcal{A}$. Applying (2.9), we find that

$$\begin{aligned}
 2a^n xf(y) + 2a^n yd(x) &= d(a^n x \circ y) \\
 &= d(x \circ a^n y) = 2xf(a^n y) + 2a^n yd(x).
 \end{aligned}$$

Hence we may rewrite this as

$$(2.10) \quad xf(y) = x \frac{f(a^n y)}{a^n}.$$

Taking the limit as $n \rightarrow \infty$ in (2.10), we obtain

$$(2.11) \quad xf(y) = xd(y).$$

Combining (2.11) with (2.9), we get

$$d(x \circ y) = 2xd(y) + 2yd(x)$$

for all $x, y \in \mathcal{A}$. So we see that $d(x^2) = 2xd(x)$ (cf. [6]). Moreover, the identity (2.11) leads to (2.5). This completes the proof. $\square$

In view of the Thomas' result [20], derivations on Banach algebras now belong to the noncommutative setting. Among various noncommutative version of the Singer-Wermer theorem, Han and Wei [7] showed the following: *Any Jordan left derivation on a semiprime algebra of characteristic not two is a derivation which maps into its center.* Also, Brešar and Vukman [6] proved the following: *Any continuous linear left derivation on a Banach algebra maps into its radical.*

Theorem 2.2. *Let $\mathcal{A}$ be a semiprime Banach algebra with unit. Suppose that $f : \mathcal{A} \rightarrow \mathcal{A}$ is a mapping for which there exist mappings $\varphi, \phi : \mathcal{A} \times \mathcal{A} \rightarrow [0, \infty)$ with (2.1) and satisfying*

$$(2.12) \quad \|f(\alpha x + \beta y) - \alpha f(x) - \beta f(y)\| \leq \varphi(x, y)$$

for all $x, y \in \mathcal{A}$ and all $\alpha, \beta \in \mathbb{U} = \{z \in \mathbb{C} : |z| = 1\}$, and the functional inequality (2.3). Then f is a linear derivation which maps $\mathcal{A}$ into the intersection of its center $Z(\mathcal{A})$ and its radical $\text{rad}(\mathcal{A})$.

Proof. We consider $\alpha = \beta = 1 \in \mathbb{U}$ in (2.12) and then f satisfies (2.2). It follows by Theorem 2.1 that there exists a unique ring Jordan left derivation d satisfying (2.4) and (2.5). Since $\mathcal{A}$ has the unit, f is a ring Jordan left derivation. Consequently, we see that

$$f(x) := \lim_{n \rightarrow \infty} \frac{f(a^n x)}{a^n}$$

for all $x \in \mathcal{A}$. Due to (2.12), we get

$$\lim_{n \rightarrow \infty} \frac{1}{a^n} \|f(a^n(\alpha x + \beta y)) - \alpha f(a^n x) - \beta f(a^n y)\| \leq \lim_{n \rightarrow \infty} \frac{\varphi(a^n x, a^n y)}{a^n} = 0,$$

which means that

$$f(\alpha x + \beta y) = \alpha f(x) + \beta f(y)$$

for all $x, y \in \mathcal{A}$ and all $\alpha, \beta \in \mathbb{U}$. Let us now assume that λ is a nonzero complex number and that L is a positive integer greater than $|\lambda|$. Then, by applying a geometric argument, there exist $\lambda_1, \lambda_2 \in \mathbb{U}$ such that $2\frac{\lambda}{L} = \lambda_1 + \lambda_2$. In particular, we see that $f(\frac{1}{2}x) = \frac{1}{2}f(x)$ for all $x \in \mathcal{A}$. Thus we have that

$$f(\lambda x) = f\left(\frac{L}{2} \cdot 2 \cdot \frac{\lambda}{L} x\right) = \frac{L}{2} f\left(2 \cdot \frac{\lambda}{L} x\right) = \frac{L}{2} (\lambda_1 + \lambda_2) f(x) = \lambda f(x)$$

for all $x \in \mathcal{A}$. Also, it is obvious that $f(0x) = 0 = 0f(x)$ for all $x \in \mathcal{A}$. So f is linear. According to the Han and Wei's result, f is a linear derivation which maps $\mathcal{A}$ into its center $Z(\mathcal{A})$. Since $Z(\mathcal{A})$ is a commutative Banach algebra, the Singer-Wermer conjecture tell us that $f|_{Z(\mathcal{A})}$ maps $Z(\mathcal{A})$ into $\text{rad}(Z(\mathcal{A})) = Z(\mathcal{A}) \cap \text{rad}(\mathcal{A})$ and thus $f^2(\mathcal{A}) \subseteq \text{rad}(\mathcal{A})$. Using the semiprimeness of $\text{rad}(\mathcal{A})$ as well as the identity

$$2f(x)yf(x) = f^2(xyx) - xf^2(yx) - f^2(xy)x + xf^2(y)x$$

for all $x, y \in \mathcal{A}$, we have $f(\mathcal{A}) \subseteq \text{rad}(\mathcal{A})$. Therefore, $f(\mathcal{A}) \subseteq Z(\mathcal{A}) \cap \text{rad}(\mathcal{A})$. The proof of the theorem is complete. $\square$

Semisimple Banach algebras are semiprime [4], and then we get the following.

Corollary 2.3. *Let $\mathcal{A}$ be a semisimple Banach algebra with unit. Suppose that $f : \mathcal{A} \rightarrow \mathcal{A}$ is a mapping for which there exist mappings $\varphi, \phi : \mathcal{A} \times \mathcal{A} \rightarrow [0, \infty)$ with (2.1) satisfying the functional inequalities (2.12) and (2.3). Then f is identically zero.*

Employing the similar way as in the proof of Theorem 2.1, we get the following result for ring left derivations.

Theorem 2.4. *Let $\mathcal{A}$ be a Banach algebra. Suppose that $f : \mathcal{A} \rightarrow \mathcal{A}$ is a mapping for which there exist mappings $\varphi, \phi : \mathcal{A} \times \mathcal{A} \rightarrow [0, \infty)$ satisfying (2.1), (2.2) and*

$$(2.13) \quad \|f(xy) - xf(y) - yf(x)\| \leq \phi(x, y)$$

for all $x, y \in \mathcal{A}$. Then there exists a unique ring left derivation $d : \mathcal{A} \rightarrow \mathcal{A}$ satisfying (2.4) and (2.5).

Corollary 2.5. *Let $\mathcal{A}$ be a Banach algebra with unit and let φ, ϕ, f be as in Theorem 2.4. Suppose that f is bounded for some open subset O of $\mathcal{A}$ and $\tilde{\varphi}$ is bounded on $\mathcal{A} \times \mathcal{A}$. Then f is a linear left derivation which maps $\mathcal{A}$ into the radical $\text{rad}(\mathcal{A})$.*

Proof. Using Theorem 2.4 and [14, Theorem 2.3], we find that there exists a unique continuous linear left derivation d satisfying (2.4) and (2.5). Since $\mathcal{A}$ contains the unit, f is a continuous linear left derivation. Therefore Brešar and Vukman's result [6] guarantees that f maps into the radical. This completes the proof. $\square$

Corollary 2.6. *Let $\mathcal{A}$ be a Banach algebra with unit and let φ, ϕ, f be as in Theorem 2.4. Suppose that f is continuous mapping and $\tilde{\varphi}$ is bounded on $\mathcal{A} \times \mathcal{A}$. Then f is a linear left derivation which maps $\mathcal{A}$ into the radical $\text{rad}(\mathcal{A})$.*

Proof. By continuity of f , we see that $f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed x . Since $\mathcal{A}$ has the unit, f is a linear left derivation on account of [14, Remarks]. Hence f maps into the radical by Brešar and Vukman's result, which completes the proof. $\square$

Let $\mathbb{R}^+$ be the set of the positive real numbers. Isac and Rassias [11] generalized the Hyers theorem by introducing $\psi : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ subject to the some conditions. With the help of [14, Theorem 2.6], we obtain the Isac and Rassias-type stability for ring Jordan left derivations which is generalization of Theorem 2.1.

Theorem 2.7. *If functions $\psi, \phi : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ satisfy*

- (1) $\psi(ts) = \psi(t)\psi(s)$ for all $t, s \in \mathbb{R}^+$
- (2) $\lim_{t \rightarrow \infty} \frac{\psi(t)}{t} = 0, \lim_{t \rightarrow \infty} \frac{\phi(t)}{t} = 0,$

and if $f : \mathcal{A} \rightarrow \mathcal{A}$ is a mapping with $\mathcal{A}$ a Banach algebra. Assume that for each fixed $x, y \in \mathcal{A}$, there exists a real number θ_{xy} such that

$$\|f(tx + ty) - f(tx) - f(ty)\| \leq \theta_{xy}[\psi(\|tx\|) + \psi(\|ty\|)]$$

for all $t \in \mathbb{R}^+$, and

$$\|f(xy) - xf(y) - f(x)y\| \leq \phi(\|x\|\|y\|)$$

for all $x, y \in \mathcal{A}$. Then there exists a unique ring Jordan left derivation $d : \mathcal{A} \rightarrow \mathcal{A}$ and a rational number $a > 1$ such that

$$\begin{aligned} \|f(tx) - d(tx)\| &\leq a^{-1} \left(1 - \frac{\psi(a)}{a}\right) \left[\theta_{(q/p)x, bx} \left(\psi \left(\left\| \frac{q}{p} tx \right\| \right) + \psi(\|btx\|) \right) \right. \\ &+ \frac{q}{p} \sum_{i=2}^p \theta_{(1/p)x, (i-1)x/p} \left(\psi \left(\left\| \frac{1}{p} tx \right\| \right) + \psi \left(\left\| \frac{i-1}{p} tx \right\| \right) \right) \\ &+ \sum_{i=2}^q \theta_{(1/p)x, (i-1)x/p} \left(\psi \left(\left\| \frac{1}{p} tx \right\| \right) + \psi \left(\left\| \frac{i-1}{p} tx \right\| \right) \right) \\ &\left. + \sum_{i=2}^b \theta_{x, (i-1)x} \left(\psi \left(\|tx\| \right) + \psi \left(\|(i-1)tx\| \right) \right) \right] \end{aligned}$$

for all $x \in \mathcal{A}$ and all $t \in \mathbb{R}^+$. Moreover, the relation (2.5) is fulfilled.

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