MINIMAXNESS OF LOCAL COHOMOLOGY MODULES DEFINED BY A PAIR OF IDEALS

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Abstract. Let R be a commutative Noetherian ring and I, J be ideals of R . We introduced the notion of (I, J) -cominimax R -modules. For an integer n and an R -module M , let $H_{I,J}^i(M)$ be an (I, J) -cominimax R -module for all $i < n$. The J -minimaxness of some Ext modules of $H_{I,J}^n(M)$ is investigated. Among of the obtaining results, there is a generalization of the main result of [1].

1. INTRODUCTION

Throughout this paper, R is a commutative Noetherian ring with identity and all modules are assumed to be unitary. Also, we assume that I and J are ideals of R . The generalized local cohomology module with respect to a pair of ideals (I, J) of R was introduced in [6]. Consider the subset

$$W(I, J) = \{\mathfrak{p} \in \operatorname{Spec}(R) \mid I^n \subseteq \mathfrak{p} + J, \text{ for an integer } n \geq 1\}$$

of $\operatorname{Spec}(R)$. Recall that for an ideal I of R the Zariski closed set $V(I)$ is the set of all prime ideals of R containing I . It is easy to see that $V(I) \subseteq W(I, J)$ and if $J = 0$, then $W(I, J)$ coincides with the $V(I)$. Note that $W(I, J)$ is stable under specialization, but in general, it is not a closed subset of $\operatorname{Spec}(R)$. For an R -module M , (I, J) -torsion submodule of M is the set of all elements x of M such that $I^n x \subseteq Jx$ for some positive integer n and is denoted by $\Gamma_{I,J}(M)$. For any R -homomorphism $f : M \rightarrow N$ one has $f(\Gamma_{I,J}(M)) \subseteq \Gamma_{I,J}(N)$ and so $\Gamma_{I,J}(f) = f|_{\Gamma_{I,J}(M)}$ is an R -homomorphism. Hence $\Gamma_{I,J}(-)$ is a covariant, R -linear functor from the category of R -modules to itself and in the case when $J = 0$, (I, J) -torsion functor $\Gamma_{I,J}(-)$ is coincides with

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the I -torsion functor $\Gamma_I(-)$. This functor is left exact and for an integer i , the local cohomology functor $H_{I,J}^i(-)$ with respect to (I, J) is defined as the i -th right derived functor of $\Gamma_{I,J}(-)$ (see [6]). For an R -module M , $H_{I,J}^i(M)$ is called the i -th local cohomology module of M with respect to (I, J) . It is obvious that if $J = 0$, then $H_{I,J}^i(-)$ coincides with the ordinary local cohomology functor $H_I^i(-)$. Let $\widetilde{W}(I, J)$ to be the set of all ideals $\mathfrak{a}$ of R such that $I^n \subseteq \mathfrak{a} + J$ for some integer n . For all $\mathfrak{a}, \mathfrak{b} \in \widetilde{W}(I, J)$ we define $\mathfrak{a} \leq \mathfrak{b}$ if $\mathfrak{a} \supseteq \mathfrak{b}$. So, $\widetilde{W}(I, J)$ is a partially ordered set. It is easy to see that $\widetilde{W}(I, J)$ is a system of ideals of R . Considering $\Phi = \widetilde{W}(I, J)$ one has $\Gamma_{I,J}(M) = \Gamma_\Phi(M)$. Hence $H_{I,J}^i(M) = H_\Phi^i(M)$ for all integers i , which studied by Bijan-Zadeh in [2].

For more informations of local cohomology module with respect to a pair of ideals (I, J) , we refer the reader to [6].

Recall that for an R -module M , the *Goldie dimension* of M is defined as the cardinal of the set of indecomposable submodules of $E(M)$ which appear in a decomposition of $E(M)$ into a direct sum of indecomposable submodules. We use $G \dim M$ to denote Goldie dimension of M (see [8]). For a prime ideal $\mathfrak{p}$ of R , let $\mu^0(\mathfrak{p}, M)$ denotes the 0-th Bass number of M with respect to the prime ideal $\mathfrak{p}$. So, it is obvious by definition of the Goldie dimension that

$$G \dim M = \sum_{\mathfrak{p} \in \text{Ass}_R M} \mu^0(\mathfrak{p}, M).$$

Also, for any ideal I of R , the *I -relative Goldie dimension* of M , which is introduced in [3], is defined as

$$G \dim_I M := \sum_{\mathfrak{p} \in V(I)} \mu^0(\mathfrak{p}, M).$$

An R -module M is called *minimax* if there is a finite submodule N of M such that M/N is an Artinian R -module. It is shown in [10] that when R is a Noetherian ring, an R -module M is minimax if and only if for any submodule N of M , $G \dim M/N < \infty$. An R -module M is said to be *I -minimax* if for any submodule N of M , $G \dim_I M/N < \infty$.

An R -module M is *I -cominimax* if $\text{Supp}(M) \subseteq V(I)$ and $\text{Ext}_R^i(R/I, M)$ is a minimax R -module, for every $i \geq 0$ (see [1]). As a generalization of the concept of I -cominimax, we define the concept of (I, J) -cominimax module (see Definition 2.1).

In this paper, among other results we will study J -minimaxness of $\text{Ext}_R^j(R/I, H_{I,J}^i(M))$, for $j = 0, 1, 2$. More precisely, we show the followings hold.

Theorem 1.1. *Let M be an R -module and n be a non-negative integer such that $H_{I,J}^i(M)$ is (I, J) -cominimax, for all $i < n$ and $\text{Ext}_R^n(R/I, M)$ is J -minimax. Then for any submodule N of $H_{I,J}^n(M)$ where $\text{Ext}_R^1(R/I, N)$ is J -minimax and for any finite R -module T with $\text{Supp } T \subseteq V(I)$, the R -module $\text{Hom}_R(T, H_{I,J}^n(M)/N)$ is J -minimax.*

Theorem 1.2. *Let M be an R -module and n be a non-negative integer such that $H_{I,J}^i(M)$ is (I, J) -cominimax, for all $i < n$. If $\text{Ext}_R^{n+1}(R/I, M)$ is a J -minimax R -module, then $\text{Ext}_R^1(R/I, H_{I,J}^n(M))$ is J -minimax.*

Theorem 1.3. *Let M be an R -module and n be a non-negative integer such that $H_{I,J}^i(M)$ is (I, J) -cominimax, for all $i < n$. If $\text{Ext}_R^i(R/I, M)$ is J -minimax, for all $i \geq 0$, then $\text{Hom}_R(R/I, H_{I,J}^{n+1}(M))$ is J -minimax if and only if $\text{Ext}_R^2(R/I, H_{I,J}^n(M))$ is J -minimax.*

2. MAIN RESULTS

We begin this section by introducing a new class of modules. Our motivation is the paper [1] in which the notion I -cominimax was introduced.

Definitions 2.1. *An R -module M is said to be (I, J) -cominimax if $\text{Supp}(M) \subseteq W(I, J)$ and $\text{Ext}_R^i(R/I, M)$ is a J -minimax R -module, for every $i \geq 0$.*

Example 2.2. *Let I, J be ideals of R and M be an R -module. It is clear that any J -minimax module with support in $W(I, J)$ is a (I, J) -cominimax R -module. In particular, any minimax R -module with support in $W(I, J)$ is (I, J) -cominimax. Also, if $J = 0$, then M is (I, J) -cominimax if and only if M is an I -cominimax R -module.*

Remark 2.3. *By [1, Proposition 2.3], the class of J -minimax modules is closed under submodules, quotients and taking extensions, i.e. it is a Serre subcategory of all R -modules. So, for R -modules M and N which are finite and J -minimax respectively, one has $\text{Ext}_R^i(M, N)$ and $\text{Tot}_i^R(M, N)$ are J -minimax for all $i \geq 0$.*

The following lemma plays a key role in what follow:

Lemma 2.4. *Let M be an (I, J) -torsion-free R -module i.e. $\Gamma_{I,J}(M) = 0$ and E be the injective hull of M . If $L = E/M$, then $\text{Ext}_R^{i-1}(R/I, L) \cong \text{Ext}_R^i(R/I, M)$ and $H_{I,J}^{i-1}(L) \cong H_{I,J}^i(M)$, for all $i > 0$.*

Proof. See [7, Remark 2.2]. □

Proposition 2.5. *Let n be an integer. If M is an R -module such that $H_{I,J}^i(M)$ is (I, J) -cominimax for all $i \leq n$ (respectively for all i), then $\text{Ext}_R^i(R/I, M)$ is J -minimax for all $i \leq n$ (respectively for all i).*

Proof. We use induction on n . If $n = 0$, since $H_{I,J}^0(M) = \Gamma_{I,J}(M)$ is (I, J) -cominimax and $\text{Hom}(R/I, M) \cong \text{Hom}_R(R/I, \Gamma_{I,J}(M))$, the result holds.

Now, suppose that $n > 0$ and the case $n - 1$ is settled. Since $\Gamma_{I,J}(M)$ is (I, J) -cominimax, $\text{Ext}_R^j(R/I, \Gamma_{I,J}(M))$ is J -minimax, for every $j \geq 0$. By using the exact sequence

$$0 \rightarrow \Gamma_{I,J}(M) \rightarrow M \rightarrow M/\Gamma_{I,J}(M) \rightarrow 0$$

it is enough to show that $\text{Ext}_R^j(R/I, M/\Gamma_{I,J}(M))$ is J -minimax, for all $j \leq n$. We show that $\text{Ext}_R^j(R/I, M)$ is J -minimax for all (I, J) -torsion-free R -modules M and all $j \leq n$. Let E be the injective hull of M and let $L = E/M$. Now, by Lemma 2.4, $H_{I,J}^i(L)$ is (I, J) -cominimax, for every $i \leq n - 1$. Thus, by induction hypothesis, $\text{Ext}_R^i(R/I, L)$ is a J -minimax R -module, for every $i \leq n - 1$. Therefore, $\text{Ext}_R^i(R/I, M)$ is J -minimax, for all $i \leq n$. $\square$

Proposition 2.6. *Let M be an R -module such that $\text{Ext}_R^i(R/I, M)$ is J -minimax, for all i . If n is a non-negative integer such that $H_{I,J}^i(M)$ is (I, J) -cominimax, for all $i \neq n$, then $H_{I,J}^n(M)$ is (I, J) -cominimax.*

Proof. We use induction on n . Put $\bar{M} = M/\Gamma_{I,J}(M)$. The exact sequence $0 \rightarrow \Gamma_{I,J}(M) \rightarrow M \rightarrow \bar{M} \rightarrow 0$ induces the long exact sequence

$$\begin{aligned} \cdots &\rightarrow \text{Ext}_R^{i-1}(R/I, \bar{M}) \rightarrow \text{Ext}_R^i(R/I, \Gamma_{I,J}(M)) \rightarrow \text{Ext}_R^i(R/I, M) \\ &\rightarrow \text{Ext}_R^i(R/I, \bar{M}) \rightarrow \text{Ext}_R^{i+1}(R/I, \Gamma_{I,J}(M)) \rightarrow \text{Ext}_R^{i+1}(R/I, M) \rightarrow \cdots (*) \end{aligned}$$

for all $i \geq 0$. It also implies that $H_{I,J}^i(M) \cong H_{I,J}^i(\bar{M})$, for all $i \geq 1$. If $n = 0$, $H_{I,J}^i(\bar{M})$ is (I, J) -cominimax, for all i (note that $H_{I,J}^0(\bar{M}) = 0$). Hence, by Proposition 2.5, $\text{Ext}_R^i(R/I, \bar{M})$ is J -minimax, for all i . So, by the exact sequence $(*)$ it follows that $H_{I,J}^0(M)$ is (I, J) -cominimax. Now, assume that $n > 0$ and the assertion has been proved for $n - 1$. Since, $H_{I,J}^0(M)$ is (I, J) -cominimax, by the exact sequence $(*)$ we may assume that $\Gamma_{I,J}(M) = 0$. Let E be the injective hull of M and put $L = E/M$. Now, the assertion holds by Lemma 2.4 and induction hypothesis. $\square$

Corollary 2.7. *Let I be a principal ideal of R and M be a J -minimax R -module. Then $H_{I,J}^i(M)$ is (I, J) -cominimax, for all $i \geq 0$.*

Proof. Put $\bar{R} = R/(J + \text{Ann}_R M)$. Then $I\bar{R}$ is a principal ideal of $\bar{R}$. So, $H_{I,J}^i(M) = 0$, for all $i > 1$ by [6, Theorem 2.7 and Proposition 4.10]. Also, $\Gamma_{I,J}(M)$ is (I, J) -cominimax by Remark 2.3. So, the assertion holds by Proposition 2.6. $\square$

The following theorem is a generalization of the main result of [1].

Theorem 2.8. *Let M be an R -module and n be a non-negative integer such that $H_{I,J}^i(M)$ is (I, J) -cominimax, for all $i < n$ and $\text{Ext}_R^n(R/I, M)$ is J -minimax. Then for any submodule N of $H_{I,J}^n(M)$ such that $\text{Ext}_R^1(R/I, N)$ is J -minimax and for any finite R -module T with $\text{Supp } T \subseteq V(I)$, $\text{Hom}_R(T, H_{I,J}^n(M)/N)$ is a J -minimax R -module.*

Proof. By Gruson's theorem [9, Theorem 4.1] we have to prove the theorem for $T = R/I$. Also, by the long exact sequence

$$\text{Hom}(R/I, H_{I,J}^t(M)) \rightarrow \text{Hom}(R/I, H_{I,J}^t(M)/N) \rightarrow \text{Ext}_R^1(R/I, N) \rightarrow \dots$$

it is enough to show that $\text{Hom}_R(R/I, H_{I,J}^n(M))$ is J -minimax. One can argue by induction on n . Since $\text{Hom}(R/I, M) \cong \text{Hom}_R(R/I, \Gamma_{I,J}(M))$ the assertion is clear for $n = 0$. Now, suppose that $n > 0$ and that the claim has been proved for $n - 1$. Since $\Gamma_{I,J}(M)$ is an (I, J) -cominimax R -module, $\text{Ext}_R^i(R/I, \Gamma_{I,J}(M))$ is J -minimax, for all $i \geq 0$. So, by the long exact sequence

$$\dots \rightarrow \text{Ext}_R^n(R/I, M) \rightarrow \text{Ext}_R^n(R/I, M/\Gamma_{I,J}(M)) \rightarrow \text{Ext}_R^{n+1}(R/I, \Gamma_{I,J}(M))$$

and Remark 2.3, $\text{Ext}_R^n(R/I, M/\Gamma_{I,J}(M))$ is J -minimax. Hence, we may assume that M is (I, J) -torsion-free. Let E be the injective hull of M and let $L = E/M$. Now, the result holds by Lemma 2.4 and induction hypothesis. $\square$

Corollary 2.9. *Let M be a J -minimax R -module and let n be a non-negative integer such that $H_{I,J}^i(M)$ is (I, J) -cominimax, for all $i < n$. If $JH_{I,J}^n(M)$ is a J -minimax R -module, then $H_{I,J}^n(M)/JH_{I,J}^n(M)$ has finite Goldie dimension. In particular, the set of associated primes of $H_{I,J}^n(M)/JH_{I,J}^n(M)$ is finite.*

Proof. By Theorem 2.8, $\text{Hom}_R(R/I, H_{I,J}^n(M)/JH_{I,J}^n(M))$ is a J -minimax R -module. By [6, Corollary 1.9], $H_{I,J}^n(M)/JH_{I,J}^n(M)$ is an I -torsion module. So, we obtain that

$$\text{Ass}_R \text{Hom}_R(R/I, H_{I,J}^n(M)/JH_{I,J}^n(M)) = \text{Ass}_R H_{I,J}^n(M)/JH_{I,J}^n(M).$$

Now, the result follows by [1, Corollary 3.6]. $\square$

Corollary 2.10. *Let M be an R -module and $n \geq 1$ be an integer such that $\text{Ext}_R^{n-1}(R/I, M)$ be a finite R -module and $\text{Supp}H_{I,J}^i(M) \subseteq \text{Max}(R)$, for all $i < n$. If $\text{Ext}_R^1(R/I, JH_{I,J}^i(M))$ is finite, for all $i < n$, then $H_{I,J}^n(M)/JH_{I,J}^n(M)$ is Artinian, for all $i < n$.*

Proof. We argue by induction on n . If $n = 1$, then $(0 :_{\Gamma_{I,J}(M)} I)$ is Artinian, since it is a finite R -module with support in $\text{Max}(R)$. Consider the exact sequence

$$0 \rightarrow J\Gamma_{I,J}(M) \rightarrow \Gamma_{I,J}(M) \rightarrow \Gamma_{I,J}(M)/J\Gamma_{I,J}(M) \rightarrow 0.$$

Then $\text{Ext}_R^1(R/I, J\Gamma_{I,J}(M))$ is Artinian, since it is a finite R -module with support in $\text{Max}(R)$. So, $\text{Hom}(R/I, \Gamma_{I,J}(M)/J\Gamma_{I,J}(M))$ is Artinian. As $\Gamma_{I,J}(M)/J\Gamma_{I,J}(M)$ is I -torsion, it yields from [4, Theorem 1.3] that $\Gamma_{I,J}(M)/J\Gamma_{I,J}(M)$ is Artinian. Assume that $n \geq 2$ and the assertion holds for $n - 1$. Since, $H_{I,J}^i(M) \cong H_{I,J}^i(M/\Gamma_{I,J}(M))$, for all $i \geq 1$ and $\Gamma_{I,J}(M/\Gamma_{I,J}(M)) = 0$ we may assume that $\Gamma_{I,J}(M) = 0$. Now, by a similar proof used in Theorem 2.8, the inductive step is complete. $\square$

An R -module M is called (I, J) -cofinite if $\text{Supp}(M) \subseteq W(I, J)$ and $\text{Ext}_R^i(R/I, M)$ is a finite R -module, for every $i \geq 0$ (see [7]).

Lemma 2.11. *If M is a J -minimax R -module with $\text{Supp}M \subseteq W(I, J)$ and if JM is an I -cofinite R -module, then M is an (I, J) -cofinite R -module if and only if $(0 :_M I)$ is a finite R -module.*

Proof. By [6, Proposition 1.7 and Corollary 1.9] we have M/JM is an I -torsion R -module. On the other hand, M/JM is an J -torsion R -module and so is minimax by assumption. Hence, by [5, Proposition 4.3], M/JM is I -cofinite if and only if $(0 :_{M/JM} I)$ is finite. Now, the result holds by the following exact sequence:

$$0 \rightarrow JM \rightarrow M \rightarrow M/JM \rightarrow 0.$$

$\square$

Proposition 2.12. *Let M be a finite R -module and n be a non-negative integer such that $H_{I,J}^i(M)$ is J -minimax, for all $i < n$. If $JH_{I,J}^i(M)$ is finite, for all $i < n$, then $\text{Hom}_R(R/I, H_{I,J}^n(M))$ is finite.*

Proof. We proceed by induction on n . The case $n = 0$ is obvious since $H_{I,J}^0(M)$ is finite. So, let $n > 0$, and the result has been proved for smaller values of n . By the induction hypothesis, $\text{Hom}_R(R/I, H_{I,J}^i(M))$ is finite R -module for all $i < n$. Therefore, by Lemma 2.11, $H_{I,J}^i(M)$

is (I, J) -cofinite for all $i < n$. Now, the result follows from [7, Theorem 2.3]. $\square$

Theorem 2.13. *Let M be an R -module and n be a non-negative integer such that $H_{I,J}^i(M)$ is (I, J) -cominimax, for all $i < n$. If $\text{Ext}_R^{n+1}(R/I, M)$ is a J -minimax R -module, then $\text{Ext}_R^1(R/I, H_{I,J}^n(M))$ is J -minimax.*

Proof. We use induction on n . If $n = 0$, then $\text{Ext}_R^1(R/I, M)$ is a J -minimax R -module. Put $\bar{M} = M/\Gamma_{I,J}(M)$. So, by the exact sequence

$$0 \rightarrow \Gamma_{I,J}(M) \rightarrow M \rightarrow \bar{M} \rightarrow 0$$

and Remark 2.3 it follows that $\text{Ext}_R^1(R/I, \Gamma_{I,J}(M))$ is J -minimax. Now, suppose that $n > 0$ and the result is true for $n - 1$. Since $\Gamma_{I,J}(M)$ is (I, J) -cominimax, the R -module $\text{Ext}_R^i(R/I, \Gamma_{I,J}(M))$ is J -minimax, for all i and so $\text{Ext}_R^{n+1}(R/I, \bar{M})$ is J -minimax. Therefore, we may assume that $\Gamma_{I,J}(M) = 0$. Hence by Lemma 2.4 and the induction hypothesis, $\text{Ext}_R^1(R/I, H_{I,J}^{n-1}(L))$ is a J -minimax R -module and so $\text{Ext}_R^1(R/I, H_{I,J}^n(M))$ is J -minimax. $\square$

Corollary 2.14. *Let M be an R -module with injective dimension $n \geq 0$. If $H_{I,J}^i(M)$ is (I, J) -cominimax, for all $i < n$, then $\text{Ext}_R^1(R/I, H_{I,J}^n(M))$ is J -minimax.*

Theorem 2.15. *Let M be an R -module and n be a non-negative integer such that $H_{I,J}^i(M)$ is (I, J) -cominimax, for all $i < n$. If $\text{Ext}_R^i(R/I, M)$ is J -minimax, for all $i \geq 0$, then $\text{Hom}_R(R/I, H_{I,J}^{n+1}(M))$ is J -minimax if and only if $\text{Ext}_R^2(R/I, H_{I,J}^n(M))$ is J -minimax.*

Proof. Put $\bar{M} = M/\Gamma_{I,J}(M)$ and $L = E/\bar{M}$ which E is the injective hull of $\bar{M}$.

$\Rightarrow$) We proceed by induction on n . Let $n = 0$. Then, by the long exact sequence

$$\cdots \rightarrow \text{Ext}_R^1(R/I, \bar{M}) \rightarrow \text{Ext}_R^2(R/I, \Gamma_{I,J}(M)) \rightarrow \text{Ext}_R^2(R/I, M) \rightarrow \cdots$$

it is enough to show that $\text{Ext}_R^1(R/I, \bar{M})$ is J -minimax. By Lemma 2.4, we conclude that

$$\begin{aligned} \text{Ext}_R^1(R/I, \bar{M}) &\cong \text{Hom}_R(R/I, L) \\ &\cong \text{Hom}_R(R/I, \Gamma_{I,J}(L)) \\ &\cong \text{Hom}_R(R/I, H_{I,J}^1(M)). \end{aligned}$$

Now, suppose that $n > 0$ and the case $n - 1$ is settled. Since $\Gamma_{I,J}(M)$ is (I, J) -cominimax, the R -module $\text{Ext}_R^i(R/I, \Gamma_{I,J}(M))$ is J -minimax, for

all i . Using the exact sequence

$$\cdots \rightarrow \text{Ext}_R^i(R/I, M) \rightarrow \text{Ext}_R^i(R/I, \bar{M}) \rightarrow \text{Ext}_R^{i+1}(R/I, \Gamma_{I,J}(M)) \rightarrow \cdots,$$

we observe that $\text{Ext}_R^i(R/I, \bar{M})$ is J -minimax, for every i . So, $\text{Ext}_R^i(R/I, L)$ is a J -minimax R -module, for all i .

Also $\text{Hom}_R(R/I, H_{I,J}^n(L)) \cong \text{Hom}_R(R/I, H_{I,J}^{n+1}(M))$ is a J -minimax R -module. Now, by the induction hypothesis which is replaced M by L , we obtain that $\text{Ext}_R^2(R/I, H_{I,J}^{n-1}(L))$ is a J -minimax R -module. Therefore, it follows that $\text{Ext}_R^2(R/I, H_{I,J}^n(M))$ is J -minimax.

$\Leftarrow$) Using the induction on n , let $n = 0$. By the exact sequence

$$\cdots \rightarrow \text{Ext}_R^1(R/I, M) \rightarrow \text{Ext}_R^1(R/I, \bar{M}) \rightarrow \text{Ext}_R^2(R/I, \Gamma_{I,J}(M)) \rightarrow \cdots,$$

it follows that $\text{Ext}_R^1(R/I, \bar{M})$ is J -minimax. Hence, $\text{Hom}_R(R/I, L)$ is a J -minimax R -module and so is $\text{Hom}_R(R/I, \Gamma_{I,J}(L))$.

Thus, $\text{Hom}_R(R/I, H_{I,J}^1(M))$ is a J -minimax R -module. Now, suppose that $n > 0$. We observe that the R -modules $\text{Ext}_R^2(R/I, H_{I,J}^{n-1}(L))$ and $\text{Ext}_R^i(R/I, L)$ are J -minimax, for all i . Now, the induction hypothesis follows that the R -module

$$\text{Hom}_R(R/I, H_{I,J}^n(L)) \cong \text{Hom}_R(R/I, H_{I,J}^{n+1}(M))$$

is J -minimax. $\square$

Corollary 2.16. *Let M be a J -minimax R -module and let n be a non-negative integer such that $H_{I,J}^i(M)$ is (I, J) -cominimax, for all $i < n$.*

Then $\text{Hom}_R(R/I, H_{I,J}^{n+1}(M))$ is J -minimax if and only if $\text{Ext}_R^2(R/I, H_{I,J}^n(M))$ is J -minimax.

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