IMPLICATIVE VAGUE IDEALS IN BCK-ALGEBRAS

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Abstract. The notion of implicative vague ideals of BCK-algebras is defined, and several properties of it are investigated. Relations between a vague ideal and an implicative vague ideal is discussed. Characterizations of an implicative vague ideal are considered.

1. Introduction

Several authors from time to time have made a number of generalizations of Zadeh's fuzzy set theory [9]. Of these, the notion of vague set theory introduced by Gau and Buehrer [3] is of interest to us. Using the vague set in the sense of Gau and Buehrer, Biswas [2] studied vague groups. Jun and Park [4,8] studied vague ideals and vague deductive systems in subtraction algebras. In [6], the concept of vague BCK/BCI-algebras is discussed. S. S. Ahn, Y. U. Cho and C. H. Park [1] studied vague quick ideals of *BCK/BCI*-algebras. Y. B. Jun and K. J. Lee ([5]) introduced the notion of positive implicative vague ideals in BCK-algebras. They established relations between a vague ideal and a positive implicative ideals.

In this paper, we also use the notion of vague set in the sense of Gau and Buehrer to discuss the vague theory in BCK-algebras. We define the notion of implicative vague ideal of BCK-algebras and investigate several properties of it. We study a relation between a vague ideal and an implicative vague ideal. We establish characterizations of an implicative vague ideal.

2. Preliminaries

We review some definitions and properties that will be useful in our results.

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By a *BCI-algebra* we mean an algebra $(X, *, 0)$ of type $(2,0)$ satisfying the following conditions:

- (a1) $(\forall x, y, z \in X) (((x * y) * (x * z)) * (z * y) = 0),$
- (a2) $(\forall x, y \in X) ((x * (x * y)) * y = 0),$
- (a3) $(\forall x \in X) (x * x = 0),$
- (a4) $(\forall x, y \in X) (x * y = 0, y * x = 0 \Rightarrow x = y).$

A BCI-algebra X satisfying the additional condition:

- (a5) $(\forall x \in X) (0 * x = 0)$

is called a *BCK-algebra*. In any BCK/BCI-algebra X one can define a partial order “ $\leq$ ” by putting $x \leq y$ if and only if $x * y = 0$.

A BCK/BCI-algebra X has the following properties:

- (b1) $(\forall x \in X) (x * 0 = x).$
- (b2) $(\forall x, y, z \in X) ((x * y) * z = (x * z) * y).$
- (b3) $(\forall x, y, z \in X) (x \leq y \Rightarrow x * z \leq y * z, z * y \leq z * x).$
- (b4) $(\forall x, y, z \in X) ((x * z) * (y * z) \leq x * y).$
- (b5) $(\forall x, y \in X) (x * (x * (x * y)) = x * y).$

A non-empty subset S of a BCK/BCI-algebra X is called a *subalgebra* of X if $x * y \in S$ whenever $x, y \in S$. A subset A of a BCK/BCI-algebra X is called an *ideal* of X if it satisfies:

- (c1) $0 \in A,$
- (c2) $(\forall x \in A) (\forall y \in X) (y * x \in A \Rightarrow y \in A).$

Note that every ideal A of a BCK/BCI-algebra X satisfies:

$$(\forall x \in A) (\forall y \in X) (y \leq x \Rightarrow y \in A).$$

A subset A of a *BCK*-algebra X is called a *positive implicative ideal* of X if it satisfies (c1) and

- (c3) $(\forall x, y, z \in A) ((x * y) * z \in A, y * z \in A \Rightarrow x * z \in A).$

Note that any positive implicative ideal is an ideal, but the converse is not true in general.

A subset A of a *BCK*-algebra X is called an *implicative ideal* of X if it satisfies (c1) and

- (c4) $(\forall x, y, z \in A) ((x * (y * x)) * z \in A, z \in A \Rightarrow x \in A).$

Note that any implicative ideal is an ideal, but the converse is not true in general.

We refer the reader to the book [7] for further information regarding BCK-algebras.

Definition 2.1.([2]) A *vague set* A in the universe of discourse U is characterized by two membership functions given by:

1. A true membership function

$$t_A : U \rightarrow [0, 1],$$

and

2. A false membership function

$$f_A : U \rightarrow [0, 1],$$

where $t_A(u)$ is a lower bound on the grade of membership of u derived from the “evidence for u ”, $f_A(u)$ is a lower bound on the negation of u derived from the “evidence against u ”, and

$$t_A(u) + f_A(u) \leq 1.$$

Thus the grade of membership of u in the vague set A is bounded by a subinterval $[t_A(u), 1 - f_A(u)]$ of $[0, 1]$. This indicates that if the actual grade of membership of u is $\mu(u)$, then

$$t_A(u) \leq \mu(u) \leq 1 - f_A(u).$$

The vague set A is written as

$$A = \{ \langle u, [t_A(u), 1 - f_A(u)] \rangle \mid u \in U \},$$

where the interval $[t_A(u), 1 - f_A(u)]$ is called the *vague value* of u in A , denoted by $V_A(u)$.

For $\alpha, \beta \in [0, 1]$ we now define (α, β) -cut and α -cut of a vague set. Recall that if $I_1 = [a_1, b_1]$ and $I_2 = [a_2, b_2]$ are two subintervals of $[0, 1]$, we can define a relation “ $\succeq$ ” by putting $I_1 \succeq I_2$ if and only if $a_1 \geq a_2$ and $b_1 \geq b_2$.

Definition 2.2.([2]) Let A be a vague set of a universe X with the true-membership function t_A and the false-membership function f_A . The (α, β) -cut of the vague set A is a crisp subset $A_{(\alpha, \beta)}$ of the set X given by

$$A_{(\alpha, \beta)} = \{x \in X \mid V_A(x) \succeq [\alpha, \beta]\}.$$

Clearly $A_{(0,0)} = X$. The (α, β) -cuts of the vague set A are also called *vague-cuts* of A .

Definition 2.3.([2]) The α -cut of the vague set A is a crisp subset A_α of the set X given by $A_\alpha = A_{(\alpha, \alpha)}$.

Note that $A_0 = X$, and if $\alpha \geq \beta$ then $A_\alpha \subseteq A_\beta$ and $A_{(\alpha, \beta)} = A_\alpha$.

Equivalently, we can define the α -cut as

$$A_\alpha = \{x \in X \mid t_A(x) \geq \alpha\}.$$

3. Implicative vague ideals

For our discussion, we shall use the following notations on interval arithmetic:

Let $I[0, 1]$ denote the family of all closed subintervals of $[0, 1]$. We define the term “imax” to mean the maximum of two intervals as

$$\text{imax}(I_1, I_2) := [\max(a_1, a_2), \max(b_1, b_2)],$$

where $I_1 = [a_1, b_1]$, $I_2 = [a_2, b_2] \in I[0, 1]$. Similarly we define “imin”. The concepts of “imax” and “imin” could be extended to define “isup” and “iinf” of infinite number of elements of $I[0, 1]$.

It is obvious that $L = \{I[0, 1], \text{isup}, \text{iinf}, \succeq\}$ is a lattice with universal bounds $[0, 0]$ and $[1, 1]$ (see [2]).

In what follows let X denote *BCK*-algebra unless specified otherwise.

Definition 3.1.([6]) A vague set A of a *BCK*/*BCI*-algebra X is called a *vague BCK/BCI-algebra* of X if the following condition is true:

$$(\forall x, y \in X)(V_A(x * y) \succeq \text{imin}\{V_A(x), V_A(y)\}),$$

that is,

$$\begin{aligned} t_A(x * y) &\geq \min\{t_A(x), t_A(y)\}, \\ 1 - f_A(x * y) &\geq \min\{1 - f_A(x), 1 - f_A(y)\} \end{aligned}$$

for all $x, y \in X$.

Definition 3.2.([6]) A vague set A of a *BCK*-algebra X is called a *vague ideal* of X if the following conditions are true:

- (d1) $(\forall x \in X)(V_A(0) \succeq V_A(x))$,
- (d2) $(\forall x, y \in X)(V_A(x) \succeq \text{imin}\{V_A(x * y), V_A(y)\})$.

that is,

$$\begin{aligned} t_A(0) &\geq t_A(x), 1 - f_A(0) \geq 1 - f_A(x), \\ \text{and } t_A(x) &\geq \min\{t_A(x * y), t_A(y)\} \\ 1 - f_A(x) &\geq \min\{1 - f_A(x * y), 1 - f_A(y)\} \end{aligned}$$

for all $x, y \in X$.

Proposition 3.3.([6]) Every vague ideal A of a *BCK*-algebra X satisfies the following properties:

- (i) $(\forall x, y \in X)(x \leq y \Rightarrow V_A(x) \succeq V_A(y))$,

$$(ii) (\forall x, y, z \in X)(V_A(x * z) \succeq \text{imin}\{V_A((x * y) * z), V_A(y)\}).$$

Definition 3.4. ([5]) A vague set A of a BCK-algebra X is called a *positive implicative vague ideal* of X if it satisfies (d1) and

$$(d3) (\forall x, y, z \in X)(V_A(x * z) \succeq \text{imin}\{V_A((x * y) * z), V_A(y * z)\})$$

that is,

$$\begin{aligned} t_A(x * z) &\geq \min\{t_A((x * y) * z), t_A(y * z)\}, \\ 1 - f_A(x * z) &\geq \min\{1 - f_A((x * y) * z), 1 - f_A(y * z)\} \end{aligned}$$

for all $x, y, z \in X$.

Definition 3.5. A vague set A of a BCK-algebra X is called an *implicative vague ideal* of a BCK-algebra X if it satisfies (d1) and

$$(d4) (\forall x, y, z \in X)(V_A(x) \succeq \text{imin}\{V_A((x * (y * x)) * z), V_A(z)\}).$$

that is,

$$\begin{aligned} t_A(x) &\geq \min\{t_A((x * (y * x)) * z), t_A(z)\}, \\ 1 - f_A(x) &\geq \min\{1 - f_A((x * (y * x)) * z), 1 - f_A(z)\} \end{aligned}$$

for all $x, y, z \in X$.

Theorem 3.6. Every implicative vague ideal of a BCK-algebra X is a vague ideal of X .

Proof. Let A be an implicative vague ideal of a BCK-algebra X . If we take $y := x$ in (d4) and use (b1), then we obtain (d2). Hence A is a vague ideal of X . $\square$

Example 3.7. Let $X := \{0, 1, 2, 3, 4\}$ be a BCK-algebra ([7]) in which the $*$ -operation is given by the following table:

$*$	0	1	2	3	4
0	0	0	0	0	0
1	1	0	0	0	0
2	2	1	0	1	0
3	3	3	3	0	0
4	4	4	4	4	0

Let A be the vague set in X defined as follows:

$$A = \{\langle 0, [0.7, 0.2] \rangle, \langle 1, [0.7, 0.2] \rangle, \langle 2, [0.7, 0.2] \rangle, \langle 3, [0.7, 0.2] \rangle, \langle 4, [0.5, 0.3] \rangle\}$$

It is routine to verify that A is an implicative vague ideal of X .

Theorem 3.8. Every implicative vague ideal of a BCK-algebra X is a positive implicative ideal of X .

Proof. Let A be an implicative vague ideal of a BCK -algebra X . Since $((x * z) * z) * (y * z) \leq (x * z) * y = (x * y) * z$, using Proposition 3.3(i) we obtain

$$V_A(((x * z) * z) * (y * z)) \succeq V_A((x * y) * z). \quad (3.1)$$

Note that

$$(x * z) * (x * (x * z)) = (x * (x * (x * z))) * z = (x * z) * z. \quad (3.2)$$

It follows from (d4), (3.1) and (3.2) that

$$\begin{aligned} V_A(x * z) &\succeq \text{imin}\{V_A(((x * z) * (x * (x * z))) * (y * z)), V_A(y * z)\} \\ &= \text{imin}\{V_A(((x * z) * z) * (y * z)), V_A(y * z)\} \\ &\succeq \text{imin}\{V_A((x * y) * z), V_A(y * z)\}. \end{aligned}$$

Hence A is a positive implicative ideal of X . $\square$

Example 3.9. Let $X := \{0, 1, 2, 3, 4\}$ be a BCK -algebra([7]) in which the $*$ -operation is given by the following table:

$*$	0	1	2	3	4
0	0	0	0	0	0
1	1	0	1	0	1
2	2	2	0	2	0
3	3	1	3	0	3
4	4	4	4	4	0

Let A be the vague set in X defined as follows:

$$A = \{\langle 0, [0.7, 0.2] \rangle, \langle 1, [0.5, 0.3] \rangle, \langle 2, [0.7, 0.2] \rangle, \langle 3, [0.5, 0.3] \rangle, \langle 4, [0.5, 0.3] \rangle\}.$$

It is routine to verify that A is a vague ideal of X . But it is not an implicative vague ideal of X , since

$$V_A(1) \not\succeq \text{imin}\{V_A((1 * (3 * 1)) * 2), V_A(2)\}.$$

Theorem 3.10.([5]) *For a vague set A in a BCK -algebra X , the following are equivalent:*

- (1) A is a positive implicative ideal of X .
- (2) $(\forall x, y \in X)(V_A(x * y) \succeq V_A((x * y) * y))$.
- (3) $(\forall x, y, z \in X)(V_A((x * z) * (y * z)) \succeq V_A((x * y) * z))$.

Theorem 3.11. *Let A be a positive implicative vague ideal of a BCK -algebra X . Then A is an implicative vague ideal of X if and only if*

$$(*) \quad (\forall x, y, z \in X)(V_A(x * (x * y)) \succeq V_A(y * (y * x))).$$

Proof. Let A be a positive implicative vague ideal of a BCK-algebra X . Assume that A is an implicative vague ideal of X . Since $x*(x*y) \leq x$, it follows that $y*x \leq y*(x*(x*y))$. Hence we have

$$\begin{aligned} (x*(x*y)) * (y*(x*(x*y))) &\leq (x*(x*y)) * (y*x) \\ &= (x*(y*x)) * (x*y) \\ &\leq y*(y*x). \end{aligned}$$

It follows from Proposition 3.3(i) that

$$V_A((x*(x*y)) * (y*(x*(x*y)))) \succeq V_A(y*(y*x)). \quad (3.3)$$

Using (d4), (b1), (d1), and (3.3), we have

$$\begin{aligned} V_A(x*(x*y)) &\succeq \text{imin}\{V_A((x*(x*y)) * (y*(x*(x*y)))) * 0, V_A(0)\} \\ &= V_A((x*(x*y)) * (y*(x*(x*y)))) \\ &\succeq V_A(y*(y*x)). \end{aligned}$$

Thus A satisfies (*).

Conversely, let A be a positive implicative vague ideal of X which satisfies (*). Since $(y*(y*x)) * (y*x) \leq x*(y*x)$, it follows from Proposition 3.3(i), Theorem 3.10(2) and (*) that

$$\begin{aligned} V_A(x*(y*x)) &\preceq V_A((y*(y*x)) * (y*x)) \\ &\preceq V_A(y*(y*x)) \\ &\preceq V_A(x*(x*y)). \end{aligned} \quad (3.4)$$

Since $(x*y)*z \leq x*y \leq x*(y*x)$ for any $x, y, z \in X$, it follows from Proposition 3.3(i) that

$$V_A(x*(y*x)) \preceq V_A((x*y)*z). \quad (3.5)$$

Using (3.4), (3.5) and Proposition 3.3(ii), we have

$$\begin{aligned} V_A(x) &\succeq \text{imin}\{V_A(x*(x*y)), V_A(x*y)\} \\ &\succeq \text{imin}\{V_A(x*(y*x)), V_A(x*y)\} \\ &\succeq \text{imin}\{V_A(x*(y*x)), V_A((x*y)*z), V_A(z)\} \\ &\succeq \text{imin}\{V_A(x*(y*x)), V_A(z)\} \\ &\succeq \text{imin}\{\text{imin}\{V_A((x*z)*(y*x)), V_A(z)\}, V_A(z)\} \\ &= \text{imin}\{V_A((x*z)*(y*x)), V_A(z)\} \\ &= \text{imin}\{V_A((x*(y*x))*z), V_A(z)\}. \end{aligned}$$

Thus A is an implicative vague ideal of X . □

Theorem 3.12. *A vague ideal A of a BCK-algebra X is implicative if and only if*

$$(**) \quad (\forall x, y \in X)(V_A(x) \succeq V_A(x * (y * x))).$$

Proof. Suppose that a vague ideal A of X satisfies $(**)$. It follows from (d2) and $(**)$ that

$$\begin{aligned} V_A(x) &\succeq V_A(x * (y * x)) \\ &\succeq \text{imin}\{V_A((x * (y * x)) * z), V_A(z)\}. \end{aligned}$$

Hence A is an implicative vague ideal of X .

Conversely, assume that a vague ideal A of X is implicative. Putting $z := 0$ in (d4), we have

$$\begin{aligned} V_A(x) &\succeq \text{imin}\{V_A((x * (y * x)) * 0), V_A(0)\} \\ &= \text{imin}\{V_A(x * (y * x)), V_A(0)\} \\ &\succeq V_A(x * (y * x)). \end{aligned}$$

This completes the proof $\square$

Lemma 3.13. ([6]) *For a vague set A in a BCK-algebra X , the following are equivalent:*

- (1) A is a vague ideal of X ,
- (2) A satisfies the following implication:

$$(\forall x, y, z \in X)((x * y) * z = 0 \Rightarrow V_A(x) \succeq \text{imin}\{V_A(y), V_A(z)\}).$$

Theorem 3.14. *For a vague set A in a BCK-algebra X , the following are equivalent:*

- (1) A is an implicative vague ideal of X ,
- (2) A satisfies the following implication:

$$(\forall x, y, a, b \in X)((x * (y * x)) * a * b = 0 \Rightarrow V_A(x) \succeq \text{imin}\{V_A(a), V_A(b)\}).$$

Proof. Assume that A is an implicative vague ideal of a BCK-algebra X . Then A is a vague ideal of X by Theorem 3.6. Let $x, y, a, b \in X$ be such that $((x * (y * x)) * a) * b = 0$. It follows from Theorem 3.12 and Lemma 3.13 that

$$V_A(x) \succeq V_A(x * (y * x)) \succeq \text{imin}\{V_A(a), V_A(b)\}.$$

Therefore A satisfies (2).

Conversely, let $x, a, b \in X$ be such that $(x * a) * b = 0$. Then $((x * (0 * x)) * a) * b = (x * a) * b = 0$. By (2), we have $V_A(x) \succeq \text{imin}\{V_A(a), V_A(b)\}$.

Thus A is a vague ideal of X by Lemma 3.13. Since $[(x * (y * x)) * (x * (y * x))] * 0 = 0$, it follow from (2) that

$$\begin{aligned} V_A(x) &\succeq \text{imin}\{V_A(x * (y * x)), V_A(0)\} \\ &= V_A(x * (y * x)). \end{aligned}$$

Thus A is an implicative vague ideal of X by Theorem 3.12. $\square$

Corollary 3.15. *Let A be a vague set of a BCK-algebra X . If A satisfies the following inequality:*

$$V_A(x) \succeq \text{imin}\{V_A(a_1), \dots, V_A(a_n)\}$$

whenever $(\dots((x * (y * x)) * a_1) * \dots) * a_n = 0$ for all $x, y, a_1, \dots, a_n \in X$, then A is an implicative vague ideal of X .

Proof. Straightforward. $\square$

Theorem 3.16. *Let A be an implicative vague ideal of a BCK-algebra X . Then for any $\alpha, \beta \in [0, 1]$, the vague-cut $A_{(\alpha, \beta)}$ of A is a crisp implicative ideal of X .*

Proof. Obviously, $0 \in A_{(\alpha, \beta)}$. Let $(x * (y * x)) * z \in A_{(\alpha, \beta)}$ and $z \in A_{(\alpha, \beta)}$. Then $V_A((x * (y * x)) * z) \succeq [\alpha, \beta]$ and $V_A(z) \succeq [\alpha, \beta]$, i.e., $t_A((x * (y * x)) * z) \geq \alpha$, $t_A(z) \geq \alpha$ and $1 - f_A((x * (y * x)) * z) \geq \beta$, $1 - f_A(z) \geq \beta$. It follows that

$$t_A(x) \geq \min\{t_A((x * (y * x)) * z), t_A(z)\} \geq \alpha$$

$$\text{and } 1 - f_A(x) \geq \min\{1 - f_A((x * (y * x)) * z), 1 - f_A(z)\} \geq \beta.$$

Hence $x \in A_{(\alpha, \beta)}$ and so $A_{(\alpha, \beta)}$ is an implicative ideal of X . $\square$

The ideals like $A_{(\alpha, \beta)}$ are also called *vague cut implicative ideals* of X .

Theorem 3.17. *Any implicative ideal I of a BCK-algebra X is a vague cut-ideal of some implicative vague ideal of X .*

Proof. Consider the vague set A of X given by

$$V_A(x) = \begin{cases} [\alpha, \alpha] & \text{if } x \in I \\ [0, 0] & \text{if } x \notin I \end{cases}$$

where $\alpha \in (0, 1)$. Since $0 \in I$, we have $V_A(0) = [\alpha, \alpha] \succeq V_A(x)$ for all $x \in X$. Let $x, y, z \in X$ be such that $(x * (y * x)) * z \in I$ and $z \in I$. If $x \notin I$, then

$$t_A(x) = 0 \leq \min\{t_A((x * (y * x)) * z), t_A(z)\}$$

$$\text{and } 1 - f_A(x) = 0 \leq \min\{1 - f_A((x * (y * x)) * z), 1 - f_A(z)\}.$$

If $x \in I$, then

$$t_A(x) = \alpha = \min\{t_A((x * (y * z)) * z), t_A(z)\}$$

$$\text{and } 1 - f_A(x) = \alpha = \min\{1 - f_A((x * (y * x)) * z), 1 - f_A(z)\}.$$

Thus A is an implicative vague ideal of X . Clearly, $I = A_{(\alpha, \alpha)}$. $\square$

Theorem 3.18. *Let A be an implicative vague ideal of a BCK-algebra X . Then the set*

$$I := \{x \in X \mid V_A(x) = V_A(0)\}$$

is a crisp implicative ideal of X .

Proof. Clearly, $0 \in I$. Let $x, y, z \in X$ be such that $(x * (y * x)) * z \in I$ and $z \in I$. Then $V_A((x * (y * x)) * z) = V_A(0)$ and $V_A(z) = V_A(0)$ and so

$$V_A(x) \succeq \text{imin}\{V_A((x * (y * x)) * z), V_A(z)\} = V_A(0).$$

Since $V_A(0) \succeq V_A(x)$ for all $x \in X$, it follows that $V_A(x) = V_A(0)$. Hence $x \in I$. Therefore I is a crisp implicative ideal of X . $\square$

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