RICCI AND SCALAR CURVATURES ON $SU(3)$

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Abstract. In this paper, we obtain the Ricci curvature and the scalar curvature on $SU(3)$ with some left invariant Riemannian metric. And then we get a necessary and sufficient condition for the scalar curvature (resp. the Ricci curvature) on the Riemannian manifold $SU(3)$ to be positive.

1. Introduction

Park (cf. [7]) studied geometric properties on $SU(3)/T(k, l)$ whose isotropy representation decomposes into four inequivalent irreducible summands. In this paper, we study curvatures on $SU(3)$ with some left invariant Riemannian metrics.

In general, for the Ricci tensor field Ric on a Riemannian manifold (M, g) ,

$$r(X) := \frac{Ric(X, X)}{\|X\|_g^2}, \quad (X \neq 0),$$

is said to be the *Ricci curvature* of (M, g) with respect to $X \in T_p M$ ($p \in M$).

First of all, we obtain the Ricci curvatures on $SU(3)$ with some left invariant Riemannian metrics $g_{(\lambda_1, \lambda_2, \lambda_3)}$ (cf. Lemma 2.1), and then estimate the Ricci curvature on the Riemannian manifold $(SU(3), g_{(\lambda_1, \lambda_2, \lambda_3)})$ (cf. Theorem 2.4). Moreover, we estimate the scalar curvature on $(SU(3), g_{(\lambda_1, \lambda_2, \lambda_3)})$ (cf. Theorem 2.2), and get a necessary and sufficient condition for the scalar curvature on $(SU(3), g_{(\lambda_1, \lambda_2, \lambda_3)})$ to be positive (resp. negative) (cf. Corollary 2.3).

Received March 16, 2012. Accepted April 3, 2012.

2000 Mathematics Subject Classification. 53C30, 53C25.

Key words and phrases. Ricci curvature, scalar curvature, Einstein manifold.

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2. Invariant connections on $SU(3)$

Let (M, g) be a Riemannian manifold with a Riemannian metric g . Then, the Riemannian connection ∇ for the Riemannian metric g is given as follows (cf. [1, 5, 6]):

$$(2.1) \quad \begin{aligned} 2g(X, \nabla_Z Y) = & Zg(X, Y) + Yg(X, Z) - Xg(Y, Z) \\ & + g(Z, [X, Y]) + g(Y, [X, Z]) - g(X, [Y, Z]) \end{aligned}$$

for $X, Y, Z \in \mathfrak{X}(M)$. And, the curvature tensor field R on (M, g) is given by (cf. [4, 8])

$$(2.2) \quad R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X, Y]} Z$$

for $X, Y, Z \in \mathfrak{X}(M)$. We denote by $\mathfrak{su}(3)$ the Lie algebra of $SU(3)$. Let E_{ij} be a real 3×3 matrix with 1 on entry (i, j) and 0 elsewhere and we put

$$(2.3) \quad \begin{aligned} Y_1 &:= (E_{12} - E_{21})/\sqrt{12\lambda_1}, & Y_2 &:= i(E_{12} + E_{21})/\sqrt{12\lambda_1}, \\ Y_3 &:= (E_{13} - E_{31})/\sqrt{12\lambda_2}, & Y_4 &:= i(E_{13} + E_{31})/\sqrt{12\lambda_2}, \\ Y_5 &:= (E_{23} - E_{32})/\sqrt{12\lambda_3}, & Y_6 &:= i(E_{23} + E_{32})/\sqrt{12\lambda_3}, \\ Y_7 &:= i \operatorname{diag} [(k+2l), -(2k+l), (k-l)]/\sqrt{36(k^2+kl+l^2)}, \\ Y_8 &:= i \operatorname{diag} [k, l, -(k+l)]/\sqrt{12(k^2+kl+l^2)} \end{aligned}$$

for positive real numbers λ_1, λ_2 and λ_3 , and $k, l \in \mathbb{R}$ such that $k^2 + l^2 \neq 0$, where $i := \sqrt{-1}$.

We take an inner product $\langle \cdot, \cdot \rangle$ on $\mathfrak{su}(3)$ such that $\{Y_i \mid i = 1, 2, \dots, 8\}$ is an orthonormal basis of $\mathfrak{su}(3)$ with respect to $\langle \cdot, \cdot \rangle$. Then the inner product $\langle \cdot, \cdot \rangle$ determines a left invariant Riemannian metric $g_{(\lambda_1, \lambda_2, \lambda_3)}$ on $SU(3)$. Here and from now on, let ∇ be the Levi-Civita connection for the metric $g_{(\lambda_1, \lambda_2, \lambda_3)}$ on $SU(3)$. Here by the help of (2.1), we get

$$(2.4) \quad 2\langle \nabla_X Y, Z \rangle = \langle Y, [Z, X] \rangle + \langle Z, [X, Y] \rangle - \langle X, [Y, Z] \rangle$$

for $X, Y, Z \in \mathfrak{su}(3)$. From (2.3), we obtain

$$\begin{aligned}
(2.5) \quad & [Y_1, Y_2] = (k+l)(2\lambda_1\sqrt{\gamma})^{-1} Y_7 + (k-l)(2\lambda_1\sqrt{3\gamma})^{-1} Y_8, \\
& [Y_1, Y_3] = -\lambda_3(2\sqrt{3\lambda_1\lambda_2\lambda_3})^{-1} Y_5, \\
& [Y_1, Y_4] = -\lambda_3(2\sqrt{3\lambda_1\lambda_2\lambda_3})^{-1} Y_6, \\
& [Y_1, Y_5] = \lambda_2(2\sqrt{3\lambda_1\lambda_2\lambda_3})^{-1} Y_3, \\
& [Y_1, Y_6] = \lambda_2(2\sqrt{3\lambda_1\lambda_2\lambda_3})^{-1} Y_4, \\
& [Y_1, Y_7] = -(k+l)(2\sqrt{\gamma})^{-1} Y_2, \\
& [Y_1, Y_8] = -(k-l)(2\sqrt{3\gamma})^{-1} Y_2, \\
& [Y_2, Y_3] = \lambda_3(2\sqrt{3\lambda_1\lambda_2\lambda_3})^{-1} Y_6, \\
& [Y_2, Y_4] = -\lambda_3(2\sqrt{3\lambda_1\lambda_2\lambda_3})^{-1} Y_5, \\
& [Y_2, Y_5] = \lambda_2(2\sqrt{3\lambda_1\lambda_2\lambda_3})^{-1} Y_4, \\
& [Y_2, Y_6] = -\lambda_2(2\sqrt{3\lambda_1\lambda_2\lambda_3})^{-1} Y_3, \\
& [Y_2, Y_7] = (k+l)(2\sqrt{\gamma})^{-1} Y_1, \\
& [Y_2, Y_8] = (k-l)(2\sqrt{3\gamma})^{-1} Y_1, \\
& [Y_3, Y_4] = l(2\lambda_2\sqrt{\gamma})^{-1} Y_7 + (2k+l)(2\lambda_2\sqrt{3\gamma})^{-1} Y_8, \\
& [Y_3, Y_5] = -\lambda_1(2\sqrt{3\lambda_1\lambda_2\lambda_3})^{-1} Y_1, \\
& [Y_3, Y_6] = \lambda_1(2\sqrt{3\lambda_1\lambda_2\lambda_3})^{-1} Y_2, \\
& [Y_3, Y_7] = -l(2\sqrt{\gamma})^{-1} Y_4, \\
& [Y_3, Y_8] = -(2k+l)(2\sqrt{3\gamma})^{-1} Y_4, \\
& [Y_4, Y_5] = -\lambda_1(2\sqrt{3\lambda_1\lambda_2\lambda_3})^{-1} Y_2, \\
& [Y_4, Y_6] = -\lambda_1(2\sqrt{3\lambda_1\lambda_2\lambda_3})^{-1} Y_1, \\
& [Y_4, Y_7] = l(2\sqrt{\gamma})^{-1} Y_3, \\
& [Y_4, Y_8] = (2k+l)(2\sqrt{3\gamma})^{-1} Y_3, \\
& [Y_5, Y_6] = -k(2\lambda_3\sqrt{\gamma})^{-1} Y_7 + (k+2l)(2\lambda_3\sqrt{3\gamma})^{-1} Y_8, \\
& [Y_5, Y_7] = k(2\sqrt{\gamma})^{-1} Y_6, \quad [Y_5, Y_8] = -(k+2l)(2\sqrt{3\gamma})^{-1} Y_6, \\
& [Y_6, Y_7] = -k(2\sqrt{\gamma})^{-1} Y_5, \quad [Y_6, Y_8] = (k+2l)(2\sqrt{3\gamma})^{-1} Y_5, \\
& [Y_7, Y_8] = 0,
\end{aligned}$$

where $\gamma = k^2 + kl + l^2$. By the help of (2.4) and (2.5), we have
 $\nabla_{Y_i} Y_i = 0, \quad (i = 1, 2, \dots, 8),$

$$\begin{aligned}
(2.6) \quad & \nabla_{Y_1} Y_2 = (k+l)(4\lambda_1\sqrt{\gamma})^{-1} Y_7 + (k-l)(4\lambda_1\sqrt{3\gamma})^{-1} Y_8, \\
& \nabla_{Y_1} Y_3 = (\lambda_1 - \lambda_2 - \lambda_3)(4\sqrt{3\lambda_1\lambda_2\lambda_3})^{-1} Y_5, \\
& \nabla_{Y_1} Y_4 = (\lambda_1 - \lambda_2 - \lambda_3)(4\sqrt{3\lambda_1\lambda_2\lambda_3})^{-1} Y_6, \\
& \nabla_{Y_1} Y_5 = (-\lambda_1 + \lambda_2 + \lambda_3)(4\sqrt{3\lambda_1\lambda_2\lambda_3})^{-1} Y_3, \\
& \nabla_{Y_1} Y_6 = (-\lambda_1 + \lambda_2 + \lambda_3)(4\sqrt{3\lambda_1\lambda_2\lambda_3})^{-1} Y_4, \\
& \nabla_{Y_1} Y_7 = -(k+l)(4\lambda_1\sqrt{\gamma})^{-1} Y_2, \\
& \nabla_{Y_1} Y_8 = -(k-l)(4\lambda_1\sqrt{3\gamma})^{-1} Y_2, \\
& \nabla_{Y_2} Y_3 = (-\lambda_1 + \lambda_2 + \lambda_3)(4\sqrt{3\lambda_1\lambda_2\lambda_3})^{-1} Y_6, \\
& \nabla_{Y_2} Y_4 = (\lambda_1 - \lambda_2 - \lambda_3)(4\sqrt{3\lambda_1\lambda_2\lambda_3})^{-1} Y_5, \\
& \nabla_{Y_2} Y_5 = (-\lambda_1 + \lambda_2 + \lambda_3)(4\sqrt{3\lambda_1\lambda_2\lambda_3})^{-1} Y_4, \\
& \nabla_{Y_2} Y_6 = (\lambda_1 - \lambda_2 - \lambda_3)(4\sqrt{3\lambda_1\lambda_2\lambda_3})^{-1} Y_3, \\
& \nabla_{Y_2} Y_7 = (k+l)(4\lambda_1\sqrt{\gamma})^{-1} Y_1, \\
& \nabla_{Y_2} Y_8 = (k-l)(4\lambda_1\sqrt{3\gamma})^{-1} Y_1, \\
& \nabla_{Y_3} Y_4 = l(4\lambda_2\sqrt{\gamma})^{-1} Y_7 + (2k+l)(4\lambda_2\sqrt{3\gamma})^{-1} Y_8, \\
& \nabla_{Y_3} Y_5 = (-\lambda_1 + \lambda_2 - \lambda_3)(4\sqrt{3\lambda_1\lambda_2\lambda_3})^{-1} Y_1, \\
& \nabla_{Y_3} Y_6 = (\lambda_1 - \lambda_2 + \lambda_3)(4\sqrt{3\lambda_1\lambda_2\lambda_3})^{-1} Y_2, \\
& \nabla_{Y_3} Y_7 = -l(4\lambda_2\sqrt{\gamma})^{-1} Y_4, \\
& \nabla_{Y_3} Y_8 = -(2k+l)(4\lambda_2\sqrt{3\gamma})^{-1} Y_4, \\
& \nabla_{Y_4} Y_5 = (-\lambda_1 + \lambda_2 - \lambda_3)(4\sqrt{3\lambda_1\lambda_2\lambda_3})^{-1} Y_2, \\
& \nabla_{Y_4} Y_6 = (-\lambda_1 + \lambda_2 - \lambda_3)(4\sqrt{3\lambda_1\lambda_2\lambda_3})^{-1} Y_1, \\
& \nabla_{Y_4} Y_7 = l(4\lambda_2\sqrt{\gamma})^{-1} Y_3, \quad \nabla_{Y_4} Y_8 = (2k+l)(4\lambda_2\sqrt{3\gamma})^{-1} Y_3, \\
& \nabla_{Y_5} Y_6 = -k(4\lambda_3\sqrt{\gamma})^{-1} Y_7 + (k+2l)(4\lambda_3\sqrt{3\gamma})^{-1} Y_8, \\
& \nabla_{Y_5} Y_7 = k(4\lambda_3\sqrt{\gamma})^{-1} Y_6, \quad \nabla_{Y_5} Y_8 = -(k+2l)(4\lambda_3\sqrt{3\gamma})^{-1} Y_6, \\
& \nabla_{Y_6} Y_7 = -k(4\lambda_3\sqrt{\gamma})^{-1} Y_5, \quad \nabla_{Y_6} Y_8 = (k+2l)(4\lambda_3\sqrt{3\gamma})^{-1} Y_5, \\
& \nabla_{Y_7} Y_8 = 0.
\end{aligned}$$

From (2.2), (2.5) and (2.6), we get

$$\begin{aligned}
 (2.7) \quad & R(Y_2, Y_1)Y_1 = (4\lambda_1 - 3)(12\lambda_1^2)^{-1} Y_2, \\
 & R(Y_3, Y_1)Y_1 = \beta(\lambda_1^2 + \lambda_2^2 - 3\lambda_3^2 - 2\lambda_1\lambda_2 + 2\lambda_2\lambda_3 + 2\lambda_3\lambda_1) Y_3, \\
 & R(Y_4, Y_1)Y_1 = \beta(\lambda_1^2 + \lambda_2^2 - 3\lambda_3^2 - 2\lambda_1\lambda_2 + 2\lambda_2\lambda_3 + 2\lambda_3\lambda_1) Y_4, \\
 & R(Y_5, Y_1)Y_1 = \beta(\lambda_1^2 - 3\lambda_2^2 + \lambda_3^2 + 2\lambda_1\lambda_2 + 2\lambda_2\lambda_3 - 2\lambda_3\lambda_1) Y_5, \\
 & R(Y_6, Y_1)Y_1 = \beta(\lambda_1^2 - 3\lambda_2^2 + \lambda_3^2 + 2\lambda_1\lambda_2 + 2\lambda_2\lambda_3 - 2\lambda_3\lambda_1) Y_6, \\
 & R(Y_7, Y_1)Y_1 = (k + l)^2(16\lambda_1^2\gamma)^{-1} Y_7 + (k^2 - l^2)(16\sqrt{3}\lambda_1^2\gamma)^{-1} Y_8, \\
 & R(Y_8, Y_1)Y_1 = (k^2 - l^2)(16\sqrt{3}\lambda_1^2\gamma)^{-1} Y_7 + (k - l)^2(48\lambda_1^2\gamma)^{-1} Y_8, \\
 & R(Y_1, Y_2)Y_2 = (4\lambda_1 - 3)(12\lambda_1^2)^{-1} Y_1, \\
 & R(Y_3, Y_2)Y_2 = \beta(\lambda_1^2 + \lambda_2^2 - 3\lambda_3^2 - 2\lambda_1\lambda_2 + 2\lambda_2\lambda_3 + 2\lambda_3\lambda_1) Y_3, \\
 & R(Y_4, Y_2)Y_2 = \beta(\lambda_1^2 + \lambda_2^2 - 3\lambda_3^2 - 2\lambda_1\lambda_2 + 2\lambda_2\lambda_3 + 2\lambda_3\lambda_1) Y_4, \\
 & R(Y_5, Y_2)Y_2 = \beta(\lambda_1^2 - 3\lambda_2^2 + \lambda_3^2 + 2\lambda_1\lambda_2 + 2\lambda_2\lambda_3 - 2\lambda_3\lambda_1) Y_5, \\
 & R(Y_6, Y_2)Y_2 = \beta(\lambda_1^2 - 3\lambda_2^2 + \lambda_3^2 + 2\lambda_1\lambda_2 + 2\lambda_2\lambda_3 - 2\lambda_3\lambda_1) Y_6, \\
 & R(Y_7, Y_2)Y_2 = (k + l)^2(16\lambda_1^2\gamma)^{-1} Y_7 + (k^2 - l^2)(16\sqrt{3}\lambda_1^2\gamma)^{-1} Y_8, \\
 & R(Y_8, Y_2)Y_2 = (k^2 - l^2)(16\sqrt{3}\lambda_1^2\gamma)^{-1} Y_7 + (k - l)^2(48\lambda_1^2\gamma)^{-1} Y_8, \\
 & R(Y_1, Y_3)Y_3 = \beta(\lambda_1^2 + \lambda_2^2 - 3\lambda_3^2 - 2\lambda_1\lambda_2 + 2\lambda_2\lambda_3 + 2\lambda_3\lambda_1) Y_1, \\
 & R(Y_2, Y_3)Y_3 = \beta(\lambda_1^2 + \lambda_2^2 - 3\lambda_3^2 - 2\lambda_1\lambda_2 + 2\lambda_2\lambda_3 + 2\lambda_3\lambda_1) Y_2, \\
 & R(Y_4, Y_3)Y_3 = (4\lambda_2 - 3)(12\lambda_2^2)^{-1} Y_4, \\
 & R(Y_5, Y_3)Y_3 = \beta(-3\lambda_1^2 + \lambda_2^2 + \lambda_3^2 + 2\lambda_1\lambda_2 - 2\lambda_2\lambda_3 + 2\lambda_3\lambda_1) Y_5, \\
 & R(Y_6, Y_3)Y_3 = \beta(-3\lambda_1^2 + \lambda_2^2 + \lambda_3^2 + 2\lambda_1\lambda_2 - 2\lambda_2\lambda_3 + 2\lambda_3\lambda_1) Y_6, \\
 & R(Y_7, Y_3)Y_3 = l^2(16\lambda_2^2\gamma)^{-1} Y_7 + (2kl + l^2)(16\sqrt{3}\lambda_2^2\gamma)^{-1} Y_8, \\
 & R(Y_8, Y_3)Y_3 = (2kl + l^2)(16\sqrt{3}\lambda_2^2\gamma)^{-1} Y_7 + (2k + l)^2(48\lambda_2^2\gamma)^{-1} Y_8, \\
 & R(Y_1, Y_4)Y_4 = \beta(\lambda_1^2 + \lambda_2^2 - 3\lambda_3^2 - 2\lambda_1\lambda_2 + 2\lambda_2\lambda_3 + 2\lambda_3\lambda_1) Y_1, \\
 & R(Y_2, Y_4)Y_4 = \beta(\lambda_1^2 + \lambda_2^2 - 3\lambda_3^2 - 2\lambda_1\lambda_2 + 2\lambda_2\lambda_3 + 2\lambda_3\lambda_1) Y_2, \\
 & R(Y_3, Y_4)Y_4 = (4\lambda_2 - 3)(12\lambda_2^2)^{-1} Y_3, \\
 & R(Y_5, Y_4)Y_4 = \beta(-3\lambda_1^2 + \lambda_2^2 + \lambda_3^2 + 2\lambda_1\lambda_2 - 2\lambda_2\lambda_3 + 2\lambda_3\lambda_1) Y_5, \\
 & R(Y_6, Y_4)Y_4 = \beta(-3\lambda_1^2 + \lambda_2^2 + \lambda_3^2 + 2\lambda_1\lambda_2 - 2\lambda_2\lambda_3 + 2\lambda_3\lambda_1) Y_6, \\
 & R(Y_7, Y_4)Y_4 = l^2(16\lambda_2^2\gamma)^{-1} Y_7 + (2kl + l^2)(16\sqrt{3}\lambda_2^2\gamma)^{-1} Y_8, \\
 & R(Y_8, Y_4)Y_4 = (2kl + l^2)(16\sqrt{3}\lambda_2^2\gamma)^{-1} Y_7 + (2k + l)^2(48\lambda_2^2\gamma)^{-1} Y_8,
 \end{aligned}$$

$$\begin{aligned}
R(Y_1, Y_5)Y_5 &= \beta(\lambda_1^2 - 3\lambda_2^2 + \lambda_3^2 + 2\lambda_1\lambda_2 + 2\lambda_2\lambda_3 - 2\lambda_3\lambda_1) Y_1, \\
R(Y_2, Y_5)Y_5 &= \beta(\lambda_1^2 - 3\lambda_2^2 + \lambda_3^2 + 2\lambda_1\lambda_2 + 2\lambda_2\lambda_3 - 2\lambda_3\lambda_1) Y_2, \\
R(Y_3, Y_5)Y_5 &= \beta(-3\lambda_1^2 + \lambda_2^2 + \lambda_3^2 + 2\lambda_1\lambda_2 - 2\lambda_2\lambda_3 + 2\lambda_3\lambda_1) Y_3, \\
R(Y_4, Y_5)Y_5 &= \beta(-3\lambda_1^2 + \lambda_2^2 + \lambda_3^2 + 2\lambda_1\lambda_2 - 2\lambda_2\lambda_3 + 2\lambda_3\lambda_1) Y_4, \\
R(Y_6, Y_5)Y_5 &= (4\lambda_3 - 3)(12\lambda_3^2)^{-1} Y_6, \\
R(Y_7, Y_5)Y_5 &= k^2(16\lambda_3^2\gamma)^{-1} Y_7 - (k^2 + 2kl)(16\sqrt{3}\lambda_3^2\gamma)^{-1} Y_8, \\
R(Y_8, Y_5)Y_5 &= -(k^2 + 2kl)(16\sqrt{3}\lambda_3^2\gamma)^{-1} Y_7 + (k + 2l)^2(48\lambda_3^2\gamma)^{-1} Y_8, \\
R(Y_1, Y_6)Y_6 &= \beta(\lambda_1^2 - 3\lambda_2^2 + \lambda_3^2 + 2\lambda_1\lambda_2 + 2\lambda_2\lambda_3 - 2\lambda_3\lambda_1) Y_1, \\
R(Y_2, Y_6)Y_6 &= \beta(\lambda_1^2 - 3\lambda_2^2 + \lambda_3^2 + 2\lambda_1\lambda_2 + 2\lambda_2\lambda_3 - 2\lambda_3\lambda_1) Y_2, \\
R(Y_3, Y_6)Y_6 &= \beta(-3\lambda_1^2 + \lambda_2^2 + \lambda_3^2 + 2\lambda_1\lambda_2 - 2\lambda_2\lambda_3 + 2\lambda_3\lambda_1) Y_3, \\
R(Y_4, Y_6)Y_6 &= \beta(-3\lambda_1^2 + \lambda_2^2 + \lambda_3^2 + 2\lambda_1\lambda_2 - 2\lambda_2\lambda_3 + 2\lambda_3\lambda_1) Y_4, \\
R(Y_5, Y_6)Y_6 &= (4\lambda_3 - 3)(12\lambda_3^2)^{-1} Y_5, \\
R(Y_7, Y_6)Y_6 &= k^2(16\lambda_3^2\gamma)^{-1} Y_7 - (k^2 + 2kl)(16\sqrt{3}\lambda_3^2\gamma)^{-1} Y_8, \\
R(Y_8, Y_6)Y_6 &= -(k^2 + 2kl)(16\sqrt{3}\lambda_3^2\gamma)^{-1} Y_7 + (k + 2l)^2(48\lambda_3^2\gamma)^{-1} Y_8, \\
R(Y_1, Y_7)Y_7 &= (k + l)^2(16\gamma\lambda_1^2)^{-1} Y_1, \\
R(Y_2, Y_7)Y_7 &= (k + l)^2(16\gamma\lambda_1^2)^{-1} Y_2, \\
R(Y_3, Y_7)Y_7 &= l^2(16\gamma\lambda_2^2)^{-1} Y_3, \quad R(Y_4, Y_7)Y_7 = l^2(16\gamma\lambda_2^2)^{-1} Y_4, \\
R(Y_5, Y_7)Y_7 &= k^2(16\gamma\lambda_3^2)^{-1} Y_5, \quad R(Y_6, Y_7)Y_7 = k^2(16\gamma\lambda_3^2)^{-1} Y_6, \\
R(Y_8, Y_7)Y_7 &= 0, \quad R(Y_1, Y_8)Y_8 = (k - l)^2(48\lambda_1^2\gamma)^{-1} Y_1, \\
R(Y_2, Y_8)Y_8 &= (k - l)^2(48\lambda_1^2\gamma)^{-1} Y_2, \\
R(Y_3, Y_8)Y_8 &= (2k + l)^2(48\lambda_2^2\gamma)^{-1} Y_3, \\
R(Y_4, Y_8)Y_8 &= (2k + l)^2(48\lambda_2^2\gamma)^{-1} Y_4, \\
R(Y_5, Y_8)Y_8 &= (k + 2l)^2(48\lambda_3^2\gamma)^{-1} Y_5, \\
R(Y_6, Y_8)Y_8 &= (k + 2l)^2(48\lambda_3^2\gamma)^{-1} Y_6, \quad R(Y_7, Y_8)Y_8 = 0,
\end{aligned}$$

where $\beta = (48\lambda_1\lambda_2\lambda_3)^{-1}$.

In general, the Ricci tensor field Ric of type (0,2) on a Riemannian manifold (M, g) is defined by (cf. [2, 3, 9])

$$(2.8) \quad Ric(Y, Z) = Trace \{X \mapsto R(X, Y)Z\}, \quad (X, Y, Z \in \mathfrak{X}(M)).$$

From (2.7) and (2.8), we obtain

Lemma 2.1. *The Ricci tensors on $(SU(3), g_{(\lambda_1, \lambda_2, \lambda_3)})$ are given as follows:*

- (a) $r_1 := Ric(Y_1, Y_1) = Ric(Y_2, Y_2)$
 $= (2\lambda_1^3 - 2\lambda_1\lambda_2^2 - 2\lambda_3^2\lambda_1 + 12\lambda_1\lambda_2\lambda_3 - 4\lambda_2\lambda_3)(24\lambda_1^2\lambda_2\lambda_3)^{-1},$
- (b) $r_2 := Ric(Y_3, Y_3) = Ric(Y_4, Y_4)$
 $= (2\lambda_2^3 - 2\lambda_1^2\lambda_2 - 2\lambda_3^2\lambda_2 + 12\lambda_1\lambda_2\lambda_3 - 4\lambda_3\lambda_1)(24\lambda_1\lambda_2^2\lambda_3)^{-1},$
- (c) $r_3 := Ric(Y_5, Y_5) = Ric(Y_6, Y_6)$
 $= (2\lambda_3^3 - 2\lambda_2^2\lambda_3 - 2\lambda_3\lambda_1^2 + 12\lambda_1\lambda_2\lambda_3 - 4\lambda_1\lambda_2)(24\lambda_1\lambda_2\lambda_3^2)^{-1},$
- (d) $r_4 := Ric(Y_7, Y_7)$
 $= \{k^2\lambda_1^2\lambda_2^2 + (k+l)^2\lambda_2^2\lambda_3^2 + l^2\lambda_3^2\lambda_1^2\}(8\lambda_1^2\lambda_2^2\lambda_3^2\gamma)^{-1},$
- (e) $r_5 := Ric(Y_8, Y_8)$
 $= \{(k+2l)^2\lambda_1^2\lambda_2^2 + (k-l)^2\lambda_2^2\lambda_3^2 + (2k+l)^2\lambda_3^2\lambda_1^2\}(24\lambda_1^2\lambda_2^2\lambda_3^2\gamma)^{-1},$
- (f) $Ric(Y_7, Y_8) = \{-(k^2 + 2kl)\lambda_1^2\lambda_2^2 + (k^2 - l^2)\lambda_2^2\lambda_3^2 + (2kl + l^2)\lambda_3^2\lambda_1^2\} \cdot$
 $(8\sqrt{3}\lambda_1^2\lambda_2^2\lambda_3^2\gamma)^{-1},$
- (g) $Ric(Y_i, Y_j) = 0$; the others,

where $\gamma = k^2 + kl + l^2$.

The trace of the Ricci tensor field Ric of a Riemannian manifold (M, g) , (i.e., $\sum_j Ric(e_j, e_j)$, where $\{e_j\}_j$ is an (locally defined) orthonormal frame on (M, g)), is called the *scalar curvature* of (M, g) (cf. [12, 13]).

From Lemma 2.1, we obtain the following

Theorem 2.2. *The scalar curvature $S_{g_{(\lambda_1, \lambda_2, \lambda_3)}}$ of $(SU(3), g_{(\lambda_1, \lambda_2, \lambda_3)})$ is given as follows:*

$$(2.9) \quad S_{g_{(\lambda_1, \lambda_2, \lambda_3)}} = -(6\lambda_1^2\lambda_2^2\lambda_3^2)^{-1}\{(\lambda_1\lambda_2)^2 + (\lambda_2\lambda_3)^2 + (\lambda_3\lambda_1)^2\} \\ - (6\lambda_1\lambda_2\lambda_3)^{-1}(\lambda_1^2 + \lambda_2^2 + \lambda_3^2 - 6\lambda_1\lambda_2 - 6\lambda_2\lambda_3 - 6\lambda_3\lambda_1).$$

From Theorem 2.2, we get the following

Corollary 2.3. *A necessary and sufficient condition for the scalar curvature $S_{g(\lambda_1, \lambda_2, \lambda_3)}$ to be positive is*

$$(2.10) \quad (\lambda_1 \lambda_2 \lambda_3) \{ \lambda_1^2 + \lambda_2^2 + \lambda_3^2 - 6(\lambda_1 \lambda_2 + \lambda_2 \lambda_3 + \lambda_3 \lambda_1) \} \\ < -\{ (\lambda_1 \lambda_2)^2 + (\lambda_2 \lambda_3)^2 + (\lambda_3 \lambda_1)^2 \}.$$

The *Ricci curvature* r of a Riemannian manifold (M, g) with respect to a nonzero vector $v \in TM$ is defined by $r(v) := Ric(v, v)/\|v\|^2$, and a manifold of constant Ricci curvature, (i.e., $Ric = cg$ for some constant c), is called an *Einstein manifold* (cf. [2, 10, 11]).

If we put $\lambda_1 = \lambda_2 = \lambda_3$ in (f) of Lemma 2.1, then $Ric(Y_7, Y_8) = 0$. By virtue of this fact and Lemma 2.1, we obtain the following

Theorem 2.4. *Assume that $\lambda_1 = \lambda_2 = \lambda_3 =: \lambda$ in $(SU(3), g_{(\lambda_1, \lambda_2, \lambda_3)})$. Then, the Ricci curvature r on $(SU(3), g_{(\lambda_1, \lambda_2, \lambda_3)})$ is estimated as follows:*

- (a) $r_1 = r_2 = r_3$ and $r_4 = r_5$,
- (b) $0 < \lambda < 1$ if and only if $r_1 \leq r \leq r_4$,
- (c) $\lambda > 1$ if and only if $r_4 \leq r \leq r_1$,
- (d) If $\lambda = 1$, then $r = \frac{1}{4}$.

Remark 2.5. Assume that $\lambda_1 = \lambda_2 = \lambda_3 =: \lambda$ in $(SU(3), g_{(\lambda_1, \lambda_2, \lambda_3)})$. Then by (d) of Theorem 2.4, we see that $(SU(3), g_{(\lambda, \lambda, \lambda)})$ is an Einstein manifold with Einstein constant $\frac{1}{4}$. This fact is well known (cf. [2]).

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