SHEAF-THEORETIC APPROACH TO THE CONVOLUTION ALGEBRAS ON QUIVER VARIETIES

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Abstract. In this paper, we study a sheaf-theoretic analysis of the convolution algebra on quiver varieties. As by-products, we reinterpret the results of H. Nakajima. We also produce a refined form of the BBD decomposition theorem for quiver varieties. Finally, we study a construction of highest weight modules through constructible functions.

1. Introduction

Let $\mathfrak{g}$ be a symmetric Kac-Moody algebra, and let $\mathfrak{U}(\mathfrak{g})$ be the universal enveloping algebra of $\mathfrak{g}$. In [9, 10], the author introduced a new class of varieties which we call *quiver varieties*. These varieties are obtained from the representations of the framed quivers corresponding to the Cartan matrix of $\mathfrak{g}$. Nakajima used his quiver varieties to construct geometrically the finite dimensional irreducible representations of $\mathfrak{U}(\mathfrak{g})$ [10]. In Nakajima's geometric approach, the key ingredient was the construction of the algebra homomorphism

$$\Phi : \mathfrak{U}(\mathfrak{g}) \longrightarrow \bigoplus_{\mathbf{v}^1, \mathbf{v}^2} H_{top} (Z(\mathbf{v}^1, \mathbf{v}^2; \mathbf{w})) .$$

Here $Z(\mathbf{v}^1, \mathbf{v}^2; \mathbf{w})$ are certain Steinberg-type varieties which are obtained from the projective morphisms $\pi_{\mathbf{v}} : \mathfrak{M}(\mathbf{v}, \mathbf{w}) \longrightarrow \mathfrak{M}_0(\mathbf{v}, \mathbf{w})$ defined in [10, 3.18].

The homomorphism Φ is neither injective nor surjective in general. However, if we restrict our attention to a finite dimensional complex simple Lie algebra, then it turns out that the homomorphism Φ is surjective.

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This implies that any simple $\bigoplus_{\mathbf{v}^1, \mathbf{v}^2} H_{top}(Z(\mathbf{v}^1, \mathbf{v}^2; \mathbf{w}))$ -module yields an irreducible representation of $\mathfrak{U}(\mathfrak{g})$.

For the underlying quivers of finite ADE type, it is known from [9, Proposition 6.7] that the based variety $\mathfrak{M}_0(\mathbf{v}, \mathbf{w})$ of the projective morphism $\pi_{\mathbf{v}}$ has a wonderful stratification $\mathfrak{M}_0(\mathbf{v}, \mathbf{w}) = \sqcup_{\mathbf{v}' \leq \mathbf{v}} \mathfrak{M}_0^{reg}(\mathbf{v}', \mathbf{w})$, and the projective morphisms $\pi_{\mathbf{v}}$ become strictly semi-small (see [10, Corollary 10.11]). Hence, we can apply the Beilinson-Bernstein-Deligne decomposition theorem to the projective morphism $\pi : \sqcup_{\mathbf{v}} \mathfrak{M}(\mathbf{v}, \mathbf{w}) \longrightarrow \sqcup_{\mathbf{v}} \mathfrak{M}_0(\mathbf{v}, \mathbf{w})$ whose restriction to each $\mathfrak{M}(\mathbf{v}, \mathbf{w})$ is $\pi_{\mathbf{v}}$. More explicitly, we obtain the following nice decomposition for the projective morphism π :

$$(1.1) \quad \pi_* \mathcal{C} = \bigoplus_{\varphi = (\mathfrak{M}_0^{reg}(\mathbf{v}^\varphi, \mathbf{w}), \chi_\varphi)} L_\varphi \otimes IC_\varphi,$$

where $\mathcal{C}$ is the constant perverse sheaf on $\sqcup_{\mathbf{v}} \mathfrak{M}(\mathbf{v}, \mathbf{w})$ such that $\mathcal{C}|_{\mathfrak{M}(\mathbf{v}, \mathbf{w})} = \underline{\mathbb{C}}_{\mathfrak{M}(\mathbf{v}, \mathbf{w})}[dim_{\mathbb{C}} \mathfrak{M}(\mathbf{v}, \mathbf{w})]$, and the direct sum is taken over the pairs φ consisting of a stratum $\mathfrak{M}_0^{reg}(\mathbf{v}^\varphi, \mathbf{w})$ of $\mathfrak{M}_0(\mathbf{v}, \mathbf{w})$ and an irreducible local system χ_φ on $\mathfrak{M}_0^{reg}(\mathbf{v}^\varphi, \mathbf{w})$. We point out that we have no shifts for the intersection cohomology complexes in the decomposition (1.1).

On the other hand, if we apply the sheaf-theoretic analysis to the convolution algebra $\bigoplus_{\mathbf{v}^1, \mathbf{v}^2} H_{top}(Z(\mathbf{v}^1, \mathbf{v}^2; \mathbf{w}))$ then we can show that

$$(1.2) \quad \bigoplus_{\mathbf{v}^1, \mathbf{v}^2} H_{top}(Z(\mathbf{v}^1, \mathbf{v}^2; \mathbf{w})) \simeq \bigoplus_{\varphi} End(L_\varphi),$$

where $\{L_\varphi\}$ are the vector spaces that appear in (1.1) (see Theorem 4.2). Hence, the vector spaces L_φ form a list of simple $\mathfrak{U}(\mathfrak{g})$ -modules. This approach can be considered as a sheaf-theoretic version of Nakajima's theory that appeared in [10]. In particular, Equation (1.2) may be viewed as a sheaf-theoretic analogue of [10, Theorem 10.15].

It is well-known that the quiver varieties $\mathfrak{M}(\mathbf{v}, \mathbf{w})$ are connected [4]. By combining this fact with our sheaf-theoretic analysis of homology spaces over quiver varieties (see Theorem 4.1), we obtain a refined form of the decomposition (1.1). In more details, in this paper we obtain the following refined form of the decomposition (1.1):

$$\pi_* \mathcal{C} = \bigoplus_{\varphi = (\mathfrak{M}_0^{reg}(\mathbf{v}^\varphi, \mathbf{w}), \underline{\mathbb{C}}_{\mathfrak{M}_0^{reg}(\mathbf{v}^\varphi, \mathbf{w})})} L_\varphi \otimes IC_\varphi,$$

where $\underline{\mathbb{C}}_{\mathfrak{M}_0^{reg}(\mathbf{v}^\varphi, \mathbf{w})}$ is the constant local systems on each stratum $\mathfrak{M}_0^{reg}(\mathbf{v}^\varphi, \mathbf{w})$, and the direct sum is taken over the pairs consisting of each stratum

$\mathfrak{M}_0^{reg}(\mathbf{v}^\varphi, \mathbf{w})$ of $\mathfrak{M}_0(\mathbf{v}, \mathbf{w})$ and the locally constant system $\mathbb{C}_{\mathfrak{M}_0^{reg}(\mathbf{v}^\varphi, \mathbf{w})}$ on $\mathfrak{M}_0^{reg}(\mathbf{v}^\varphi, \mathbf{w})$.

In the last section of this paper, we treat constructible functions to study the representation theory of $\mathfrak{U}(\mathfrak{g})$. In [9], Nakajima constructed irreducible representations of $\mathfrak{U}(\mathfrak{g})$ by using the vector space of constructible functions over the fiber $\pi^{-1}(0)$. However, in this paper we take an arbitrary fiber $\pi^{-1}(x)$ and construct irreducible representations of $\mathfrak{U}(\mathfrak{g})$ by using the constructible functions defined over the fiber $\pi^{-1}(x)$.

2. Quiver varieties

Let $\mathfrak{g}$ be a finite dimensional complex simple Lie algebra of type ADE , and let (I, E) be the corresponding Dynkin diagram. Here I and E denote the sets of vertices and edges of the Dynkin diagram, respectively. Let $V = (V_i)_{i \in I}$ and $W = (W_i)_{i \in I}$ be collections of complex finite dimensional vector spaces attached to the set I . We write $\mathbf{v}$ and $\mathbf{w}$ for row vectors $(\dim V_i)_{i \in I}$ and $(\dim W_i)_{i \in I}$, respectively. Next, we denote by H the set of pairs consisting of an edge and its orientation. For $h \in H$, we write $out(h)$ (resp., $in(h)$) for outgoing (resp., ingoing) vertex of h . For the above two collections V and W of vector spaces, we define the following vector space:

$$\begin{aligned} M(\mathbf{v}, \mathbf{w}) = & \left(\bigoplus_{h \in H} \text{Hom}(V_{out(h)}, V_{in(h)}) \right) \oplus \left(\bigoplus_{k \in I} \text{Hom}(W_k, V_k) \right) \\ & \oplus \left(\bigoplus_{k \in I} \text{Hom}(V_k, W_k) \right). \end{aligned}$$

Each of three components of an element of $M(\mathbf{v}, \mathbf{w})$ will be denoted by $B = (B_h)_{h \in H}$, $i = (i_k)_{k \in I}$ and $j = (j_k)_{k \in I}$.

For $h \in H$, we write $\bar{h}$ to denote the same edge as h with the reverse orientation. By the orientation of the Dynkin diagram (I, E) we mean a subset $\Omega \subset H$ satisfying $\bar{\Omega} \cup \Omega = H$ and $\bar{\Omega} \cap \Omega = \emptyset$. For a fixed orientation Ω of (I, E) , we define a map $\epsilon : H \rightarrow \{\pm 1\}$ by

$$\epsilon(h) = \begin{cases} 1, & \text{if } h \in \Omega, \\ -1, & \text{if } h \in \bar{\Omega}. \end{cases}$$

Define an action of $G_{\mathbf{v}} = \prod_{k \in I} GL(V_k)$ on $M(\mathbf{v}, \mathbf{w})$ by

$$g \cdot (B, i, j) = (gBg^{-1}, gi, jg^{-1}),$$

and let $\mu : M(\mathbf{v}, \mathbf{w}) \longrightarrow \bigoplus_{k \in I} \text{End}(V_k)$ be a map whose k th component is given by $\mu_k(B, i, j) = \sum_{in(h)=k} \epsilon(h) B_h B_{\bar{h}} + i_k j_k$. Then, we define $\mathfrak{M}_0(\mathbf{v}, \mathbf{w})$ as the affine algebro-geometric quotient $\mu^{-1}(0)/G_{\mathbf{v}}$. This is an affine variety, and its geometric points are exactly closed G -orbits on $\mu^{-1}(0)$.

On the other hand, it is known that we may have another quotient of $\mu^{-1}(0)$ through the geometric invariant theory quotient [6, 11]. In more details, we first fix a character χ on $G_{\mathbf{v}}$ given by $\chi(g) = \prod_k \det(g_k^{-1})$. Then, we define $\mathfrak{M}(\mathbf{v}, \mathbf{w})$ as the geometric invariant theory quotient $\mu^{-1}(0)/_{\chi} G_{\mathbf{v}}$ of $\mu^{-1}(0)$. We call this quotient a *quiver variety*. It is known that the quiver variety $\mathfrak{M}(\mathbf{v}, \mathbf{w})$ is a nonsingular quasi projective variety.

We say that an element $(B, i, j) \in \mu^{-1}(0)$ is *stable* if each subspace $S = (S_k)$ of $V = (V_k)$, which is invariant under (B_h) (i.e., $B_h(S_{out(h)}) \subset S_{in(h)}$) and contained in $\ker j$ (i.e., $S_k \subset \ker j_k$), is actually equal to 0. We denote by $\mu^{-1}(0)^s$ the set of all stable points in $\mu^{-1}(0)$. We recall that the geometric points of $\mathfrak{M}(\mathbf{v}, \mathbf{w})$ consists of $\mu^{-1}(0)^s/G$, that is G -orbit spaces on $\mu^{-1}(0)^s$.

If $(B, i, j) \in \mu^{-1}(0)$, then we denote by $[B, i, j]$ the corresponding geometric point in $\mathfrak{M}_0(\mathbf{v}, \mathbf{w})$. We will use the same notation $[B, i, j]$ to denote the geometric point of $\mathfrak{M}(\mathbf{v}, \mathbf{w})$ if $(B, i, j) \in \mu^{-1}(0)^s$.

Notice that we have a natural projective morphism

$$(2.1) \quad \pi_{\mathbf{v}} : \mathfrak{M}(\mathbf{v}, \mathbf{w}) \longrightarrow \mathfrak{M}_0(\mathbf{v}, \mathbf{w})$$

which sends an orbit $[B, i, j]$ to the unique closed orbit $[B_0, i_0, j_0]$ contained in $\overline{[B, i, j]}$.

Let us now consider two quiver varieties $\mathfrak{M}(\mathbf{v}^1, \mathbf{w})$ and $\mathfrak{M}(\mathbf{v}^2, \mathbf{w})$. Then, we obtain projective morphisms $\pi_{\mathbf{v}^i} : \mathfrak{M}(\mathbf{v}^i, \mathbf{w}) \longrightarrow \mathfrak{M}_0(\mathbf{v}^i, \mathbf{w})$ for $i = 1, 2$. Since we have natural inclusions $\mathfrak{M}_0(\mathbf{v}^i, \mathbf{w}) \subset \mathfrak{M}_0(\mathbf{v}^1 + \mathbf{v}^2, \mathbf{w})$ for $i = 1, 2$, each $\pi_{\mathbf{v}^i}$ can be regarded as a morphism

$$(2.2) \quad \pi_{\mathbf{v}^i} : \mathfrak{M}(\mathbf{v}^i, \mathbf{w}) \longrightarrow \mathfrak{M}_0(\mathbf{v}^1 + \mathbf{v}^2, \mathbf{w}).$$

For the projective morphism $\pi_{\mathbf{v}}$ in (2.1), we write $\mathfrak{M}(\mathbf{v}, \mathbf{w})_x$ for $\pi_{\mathbf{v}}^{-1}(x)$. In particular, we denote by $\mathfrak{L}(\mathbf{v}, \mathbf{w})$ the fiber $\pi_{\mathbf{v}}^{-1}(0)$.

For the affine varieties $\mathfrak{M}_0(\mathbf{v}, \mathbf{w})$ arising from the Dynkin diagrams of finite dimensional simple Lie algebras of type *ADE*, it is known that there exists a stratification

$$(2.3) \quad \mathfrak{M}_0(\mathbf{v}, \mathbf{w}) = \sqcup_{\mathbf{v}' \leq \mathbf{v}} \mathfrak{M}_0^{reg}(\mathbf{v}', \mathbf{w}),$$

where $\mathfrak{M}_0^{reg}(\mathbf{v}', \mathbf{w})$ is an open subset of $\mathfrak{M}_0(\mathbf{v}', \mathbf{w})$ consisting of the points $[B, i, j]$ that have the trivial stabilizer in $G_{\mathbf{v}'}$ (see [10, Lemma 3.27]).

For a fixed $\mathbf{w}$, let us define the following projective morphism

$$(2.4) \quad \pi : \sqcup_{\mathbf{v}} \mathfrak{M}(\mathbf{v}, \mathbf{w}) \longrightarrow \cup_{\mathbf{v}} \mathfrak{M}_0(\mathbf{v}, \mathbf{w}) = \sqcup_{\mathbf{v}'} \mathfrak{M}_0^{reg}(\mathbf{v}', \mathbf{w})$$

whose restriction to each $\mathfrak{M}(\mathbf{v}, \mathbf{w})$ is $\pi_{\mathbf{v}}$.

Then, π is a topological fiber bundle over each stratum $\mathfrak{M}_0^{reg}(\mathbf{v}', \mathbf{w})$ [10]. Furthermore, according to [10, Corollary 10.11] the projective morphism π is a strictly semi-small morphism with respect to the stratification given by (2.3).

3. Nakajima's construction

In the remaining part of this paper, we fix a dimension vector $\mathbf{w} = (\dim W_i)_{i \in I}$. We also assume that all quiver varieties $\mathfrak{M}(\mathbf{v}, \mathbf{w})$ are obtained from the Dynkin diagrams (I, E) of finite dimensional complex simple Lie algebras of type ADE . Also, $H_i(X)$ will always denote the Borel-Moore homology with $\mathbb{C}$ -coefficients and $H_i^{ord}(X)$ stands for the ordinary homology. In particular, $H_{top}(X)$ indicates the Borel-Moore homology at the degree $\dim_{\mathbb{R}} X$. We refer to [2, 7] for the details of the Borel-Moore homology.

Let $\mathfrak{g}$ be a finite dimensional complex simple Lie algebra of type ADE , and let $\text{rank}(\mathfrak{g}) = n$. Then, the following data are attached to $\mathfrak{g}$ with respect to a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$.

1. simple roots $\alpha_k \in \mathfrak{h}^*$ ($k = 1, 2, \dots, n$).
2. simple coroots $h_k \in \mathfrak{h}$ ($k = 1, 2, \dots, n$).
3. fundamental weights $\Lambda_k \in \mathfrak{h}^*$ ($k = 1, 2, \dots, n$) such that $\Lambda_k(h_l) = \delta_{kl}$.
4. Cartan matrix $C = (\alpha_j(h_i))$.

We set $\alpha_{\mathbf{v}} = \sum_{k=1}^n (\dim V_k) \alpha_k$ and $\Lambda_{\mathbf{w}} = \sum_{k=1}^n (\dim W_k) \Lambda_k$ for given dimension vectors $\mathbf{v} = (\dim V_1, \dots, \dim V_n)$ and $\mathbf{w} = (\dim W_1, \dots, \dim W_n)$. We also write $\mathbf{e}^k$ for the dimension vector such that k th entry is 1 and the other entries are 0.

Lemma 3.1. *Let $\mathfrak{g}$ be a finite dimensional complex simple Lie algebra of type ADE , and let $\mathfrak{M}(\mathbf{v}, \mathbf{w})$ be a quiver variety obtained from the Dynkin diagram of $\mathfrak{g}$. Then, $\mathfrak{M}(\mathbf{v}, \mathbf{w})$ is empty for all but finitely many choices of $\mathbf{v}$.*

Proof. We first notice that $\mathfrak{M}(\mathbf{v}, \mathbf{w}) \neq \emptyset$ if and only if $\mathfrak{L}(\mathbf{v}, \mathbf{w}) \neq \emptyset$.

Recall that $\bigoplus_{\mathbf{v}'} H_{top}(\mathfrak{L}(\mathbf{v}', \mathbf{w}))$ is the irreducible highest weight $\mathfrak{U}(\mathfrak{g})$ -module, and each direct summand $H_{top}(\mathfrak{L}(\mathbf{v}', \mathbf{w}))$ has a weight $\Lambda_{\mathbf{w}} - \alpha_{\mathbf{v}'}$ [9, 10].

So, $\bigoplus_{\mathbf{v}'} H_{top}(\mathfrak{L}(\mathbf{v}', \mathbf{w}))$ has the weight space $H_{top}(\mathfrak{L}(\mathbf{v}, \mathbf{w}))$ of weight $\Lambda_{\mathbf{w}} - \alpha_{\mathbf{v}}$ if $\mathfrak{L}(\mathbf{v}, \mathbf{w}) \neq \emptyset$. According to the basic representation theory of finite dimensional simple Lie algebras, every integrable highest weight $\mathfrak{U}(\mathfrak{g})$ -module is finite dimensional. Thus the direct sum $\bigoplus_{\mathbf{v}'} H_{top}(\mathfrak{L}(\mathbf{v}', \mathbf{w}))$ is a finite direct sum. The lemma now follows. $\square$

For two dimension vectors $\mathbf{v}^1$ and $\mathbf{v}^2$ satisfying $\mathbf{v}^1 = \mathbf{v}^2 - \mathbf{e}^k$, we consider two quiver varieties $\mathfrak{M}(\mathbf{v}^1, \mathbf{w})$ and $\mathfrak{M}(\mathbf{v}^2, \mathbf{w})$. From these two quiver varieties, we define the Hecke correspondence $\mathfrak{B}_k(\mathbf{v}^2, \mathbf{w})$ as follows: a point $([B^1, i^1, j^1], [B^2, i^2, j^2])$ of $\mathfrak{M}(\mathbf{v}^1, \mathbf{w}) \times \mathfrak{M}(\mathbf{v}^2, \mathbf{w})$ is an element of $\mathfrak{B}_k(\mathbf{v}^2, \mathbf{w})$ if and only if there exists $\zeta \in \bigoplus_{k \in I} \text{Hom}(V_k^1, V_k^2)$ such that

$$(3.1) \quad \zeta B^1 = B^2 \zeta, \quad \zeta i^1 = i^2, \quad j^1 = j^2 \zeta.$$

The stability condition implies that such ζ exists uniquely if it exists [10].

In virtue of the morphisms (2.2), we have the following Steinberg-type variety for given two dimension vectors $\mathbf{v}^1 = \mathbf{v}^2 - \mathbf{e}^k$ and $\mathbf{v}^2$:

$$Z(\mathbf{v}^1, \mathbf{v}^2; \mathbf{w}) = \{(x^1, x^2) \in \mathfrak{M}(\mathbf{v}^1, \mathbf{w}) \times \mathfrak{M}(\mathbf{v}^2, \mathbf{w}) \mid \pi_{\mathbf{v}^1}(x^1) = \pi_{\mathbf{v}^2}(x^2)\}.$$

From Equations (3.1), it is easy to see that $\mathfrak{B}_k(\mathbf{v}^2, \mathbf{w})$ is a subvariety of $Z(\mathbf{v}^1, \mathbf{v}^2; \mathbf{w})$. In fact, $\mathfrak{B}_k(\mathbf{v}, \mathbf{w})$ is an irreducible component of $Z(\mathbf{v}^1, \mathbf{v}^2; \mathbf{w})$.

Let $\omega : \mathfrak{M}(\mathbf{v}^1, \mathbf{w}) \times \mathfrak{M}(\mathbf{v}^2, \mathbf{w}) \rightarrow \mathfrak{M}(\mathbf{v}^2, \mathbf{w}) \times \mathfrak{M}(\mathbf{v}^1, \mathbf{w})$ be the morphism which interchange the two factors, and let

$$r_k(\mathbf{v}, \mathbf{w}) = \frac{1}{2} (\dim_{\mathbb{C}} \mathfrak{M}(\mathbf{v}^1, \mathbf{w}) - \dim_{\mathbb{C}} \mathfrak{M}(\mathbf{v}^2, \mathbf{w})).$$

Then, we define

$$\begin{aligned} E_k &= \sum_{\mathbf{v}} [\mathfrak{B}_k(\mathbf{v}, \mathbf{w})], \\ F_k &= \sum_{\mathbf{v}} (-1)^{r_k(\mathbf{v}, \mathbf{w})} [\omega(\mathfrak{B}_k(\mathbf{v}, \mathbf{w}))], \\ H_k &= \sum_{\mathbf{v}} (\Lambda_{\mathbf{w}} - \alpha_{\mathbf{v}})(h_k) [\Delta(\mathbf{v}, \mathbf{w})], \end{aligned}$$

where $\Delta(\mathbf{v}, \mathbf{w})$ denotes the diagonal in $\mathfrak{M}(\mathbf{v}, \mathbf{w}) \times \mathfrak{M}(\mathbf{v}, \mathbf{w})$.

According to Lemma 3.1, the elements E_k, F_k and H_k are finite sums so that they are well-defined elements in $\bigoplus_{\mathbf{v}^1, \mathbf{v}^2} H_{top}(Z(\mathbf{v}^1, \mathbf{v}^2; \mathbf{w}))$.

We recall that $\bigoplus_{\mathbf{v}^1, \mathbf{v}^2} H_{top}(Z(\mathbf{v}^1, \mathbf{v}^2; \mathbf{w}))$ is a $\mathbb{C}$ -algebra equipped with the convolution product, and this algebra yields the following convolution action

$$(3.2) \quad \star : H_{top}(Z(\mathbf{v}^1, \mathbf{v}^2, \mathbf{w})) \otimes H_{top}(\mathfrak{M}(\mathbf{v}^2, \mathbf{w})_x) \longrightarrow H_{top}(\mathfrak{M}(\mathbf{v}^1, \mathbf{w})_x)$$

which is due to the strictly semi-small morphism π in (2.4). We refer to [10, Section 8] for the details of the action (3.2).

- Theorem 3.2.** 1. *There exists a unique algebra homomorphism $\Phi : \mathfrak{U}(\mathfrak{g}) \longrightarrow \bigoplus_{\mathbf{v}^1, \mathbf{v}^2} H_{top}(Z(\mathbf{v}^1, \mathbf{v}^2; \mathbf{w}))$ such that $\Phi(e_k) = E_k$, $\Phi(f_k) = F_k$ and $\Phi(h_k) = H_k$, where $\{e_k, f_k, h_k\}_{k=1, \dots, n}$ is the set of Chevalley generators of $\mathfrak{g}$.*
2. *Let $x \in \mathfrak{M}_0^{reg}(\mathbf{v}^0, \mathbf{w})$. Then, $\bigoplus_{\mathbf{v}} H_{top}(\mathfrak{M}(\mathbf{v}, \mathbf{w})_x)$ is the irreducible highest weight $\mathfrak{U}(\mathfrak{g})$ -module with the highest weight vector $[\mathfrak{M}(\mathbf{v}^0, \mathbf{w})_x]$.*
3. *The algebra homomorphism Φ in (1) is surjective.*

Proof. See [10, Theorem 9.4, Theorem 10.2, Theorem 10.15] for the proofs. $\square$

Since the homomorphism Φ is surjective, we obtain the following corollary.

Corollary 3.3. *Any simple $\bigoplus_{\mathbf{v}^1, \mathbf{v}^2} H_{top}(Z(\mathbf{v}^1, \mathbf{v}^2; \mathbf{w}))$ -module yields an irreducible representation of $\mathfrak{U}(\mathfrak{g})$.*

4. sheaf-theoretic approach

In this section, we apply the Beilinson-Bernstein-Deligne decomposition theorem to the morphism π in (2.4) (see [1] for the details of the decomposition theorem). In particular, in our case where π is strictly semi-small, we have the following nice form of the Beilinson-Bernstein-Deligne decomposition theorem:

$$(4.1) \quad \pi_* \mathcal{C} = \bigoplus_{\varphi = (\mathfrak{M}_0^{reg}(\mathbf{v}^\varphi, \mathbf{w}), \chi_\varphi)} L_\varphi \otimes IC_\varphi,$$

where $\mathcal{C}$ is the constant perverse sheaf on $\sqcup_{\mathbf{v}} \mathfrak{M}(\mathbf{v}, \mathbf{w})$ such that $\mathcal{C}|_{\mathfrak{M}(\mathbf{v}, \mathbf{w})} = \underline{\mathbb{C}}_{\mathfrak{M}(\mathbf{v}, \mathbf{w})}[\dim_{\mathbb{C}} \mathfrak{M}(\mathbf{v}, \mathbf{w})]$, and the direct sum is taken over the pairs

consisting of a stratum $\mathfrak{M}_0^{reg}(\mathbf{v}^\varphi, \mathbf{w})$ of $\mathfrak{M}_0(\mathbf{v}, \mathbf{w})$ and an irreducible local system χ_φ on $\mathfrak{M}_0^{reg}(\mathbf{v}^\varphi, \mathbf{w})$.

According to [9, Corollary 5.5], we have a homotopy equivalence $i^\mathbf{v} : \mathfrak{L}(\mathbf{v}, \mathbf{w}) \hookrightarrow \mathfrak{M}(\mathbf{v}, \mathbf{w})$. This homotopy equivalence is proper because $\mathfrak{L}(\mathbf{v}, \mathbf{w})$ is compact. Thus, $i^\mathbf{v}$ yields the direct image map between induced Borel-Moore homology spaces.

Let $i_* : \bigoplus_{\mathbf{v}} H_{2d_{\mathbf{v}}}(\mathfrak{L}(\mathbf{v}, \mathbf{w})) \longrightarrow \bigoplus_{\mathbf{v}} H_{2d_{\mathbf{v}}}(\mathfrak{M}(\mathbf{v}, \mathbf{w}))$ be a direct sum of the direct image maps between the Borel-Moore homologies. That is, we have $i_* = \bigoplus_{\mathbf{v}} i_*^\mathbf{v}$ for $i_*^\mathbf{v} : H_{2d_{\mathbf{v}}}(\mathfrak{L}(\mathbf{v}, \mathbf{w})) \longrightarrow H_{2d_{\mathbf{v}}}(\mathfrak{M}(\mathbf{v}, \mathbf{w}))$.

Theorem 4.1. *Let us keep the same notations as above. Then, we have*

$$Im(i_*) = \bigoplus_{\mathbf{v}} H_{2d_{\mathbf{v}}}(\mathfrak{L}(\mathbf{v}, \mathbf{w})) = L_{(\{0\}, \mathbb{C}_{\{0\}})}.$$

Proof. We notice that $\mathfrak{M}_0^{reg}(0, \mathbf{w}) = \{0\}$. That is, $\mathfrak{M}_0^{reg}(0, \mathbf{w})$ is an one point stratum. Let $i_{\{0\}} : \{0\} \hookrightarrow \mathfrak{M}_0(\mathbf{v}, \mathbf{w})$ be the embedding of the one point stratum $\mathfrak{M}_0^{reg}(0, \mathbf{w})$. Then, by [3, Lemma 8.5.4] and the Beilinson-Bernstein-Deligne decomposition theorem we have

$$(4.2) \quad H_k(\mathfrak{L}(\mathbf{v}, \mathbf{w})) = H^{2d_{\mathbf{v}}-k} \left(i_{\{0\}}^! \pi_{\mathbf{v}*} \mathbb{C}_{\mathfrak{M}(\mathbf{v}, \mathbf{w})} [2d_{\mathbf{v}}] \right)$$

$$(4.3) \quad = \bigoplus_{\varphi} L_{\varphi} \otimes H^{2d_{\mathbf{v}}-k} \left(i_{\{0\}}^! IC_{\varphi} \right),$$

where $\varphi = (\mathfrak{M}_0^{reg}(\mathbf{v}^\varphi, \mathbf{w}), \chi_{\varphi})$ is a pair associated with both $\mathbf{v}^\varphi \leq \mathbf{v}$ and an irreducible local system χ_{φ} on $\mathfrak{M}_0^{reg}(\mathbf{v}^\varphi, \mathbf{w})$.

Similarly, due to [3, Lemma 8.5.4] and the Beilinson-Bernstein-Deligne decomposition theorem, we obtain

$$\begin{aligned} H_k(\mathfrak{M}(\mathbf{v}, \mathbf{w})) &= H^{4d_{\mathbf{v}}-k}(\mathfrak{M}(\mathbf{v}, \mathbf{w})) \\ &= H^{4d_{\mathbf{v}}-k}(\mathfrak{L}(\mathbf{v}, \mathbf{w})) \\ &= \bigoplus_{\varphi} L_{\varphi} \otimes H^{2d_{\mathbf{v}}-k} \left(i_{\{0\}}^* IC_{\varphi} \right), \end{aligned}$$

where φ is the same as that of (4.3). Thus, we can identify

$$i_*^\mathbf{v} : H_{2d_{\mathbf{v}}}(\mathfrak{L}(\mathbf{v}, \mathbf{w})) \longrightarrow H_{2d_{\mathbf{v}}}(\mathfrak{M}(\mathbf{v}, \mathbf{w}))$$

with a direct sum of morphisms

$$\Phi_{\varphi} : L_{\varphi} \otimes H^0 \left(i_{\{0\}}^! IC_{\varphi} \right) \longrightarrow L_{\varphi} \otimes H^0 \left(i_{\{0\}}^* IC_{\varphi} \right),$$

where $\varphi = (\mathfrak{M}_0^{reg}(\mathbf{v}^\varphi, \mathbf{w}), \chi_{\varphi})$ for $\mathbf{v}^\varphi \leq \mathbf{v}$.

Recall from [3, Lemma 8.5.3] that $H^0(i_{\{0\}}^! IC_\varphi) = 0$ for $\varphi = (\mathfrak{M}_0^{reg}(\mathbf{v}^\varphi, \mathbf{w}), \chi_\varphi)$ satisfying $\mathfrak{M}_0^{reg}(\mathbf{v}^\varphi, \mathbf{w}) \neq \mathfrak{M}_0^{reg}(0, \mathbf{w})$. Moreover, if $\mathfrak{M}_0^{reg}(\mathbf{v}^\varphi, \mathbf{w}) = \mathfrak{M}_0^{reg}(0, \mathbf{w})$ then the corresponding morphisms Φ_φ is an isomorphism.

Hence, we obtain that $\ker(i_*^\mathbf{v}) = \{0\}$. Furthermore, we notice that $H^0(i_{\{0\}}^* IC_\varphi) = \mathbb{C}$ for $\varphi = (\mathfrak{M}_0^{reg}(0, \mathbf{w}), \mathbb{C}_{\{0\}})$. The theorem is now immediate. $\square$

Henceforth, we write $\mathfrak{M}(\mathbf{w}) = \sqcup_{\mathbf{v}} \mathfrak{M}(\mathbf{v}, \mathbf{w})$, $\mathfrak{L}(\mathbf{w}) = \sqcup_{\mathbf{v}} \mathfrak{L}(\mathbf{v}, \mathbf{w})$ and $Z(\mathbf{w}) = \sqcup_{\mathbf{v}^1, \mathbf{v}^2} Z(\mathbf{v}^1, \mathbf{v}^2; \mathbf{w})$.

We now introduce a new $\mathbb{Z}$ -grading on the Borel-Moore homology to preserve gradings under the convolution product. More explicitly, we set

$$H_{[p]}(\mathfrak{L}(\mathbf{v}, \mathbf{w})) := H_{2d_{\mathbf{v}}-p}(\mathfrak{L}(\mathbf{v}, \mathbf{w})),$$

$$H_{[p]}(\mathfrak{L}(\mathbf{w})) := \bigoplus_{\mathbf{v}} H_{[p]}(\mathfrak{L}(\mathbf{v}, \mathbf{w})),$$

$$H_{[p]}(Z(\mathbf{v}^1, \mathbf{v}^2; \mathbf{w})) := H_{m_{\mathbf{v}^1}+m_{\mathbf{v}^2}-p}(Z(\mathbf{v}^1, \mathbf{v}^2; \mathbf{w}))$$

and

$$H_{[p]}(Z(\mathbf{w})) := \bigoplus_{\mathbf{v}^1, \mathbf{v}^2} H_{[p]}(Z(\mathbf{v}^1, \mathbf{v}^2; \mathbf{w})),$$

where $d_{\mathbf{v}} = \dim_{\mathbb{C}} \mathfrak{L}(\mathbf{v}, \mathbf{w})$ and $m_{\mathbf{v}^i} = \dim_{\mathbb{C}} \mathfrak{M}(\mathbf{v}^i, \mathbf{w})$.

Then, by the definition of the convolution product we have

$$H_{[p]}(Z(\mathbf{v}^1, \mathbf{v}^2; \mathbf{w})) \star H_{[q]}(Z(\mathbf{v}^2, \mathbf{v}^3; \mathbf{w})) \subset H_{[p+q]}(Z(\mathbf{v}^1, \mathbf{v}^3; \mathbf{w})),$$

and this implies that

$$H_{[p]}(Z(\mathbf{w})) \star H_{[q]}(Z(\mathbf{w})) \subset H_{[p+q]}(Z(\mathbf{w})).$$

Let $D^b(sh(X))$ be the bounded derived category of sheaves on an algebraic variety X . Then, we write $D^b(X)$ for the full subcategory of $D^b(sh(X))$ formed by constructible complexes. For any complexes $A^\bullet, B^\bullet \in D^b(X)$, $Ext_{D^b(X)}^k(A^\bullet, B^\bullet)$ is defined as $Hom_{D^b(X)}(A^\bullet, B^\bullet[k])$.

With the new gradings defined in above, we obtain the following theorem.

Theorem 4.2. 1. *The convolution action makes $\bigoplus_{p \geq 0} H_{[p]}(\mathfrak{L}(\mathbf{w}))$ a graded $\bigoplus_{p \geq 0} H_{[p]}(Z(\mathbf{w}))$ -module with respect to the new gradings defined in above.*

2. We have a graded algebra isomorphism

$$\bigoplus_{p \geq 0} (H_{[p]}(Z(\mathbf{w}))) \simeq \bigoplus_{p \geq 0} \left(\bigoplus_{\varphi, \psi} \text{Hom}_{\mathbb{C}}(L_{\varphi}, L_{\psi}) \otimes \text{Ext}_{D^b(\cup_{\mathbf{v}} \mathfrak{M}_0(\mathbf{v}, \mathbf{w}))}^p(IC_{\varphi}, IC_{\psi}) \right),$$

where φ and ψ are the pairs arising from the decomposition (4.1).
In particular, we obtain that

$$(4.4) \quad \bigoplus_{\mathbf{v}^1, \mathbf{v}^2} H_{top}(Z(\mathbf{v}^1, \mathbf{v}^2; \mathbf{w})) \simeq \bigoplus_{\varphi} \text{End}(L_{\varphi}).$$

Proof. The first part of the theorem is obtained directly from the definition of the convolution product. In more details, we consider the convolution product under the circumstances $M_1 = \mathfrak{M}(\mathbf{v}^1, \mathbf{w})$, $M_2 = \mathfrak{M}(\mathbf{v}^2, \mathbf{w})$, $M_3 = \{\text{point}\}$, $Z(\mathbf{v}^1, \mathbf{v}^2; \mathbf{w}) \subset M_1 \times M_2$ and $\mathfrak{L}(\mathbf{v}^2, \mathbf{w}) \subset M_2 \times M_3$. Then, by the definition of the convolution product, we obtain that

$$H_{m_{\mathbf{v}^1} + m_{\mathbf{v}^2} - p}(Z(\mathbf{v}^1, \mathbf{v}^2; \mathbf{w})) \star H_{m_{\mathbf{v}^2} - q}(\mathfrak{L}(\mathbf{v}^2, \mathbf{w})) \subset H_{m_{\mathbf{v}^1} - (p+q)}(\mathfrak{L}(\mathbf{v}^1, \mathbf{w})).$$

This implies that

$$H_{[p]}(Z(\mathbf{v}^1, \mathbf{v}^2; \mathbf{w})) \star H_{[q]}(\mathfrak{L}(\mathbf{v}^2, \mathbf{w})) \subset H_{[p+q]}(\mathfrak{L}(\mathbf{v}^1, \mathbf{w})).$$

We now prove the second part of the theorem.

Let $\pi : \mathfrak{M}(\mathbf{w}) \rightarrow \cup_{\mathbf{v}} \mathfrak{M}_0(\mathbf{v}, \mathbf{w})$ be the projective morphism defined in (2.4). Then, by [3, Lemma 8.6.1] and the decomposition (4.1), we obtain that

$$\begin{aligned} H_{[p]}(Z(\mathbf{w})) &\simeq \text{Ext}_{D^b(\cup_{\mathbf{v}} \mathfrak{M}_0(\mathbf{v}, \mathbf{w}))}^p(\pi_* \mathcal{C}, \pi_* \mathcal{C}) \\ &\simeq \text{Ext}_{D^b(\cup_{\mathbf{v}} \mathfrak{M}_0(\mathbf{v}, \mathbf{w}))}^p \left(\bigoplus_{\varphi} L_{\varphi} \otimes IC_{\varphi}, \bigoplus_{\psi} L_{\psi} \otimes IC_{\psi} \right) \\ &\simeq \bigoplus_{\varphi, \psi} \text{Hom}_{\mathbb{C}}(L_{\varphi}, L_{\psi}) \otimes \text{Ext}_{D^b(\cup_{\mathbf{v}} \mathfrak{M}_0(\mathbf{v}, \mathbf{w}))}^p(IC_{\varphi}, IC_{\psi}), \end{aligned}$$

where $\varphi = (\mathfrak{M}_0^{reg}(\mathbf{v}^{\varphi}, \mathbf{w}), \chi_{\varphi})$ (resp., $\psi = (\mathfrak{M}_0^{reg}(\mathbf{v}^{\psi}, \mathbf{w}), \chi_{\psi})$) is a pair associated with an irreducible local system χ_{φ} (resp., χ_{ψ}) on $\mathfrak{M}_0^{reg}(\mathbf{v}^{\varphi}, \mathbf{w})$ (resp., $\mathfrak{M}_0^{reg}(\mathbf{v}^{\psi}, \mathbf{w})$).

Since the Yoneda product is compatible with the convolution, the above isomorphisms yield a graded algebra isomorphism. In addition, we also have

$$\begin{aligned} \bigoplus_{\mathbf{v}^1, \mathbf{v}^2} H_{top}(Z(\mathbf{v}^1, \mathbf{v}^2; \mathbf{w})) &= \bigoplus_{\mathbf{v}^1, \mathbf{v}^2} H_{m_{\mathbf{v}^1} + m_{\mathbf{v}^2}}(Z(\mathbf{v}^1, \mathbf{v}^2; \mathbf{w})) \\ &= H_{[0]}(Z(\mathbf{w})). \end{aligned}$$

The theorem is now immediate because $Ext_{D^b(\cup_{\mathbf{v}} \mathfrak{M}_0(\mathbf{v}, \mathbf{w}))}^0(IC_\varphi, IC_\psi) = \mathbb{C}\delta_{\varphi, \psi}$. $\square$

By Equation (4.4), each L_φ is equipped with $\bigoplus_{\mathbf{v}^1, \mathbf{v}^2} H_{top}(Z(\mathbf{v}^1, \mathbf{v}^2; \mathbf{w}))$ -module structure.

Corollary 4.3. *The vector spaces L_φ in (4.4) are simple $\mathfrak{U}(\mathfrak{g})$ -modules.*

Proof. Notice that all simple modules of the semisimple algebra $\bigoplus_\varphi End(L_\varphi)$ are of the form L_φ . Thus, the collection $\{L_\varphi\}$ is a complete list of simple $\bigoplus_{\mathbf{v}^1, \mathbf{v}^2} H_{top}(Z(\mathbf{v}^1, \mathbf{v}^2; \mathbf{w}))$ -modules. The corollary now follows from Corollary 3.3. $\square$

Remark 4.4. 1. Equation (4.4) may be viewed as a sheaf-theoretic analogue of [10, Theorem 10.15].

2. Let Φ be the surjective algebra homomorphism in Theorem 3.2, and let us denote by $J(R)$ the Jacobson radical of a ring R . Then, we obtain that $J(\mathfrak{U}(\mathfrak{g})) \subset Ker(\Phi)$ because

$$\Phi(J(\mathfrak{U}(\mathfrak{g}))) \subset J\left(\bigoplus_{\mathbf{v}^1, \mathbf{v}^2} H_{top}(Z(\mathbf{v}^1, \mathbf{v}^2; \mathbf{w}))\right) = \{0\}.$$

For this reason, in general we may have a simple $\mathfrak{U}(\mathfrak{g})$ -module that does not appear on the complete list $\{L_\varphi\}$ of simple $\bigoplus_{\mathbf{v}^1, \mathbf{v}^2} H_{top}(Z(\mathbf{v}^1, \mathbf{v}^2; \mathbf{w}))$ -modules.

In the following theorem, we provide a refined form of the decomposition (4.1).

Theorem 4.5. *Let π be the projective morphism defined in (2.4). Then, the intersection cohomology complexes that appear in the decomposition (4.1) are exactly those associated with the constant local system $\mathbb{C}_{\mathfrak{M}_0^{reg}(\mathbf{v}^0, \mathbf{w})}$ on each stratum $\mathfrak{M}_0^{reg}(\mathbf{v}^0, \mathbf{w})$.*

In other words, the decomposition (4.1) can be refined as

$$\pi_* \mathcal{C} = \bigoplus_{\varphi = \left(\mathfrak{M}_0^{reg}(\mathbf{v}^\varphi, \mathbf{w}), \mathbb{C}_{\mathfrak{M}_0^{reg}(\mathbf{v}^\varphi, \mathbf{w})} \right)} L_\varphi \otimes IC_\varphi.$$

Proof. We first notice that any nonempty stratum $\mathfrak{M}_0^{reg}(\mathbf{v}, \mathbf{w})$ is irreducible because $\mathfrak{M}_0^{reg}(\mathbf{v}, \mathbf{w})$ is an open subset of $\mathfrak{M}_0(\mathbf{v}, \mathbf{w})$ [4]. So, $\mathbb{C}_{\mathfrak{M}_0^{reg}(\mathbf{v}^\varphi, \mathbf{w})}$ is an irreducible local system on each nonempty stratum $\mathfrak{M}_0^{reg}(\mathbf{v}^\varphi, \mathbf{w})$.

Let us now take $x \in \mathfrak{M}_0^{reg}(\mathbf{v}^0, \mathbf{w})$.

By applying the same argument as the proof of Theorem 4.1 to the embedding $j : \mathfrak{M}_0^{reg}(\mathbf{v}, \mathbf{w}) \hookrightarrow \mathfrak{M}_0(\mathbf{v}, \mathbf{w})$, we have

$$(4.5) \quad \bigoplus_{\mathbf{v}} H_{top}(\mathfrak{M}(\mathbf{v}, \mathbf{w})_x) = \bigoplus_{\varphi} L_{\varphi},$$

where L_{φ} are vector spaces that appear in the decomposition (4.1) and φ runs over such $(\mathfrak{M}_0^{reg}(\mathbf{v}^0, \mathbf{w}), \chi)$ that consist of the fixed stratum $\mathfrak{M}_0^{reg}(\mathbf{v}^0, \mathbf{w})$ containing x and irreducible local systems χ attached to $\mathfrak{M}_0^{reg}(\mathbf{v}^0, \mathbf{w})$.

However, by Theorem 3.2, the left hand side of (4.5) is an irreducible $\mathfrak{U}(\mathfrak{g})$ -module. This implies that we should have only one direct summand on the right hand side of (4.5) because $\{L_{\varphi}\}$ is a set of simple $\mathfrak{U}(\mathfrak{g})$ -modules due to Corollary 4.3. Hence, we obtain

$$\bigoplus_{\mathbf{v}} H_{top}(\mathfrak{M}(\mathbf{v}, \mathbf{w})_x) = L_{\left(\mathfrak{M}_0^{reg}(\mathbf{v}^0, \mathbf{w}), \mathbb{C}_{\mathfrak{M}_0^{reg}(\mathbf{v}^0, \mathbf{w})}\right)}.$$

Therefore, the intersection cohomology complexes that appear in the decomposition of $\pi_* \mathcal{C}$ are those coming from the pairs $(\mathfrak{M}_0^{reg}(\mathbf{v}^0, \mathbf{w}), \mathbb{C}_{\mathfrak{M}_0^{reg}(\mathbf{v}^0, \mathbf{w})})$ for each nonempty stratum $\mathfrak{M}_0^{reg}(\mathbf{v}^0, \mathbf{w})$. The theorem now follows. $\square$

5. constructible functions

Let X be an algebraic variety. A *constructible subset* of X is a finite union of locally closed subsets of X . A function $f : X \rightarrow \mathbb{C}$ is called a *constructible function* if $f(X)$ is finite and $f^{-1}(a)$ is a constructible subset of X for each $a \in \mathbb{C}$.

We write $A(X)$ for the $\mathbb{C}$ -vector space consisting of all $\mathbb{C}$ -valued constructible functions on X . If Y is a subvariety of X , then we consider $A(Y)$ as a subspace of $A(X)$ by extending with zero on a set $X - Y$.

By the *characteristic function* $\langle Z \rangle$ of a constructible subset $Z \subset X$ we mean

$$\langle Z \rangle(x) = \begin{cases} 1, & \text{if } x \in Z, \\ 0, & \text{if } x \in X - Z. \end{cases}$$

Then, the characteristic function $\langle Z \rangle$ is an element of $A(X)$. For a given morphism $p : X \rightarrow Y$ between algebraic varieties, we define the *push-forward* $p_* : A(X) \rightarrow A(Y)$ by

$$(p_* f)(y) = \sum_{c \in \mathbb{C}} c \chi(p^{-1}(y) \cap f^{-1}(c)),$$

where χ denotes the Euler characteristic. We also define the *pull-back* $p^* : A(Y) \longrightarrow A(X)$ by

$$(p^*g)(x) = g(p(x)).$$

Let M_1, M_2 and M_3 be algebraic varieties, and let $p_{ij} : M_1 \times M_2 \times M_3 \longrightarrow M_i \times M_j$ ($i, j = 1, 2, 3$) be the projections. Then, for $f \in A(M_1 \times M_2)$ and $g \in A(M_2 \times M_3)$, we define the convolution product $f \star g$ by

$$(5.1) \quad f \star g = (p_{13})_* ((p_{12}^* f) (p_{23}^* g)) \in A(M_1 \times M_3).$$

Suppose that we have morphisms $p_i : M_i \longrightarrow M_0$ ($i = 1, 2, 3$) for a fixed variety M_0 . By considering $A(M_i \times_{M_0} M_j)$ as a subspaces of $A(M_i \times M_j)$, we define the convolution product $f \star g \in A(M_1 \times M_3)$ for $f \in A(M_1 \times_{M_0} M_2)$ and $g \in A(M_2 \times_{M_0} M_3)$. Notice that the convolution $f \star g$ is also contained in $A(M_1 \times_{M_0} M_3)$. In particular, we have the convolution product

$$A(Z(\mathbf{v}^1, \mathbf{v}^2; \mathbf{w})) \otimes A(Z(\mathbf{v}^2, \mathbf{v}^3; \mathbf{w})) \longrightarrow A(Z(\mathbf{v}^1, \mathbf{v}^3; \mathbf{w})),$$

and this convolution yields a $\mathbb{C}$ -algebra structure on $\bigoplus_{\mathbf{v}^1, \mathbf{v}^2} A(Z(\mathbf{v}^1, \mathbf{v}^2; \mathbf{w}))$.

Using Lemma 3.1, we define the following elements:

$$\begin{aligned} \widetilde{E}_k &= \sum_{\mathbf{v}} \langle \mathfrak{B}_k(\mathbf{v}, \mathbf{w}) \rangle \in \bigoplus_{\mathbf{v}^1, \mathbf{v}^2} A(Z(\mathbf{v}^1, \mathbf{v}^2; \mathbf{w})), \\ \widetilde{F}_k &= \sum_{\mathbf{v}} \langle \omega(\mathfrak{B}_k(\mathbf{v}, \mathbf{w})) \rangle \in \bigoplus_{\mathbf{v}^1, \mathbf{v}^2} A(Z(\mathbf{v}^1, \mathbf{v}^2; \mathbf{w})), \\ \widetilde{H}_k &= \sum_{\mathbf{v}} (\Lambda_{\mathbf{w}} - \alpha_{\mathbf{v}})(h_k) \langle \triangle(\mathbf{v}, \mathbf{w}) \rangle \in \bigoplus_{\mathbf{v}^1, \mathbf{v}^2} A(Z(\mathbf{v}^1, \mathbf{v}^2; \mathbf{w})). \end{aligned}$$

The following theorem is an analogue of Theorem 3.2 through constructible functions.

Theorem 5.1. *There exists a unique algebra homomorphism*

$$\Psi : \mathfrak{U}(\mathfrak{g}) \longrightarrow \bigoplus_{\mathbf{v}^1, \mathbf{v}^2} A(Z(\mathbf{v}^1, \mathbf{v}^2; \mathbf{w}))$$

such that $\Psi(e_k) = \widetilde{E}_k$, $\Psi(f_k) = \widetilde{F}_k$ and $\Psi(h_k) = \widetilde{H}_k$.

Proof. See [12, Theorem 6.5]. □

Put $M_1 = \mathfrak{M}(\mathbf{v}^1, \mathbf{w})$, $M_2 = \mathfrak{M}(\mathbf{v}^2, \mathbf{w})$, $M_3 = \{\text{point}\}$, $Z_{12} = Z(\mathbf{v}^1, \mathbf{v}^2; \mathbf{w}) \subset M_1 \times M_2$ and $Z_{23} = \mathfrak{M}(\mathbf{v}^2, \mathbf{w})_x \subset M_2 \times M_3$ in the convolution set-up of constructible functions.

Then, we obtain the convolution action

$$(5.2) \quad A(Z(\mathbf{v}^1, \mathbf{v}^2; \mathbf{w})) \otimes A(\mathfrak{M}(\mathbf{v}^2, \mathbf{w})_x) \longrightarrow A(\mathfrak{M}(\mathbf{v}^1, \mathbf{w})_x).$$

So, the enveloping algebra $\mathfrak{U}(\mathfrak{g})$ acts on $\bigoplus_{\mathbf{v}} A(\mathfrak{M}(\mathbf{v}, \mathbf{w})_x)$ via the algebra homomorphism in Theorem 5.1.

In the following theorem, we give a constructible function analogue of [10, Theorem 10.2].

Theorem 5.2. *Let $x \in \mathfrak{M}_0^{reg}(\mathbf{v}^0, \mathbf{w})$, and let*

$$\mathbb{L}(\mathbf{v}, \mathbf{w})_x = A(\mathfrak{M}(\mathbf{v}, \mathbf{w})_x) \cap (\mathfrak{U}(\mathfrak{g}) \langle \mathfrak{M}(\mathbf{v}^0, \mathbf{w})_x \rangle).$$

Then $\bigoplus_{\mathbf{v}} \mathbb{L}(\mathbf{v}, \mathbf{w})_x$ is the irreducible highest weight module with the highest weight vector $\langle \mathfrak{M}(\mathbf{v}^0, \mathbf{w})_x \rangle$ of the highest weight $\Lambda_{\mathbf{w}} - \alpha_{\mathbf{v}^0}$.

Proof. By the convolution action defined in (5.2), we have

$$(5.3) \quad \widetilde{F}_k^N \cdot A(\mathfrak{M}(\mathbf{v}^0, \mathbf{w})_x) \subset A(\mathfrak{M}(\mathbf{v}^0 + N\mathbf{e}^k, \mathbf{w})_x)$$

and

$$(5.4) \quad \widetilde{E}_k^N \cdot A(\mathfrak{M}(\mathbf{v}^0, \mathbf{w})_x) \subset A(\mathfrak{M}(\mathbf{v}^0 - N\mathbf{e}^k, \mathbf{w})_x).$$

From (5.3) and (5.4), we have $\bigoplus_{\mathbf{v}} \mathbb{L}(\mathbf{v}, \mathbf{w})_x = \mathfrak{U}(\mathfrak{g}) \langle \mathfrak{M}(\mathbf{v}^0, \mathbf{w})_x \rangle$.

Also, by Lemma 3.1, $\widetilde{F}_k^N$ and $\widetilde{E}_k^N$ are integrable operators on $\bigoplus_{\mathbf{v}} A(\mathfrak{M}(\mathbf{v}, \mathbf{w})_x)$. This implies that $\bigoplus_{\mathbf{v}} \mathbb{L}(\mathbf{v}, \mathbf{w})_x$ is an integrable $\mathfrak{U}(\mathfrak{g})$ -module.

On the other hand, we note that

$$\widetilde{E}_k \cdot \langle \mathfrak{M}(\mathbf{v}^0, \mathbf{w})_x \rangle \in A(\mathfrak{M}(\mathbf{v}^0 - \mathbf{e}^k, \mathbf{w})_x).$$

Since $x \in \mathfrak{M}_0^{reg}(\mathbf{v}^0, \mathbf{w})$, we obtain $x \notin \mathfrak{M}_0(\mathbf{v}^0 - \mathbf{e}^k, \mathbf{w})$ due to the stratification (2.3). Thus, we have $\mathfrak{M}(\mathbf{v}^0 - \mathbf{e}^k, \mathbf{w})_x = \emptyset$ which implies that $\widetilde{E}_k \cdot \langle \mathfrak{M}(\mathbf{v}^0, \mathbf{w})_x \rangle = 0$.

Let us now take $\alpha \in \mathbb{L}(\mathbf{v}, \mathbf{w})_x$. Then, we have

$$\widetilde{H}_k \cdot \alpha = (\Lambda_{\mathbf{w}} - \alpha_{\mathbf{v}})(h_k) \alpha,$$

because $\alpha \in A(\mathfrak{M}(\mathbf{v}, \mathbf{w})_x)$ and $\langle \Delta(\mathbf{v}, \mathbf{w}) \rangle \cdot \alpha = \alpha$.

The theorem now follows. $\square$

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