RELATIONSHIPS BETWEEN INTEGRAL TRANSFORMS AND CONVOLUTIONS ON AN ANALOGUE OF WIENER SPACE*

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Abstract. In the present paper, we evaluate the analytic conditional Fourier-Feynman transforms and convolution products of unbounded function which is the product of the cylinder function and the function in a Banach algebra which is defined on an analogue of Wiener space and useful in the Feynman integration theories and quantum mechanics. We then investigate the inverse transforms of the function with their relationships and finally prove that the analytic conditional Fourier-Feynman transforms of the conditional convolution products for the functions, can be expressed in terms of the product of the conditional Fourier-Feynman transforms of each function.

1. Introduction and preliminaries

Let $C_0[0, t]$ denote the Wiener space, that is, the space of real-valued continuous paths x on the closed interval $[0, t]$ with $x(0) = 0$. On the space $C_0[0, t]$, Chang and Skoug [2] introduced the concepts of conditional Fourier-Feynman transform and conditional convolution product and then, examined the effects that drift has on the conditional Fourier-Feynman transform, the conditional convolution product, and various relationships that occur between them. Moreover, on $C[0, t]$, the space of real-valued continuous paths on $[0, t]$, Kim [13] extended the relationships between the conditional convolution product and the

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$L_p(1 \leq p \leq \infty)$ -analytic conditional Fourier-Feynman transform of the functions in a Banach algebra which corresponds to the Cameron-Storvick's Banach algebra $\mathcal{S}$ [1]. The author and his coauthors [3, 4, 5, 6, 7, 14] also established relationships between them for various functions on $C[0, t]$. In particular, he [3] derived an evaluation formula for the L_p -analytic conditional Fourier-Feynman transforms and convolution products of unbounded functions with the conditioning functions X_n and X_{n+1} on $C[0, t]$ given by $X_n(x) = (x(t_0), x(t_1), \dots, x(t_n))$ and $X_{n+1}(x) = (x(t_0), x(t_1), \dots, x(t_n), x(t_{n+1}))$, where n is a positive integer and $0 = t_0 < t_1 < \dots < t_n < t_{n+1} = t$ is a partition of $[0, t]$, and then, derived their relationships. Note that X_n is independent of the present positions of the paths in $C[0, t]$ while X_{n+1} wholly depends on the present positions.

In this paper, when $n = 0$, we further develop the relationships on the space $(C[0, t], w_\varphi)$, an analogue of Wiener space associated with a probability measure φ on the Borel class $\mathcal{B}(\mathbb{R})$ of $\mathbb{R}$ [11, 15, 16]. In fact, using simple formulas for the conditional expectations given X_0 and X_1 , we proceed to evaluate the L_p -analytic conditional Fourier-Feynman transforms and convolution products for the functions of the form

$$(1.1) \quad f_r((v_1, x), \dots, (v_r, x)) \int_{L_2[0, t]} \exp\{i(v, x)\} d\sigma(v)$$

for w_φ -a.e. $x \in C[0, t]$, where $f_r \in L_p(\mathbb{R}^r)$, $\{v_1, \dots, v_r\}$ is an orthonormal set in $L_2[0, t]$ and σ is a complex Borel measure of bounded variation on $L_2[0, t]$. We then investigate the inverse transforms and the relationships between them of the functions given by (1.1). Finally, we show that the L_p -analytic conditional Fourier-Feynman transform of the conditional convolution product for the functions Ψ_1 and Ψ_2 given by (1.1), can be expressed by the formula

$$\begin{aligned} & T_q^{(p)}[(\Psi_1 * \Psi_2)_q | X_0](\cdot, \xi_0) | X_0(y, \zeta_0) \\ &= \left[T_q^{(p)}[\Psi_1 | X_0] \left(\frac{1}{\sqrt{2}} y, 0 \right) \right] \left[T_q^{(p)}[\Psi_2 | X_0] \left(\frac{1}{\sqrt{2}} y, 0 \right) \right] \end{aligned}$$

for a nonzero real q , w_φ -a.e. $y \in C[0, t]$ and P_{X_0} -a.e. $\xi_0, \zeta_0 \in \mathbb{R}$. Thus the analytic conditional Fourier-Feynman transforms of the conditional convolution products for the functions, can be interpreted as the product of the analytic conditional Fourier-Feynman transforms of each function. Note that they are independent of the initial positions of paths in $C[0, t]$ while they depend only on the present time.

Let $\mathbb{C}$, $\mathbb{C}_+$ and $\mathbb{C}_+^\sim$ denote the sets of complex numbers, complex numbers with positive real parts and nonzero complex numbers with nonnegative real parts, respectively. Let $C[0, t]$ be the space of all real-valued continuous functions on the closed interval $[0, t]$ with its Borel class $\mathcal{B}(C[0, t])$. For a probability measure φ on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$, let w_φ be an analogue of the Wiener measure on $\mathcal{B}(C[0, t])$ associated with φ [11, 15, 16]. Let $F : C[0, t] \rightarrow \mathbb{C}$ be integrable and X be a random vector on $C[0, t]$ assuming that the value space of X is a normed space equipped with the Borel σ -algebra. Then, we have the conditional expectation $E[F|X]$ of F given X from a well known probability theory. Furthermore, there exists a P_X -integrable $\mathbb{C}$ -valued function ψ on the value space of X such that $E[F|X](x) = (\psi \circ X)(x)$ for w_φ -a.e. $x \in C[0, t]$, where P_X is the probability distribution of X . The function ψ is called the conditional w_φ -integral of F given X and it is also denoted by $E[F|X]$.

Throughout this paper, let $X_0 : C[0, t] \rightarrow \mathbb{R}$ and $X_1 : C[0, t] \rightarrow \mathbb{R}^2$ be given by $X_0(x) = x(0)$ and $X_1(x) = (x(0), x(t))$. Let $F : C[0, t] \rightarrow \mathbb{C}$ be a function such that $F(\lambda^{-\frac{1}{2}} \cdot)$ is integrable for $\lambda > 0$. Then we have the following formula from [9]

$$\begin{aligned} & E[F(\lambda^{-\frac{1}{2}} \cdot) | X_1(\lambda^{-\frac{1}{2}} \cdot)](\xi_0, \xi_1) \\ &= E \left[F \left(\lambda^{-\frac{1}{2}} \left(x - x(0) - \frac{\dot{}}{t}(x(t) - x(0)) \right) + \xi_0 + \frac{\dot{}}{t}(\xi_1 - \xi_0) \right) \right] \end{aligned}$$

for $P_{X_1^\lambda}$ -a.e. $(\xi_0, \xi_1) \in \mathbb{R}^2$, where $P_{X_1^\lambda}$ is the probability distribution of $X_1(\lambda^{-\frac{1}{2}} \cdot)$ on $(\mathbb{R}^2, \mathcal{B}(\mathbb{R}^2))$. For $y \in C[0, t]$, let

$$I_F^\lambda(y, \xi_0, \xi_1) = E \left[F \left(\lambda^{-\frac{1}{2}} \left(x - x(0) - \frac{\dot{}}{t}(x(t) - x(0)) \right) + y + \xi_0 + \frac{\dot{}}{t}(\xi_1 - \xi_0) \right) \right],$$

where the expectation is taken over the variable x . Moreover, we have from [8]

$$E[F(\lambda^{-\frac{1}{2}} \cdot) | X_0(\lambda^{-\frac{1}{2}} \cdot)](\xi_0) = \left(\frac{\lambda}{2\pi t} \right)^{\frac{1}{2}} \int_{\mathbb{R}} I_F^\lambda(0, \xi_0, \xi_1) \exp \left\{ -\frac{\lambda(\xi_1 - \xi_0)^2}{2t} \right\} d\xi_1$$

for $P_{X_0^\lambda}$ -a.e. $\xi_0 \in \mathbb{R}$, where $P_{X_0^\lambda}$ is the probability distribution of $X_0(\lambda^{-\frac{1}{2}} \cdot)$ on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$. For $y \in C[0, t]$, let

$$K_F^\lambda(y, \xi_0) = \left(\frac{\lambda}{2\pi t} \right)^{\frac{1}{2}} \int_{\mathbb{R}} I_F^\lambda(y, \xi_0, \xi_1) \exp \left\{ -\frac{\lambda(\xi_1 - \xi_0)^2}{2t} \right\} d\xi_1.$$

If $I_F^\lambda(0, \xi_0, \xi_1)$ has the analytic extension $J_\lambda^*(F)(\xi_0, \xi_1)$ on $\mathbb{C}_+$ as a function of λ , then it is called the analytic conditional Wiener w_φ -integral

of F given X_1 with parameter λ and denoted by $E^{anw\lambda}[F|X_1](\xi_0, \xi_1) = J_\lambda^*(F)(\xi_0, \xi_1)$ for $(\xi_0, \xi_1) \in \mathbb{R}^2$. Moreover, if for a nonzero real q , $E^{anw\lambda}[F|X_1](\xi_0, \xi_1)$ has the limit as λ approaches to $-iq$ through $\mathbb{C}_+$, then it is called the conditional analytic Feynman w_φ -integral of F given X_1 with parameter q and denoted by

$$E^{anf_q}[F|X_1](\xi_0, \xi_1) = \lim_{\lambda \rightarrow -iq} E^{anw\lambda}[F|X_1](\xi_0, \xi_1).$$

Similarly, the definitions of $E^{anw\lambda}[F|X_0](\xi_0)$ and $E^{anf_q}[F|X_0](\xi_0)$ are understood with $K_F^\lambda(0, \xi_0)$ if X_1 is replaced by X_0 .

For a given extended real number p with $1 < p \leq \infty$, suppose that p and p' are related by $\frac{1}{p} + \frac{1}{p'} = 1$ (possibly $p' = 1$ if $p = \infty$). Let F_n and F be measurable functions such that $\lim_{n \rightarrow \infty} \int_{C[0,t]} |F_n(x) - F(x)|^{p'} dw_\varphi(x) = 0$. Then we write $\text{l.i.m.}_{n \rightarrow \infty} (w^{p'}) (F_n) = F$ and call F the limit in the mean of order p' . A similar definition is understood when n is replaced by a continuously varying parameter.

Let F and G be defined on $C[0, t]$. For $\lambda \in \mathbb{C}_+$ and w_φ -a.e. $y \in C[0, t]$, let

$$T_\lambda[F|X_1](y, \xi_0, \xi_1) = E^{anw\lambda}[F(y + \cdot)|X_1](\xi_0, \xi_1)$$

for P_{X_1} -a.e. $(\xi_0, \xi_1) \in \mathbb{R}^2$ if it exists. For a nonzero real q and w_φ -a.e. $y \in C[0, t]$, define the L_1 -analytic conditional Fourier-Feynman transform $T_q^{(1)}[F|X_1]$ of F given X_1 by the formula

$$T_q^{(1)}[F|X_1](y, \xi_0, \xi_1) = E^{anf_q}[F(y + \cdot)|X_1](\xi_0, \xi_1)$$

for P_{X_1} -a.e. $(\xi_0, \xi_1) \in \mathbb{R}^2$ if it exists. For $1 < p \leq \infty$, define the L_p -analytic conditional Fourier-Feynman transform $T_q^{(p)}[F|X_1]$ of F given X_1 by the formula

$$T_q^{(p)}[F|X_1](\cdot, \xi_0, \xi_1) = \text{l.i.m.}_{\lambda \rightarrow -iq} (w^{p'}) (T_\lambda[F|X_1](\cdot, \xi_0, \xi_1))$$

for P_{X_1} -a.e. $(\xi_0, \xi_1) \in \mathbb{R}^2$, where λ approaches to $-iq$ through $\mathbb{C}_+$. We also define the conditional convolution product $[(F * G)_\lambda|X_1]$ of F and G given X_1 by the formula, for w_φ -a.e. $y \in C[0, t]$

$$\begin{aligned} & [(F * G)_\lambda|X_1](y, \xi_0, \xi_1) \\ = & \begin{cases} E^{anw\lambda} \left[F \left(\frac{y + \cdot}{\sqrt{2}} \right) G \left(\frac{y - \cdot}{\sqrt{2}} \right) \middle| X_1 \right] (\xi_0, \xi_1), & \lambda \in \mathbb{C}_+; \\ E^{anf_q} \left[F \left(\frac{y + \cdot}{\sqrt{2}} \right) G \left(\frac{y - \cdot}{\sqrt{2}} \right) \middle| X_1 \right] (\xi_0, \xi_1), & \lambda = -iq \end{cases} \end{aligned}$$

if they exist for P_{X_1} -a.e. $(\xi_0, \xi_1) \in \mathbb{R}^2$. If $\lambda = -iq$, we replace $[(F * G)_\lambda|X_1]$ by $[(F * G)_q|X_1]$. Similar definitions and notations are understood with $\xi_0 \in \mathbb{R}$ if X_1 is replaced by X_0 .

2. The initial position-dependent conditional transform and convolution

Let $\|\cdot\|_2$ denote the L_2 -norm on the real Hilbert space $L_2[0, t]$. For v in $L_2[0, t]$ and x in $C[0, t]$, let (v, x) denote the Paley-Wiener-Zygmund integral of v according to x [11]. Note that by $\langle \cdot, \cdot \rangle_{\mathbb{R}^r}$, the dot product on the r -dimensional Euclidean space $\mathbb{R}^r$, is also denoted.

Throughout this paper, let $\{v_l : l = 1, \dots, r\}$ be an orthonormal subset of $L_2[0, t]$ such that $\mathcal{G} \equiv \{v_l - \frac{1}{t} \int_0^t v_l(s) ds : l = 1, \dots, r\}$ is an independent set. Let $\{e_l : l = 1, \dots, r\}$ be the orthonormal set obtained from $\mathcal{G}$ by the Gram-Schmidt orthonormalization process. Now, for $l = 1, \dots, r$, let $v_l - \frac{1}{t} \int_0^t v_l(s) ds = \sum_{j=1}^r \beta_{lj} e_j$ be the linear combinations of the e_j s and let $A = [\beta_{lj}]_{r \times r}$ be the coefficient matrix of the combinations. Define the linear transformation $T_A : \mathbb{R}^r \rightarrow \mathbb{R}^r$ by $T_A \vec{z} = \vec{z} A^T$ where A^T is the transpose of A and $\vec{z}$ is any row-vector in $\mathbb{R}^r$. Note that A is invertible so that T_A is an isomorphism. For $v \in L_2[0, t]$, let

$$(2.1) \quad c_l(v) = \int_0^t v(s) e_l(s) ds - \frac{1}{t} \int_0^t v(s) ds \int_0^t e_l(s) ds$$

for $l = 1, \dots, r$ and let $(\vec{v}, x) = ((v_1, x), \dots, (v_r, x))$ for $x \in C[0, t]$. Furthermore, for $\lambda \in \mathbb{C}_+^\sim$, $v \in L_2[0, t]$ and $\vec{u} = (u_1, \dots, u_r) \in \mathbb{R}^r$, let

$$(2.2) \quad A_1(\lambda, v, \vec{u}) = \exp \left\{ \frac{1}{2\lambda} \left[\sum_{j=1}^r [\lambda i u_j + c_j(v)]^2 - \left\| v - \frac{1}{t} \int_0^t v(s) ds \right\|_2^2 \right] \right\}.$$

Let $\vec{V}_t = \frac{1}{t} (\int_0^t v_1(s) ds, \dots, \int_0^t v_r(s) ds)$. For $\vec{z} \in \mathbb{R}^r$ and $(\xi_0, \xi_1) \in \mathbb{R}^2$, let

$$(2.3) \quad A_2(f, g; \lambda, \vec{z}, \xi_0, \xi_1, v) = \left(\frac{\lambda}{2\pi} \right)^{\frac{r}{2}} \int_{\mathbb{R}^r} f(\vec{z} + (\xi_1 - \xi_0) \vec{V}_t + T_A \vec{u}) \\ \times g(\vec{z} - (\xi_1 - \xi_0) \vec{V}_t - T_A \vec{u}) A_1(\lambda, v, \vec{u}) d\vec{u}$$

if exists, where f and g are Borel measurable on $\mathbb{R}^r$. Note that by the Bessel's inequality,

$$(2.4) \quad |A_1(\lambda, v, \vec{u})| \leq \exp \left\{ -\frac{\operatorname{Re} \lambda}{2} \|\vec{u}\|_{\mathbb{R}^r}^2 \right\} \leq 1.$$

Using the same method as used in the proof of Theorem 3.1 in [10], we can prove the following lemma.

Lemma 2.1. *Let f be Borel measurable on $\mathbb{R}^r$. Then, for $\lambda > 0$ and $v \in L_2[0, t]$,*

$$\begin{aligned} & \int_{C[0, t]} f \left(\lambda^{-\frac{1}{2}} \left(\vec{v}, x - x(0) - \frac{\dot{\cdot}}{t} (x(t) - x(0)) \right) \right) \\ & \times \exp \left\{ i \lambda^{-\frac{1}{2}} \left(v, x - x(0) - \frac{\dot{\cdot}}{t} (x(t) - x(0)) \right) \right\} dw_\varphi(x) \\ & \stackrel{*}{=} A_2(f, 1; \lambda, \vec{0}, 0, 0, v), \end{aligned}$$

where $\stackrel{*}{=}$ means that if either side exists, then both sides exist and they are equal.

For $1 \leq p \leq \infty$, let $\mathcal{A}_r^{(p)}$ be the space of the cylinder functions F_r of the form given by

$$(2.5) \quad F_r(x) = f_r(\vec{v}, x)$$

for w_φ -a.e. $x \in C[0, t]$, where $f_r \in L_p(\mathbb{R}^r)$. Note that, without loss of generality, we can take f_r to be Borel measurable. Let $\mathcal{M} = \mathcal{M}(L_2[0, t])$ be the class of all $\mathbb{C}$ -valued Borel measures of bounded variation over $L_2[0, t]$ and let $\mathcal{S}_{w_\varphi}$ be the space of all functions F which for $\sigma \in \mathcal{M}$ have the form

$$(2.6) \quad F(x) = \int_{L_2[0, t]} \exp\{i(v, x)\} d\sigma(v)$$

for w_φ -a.e. $x \in C[0, t]$. Note that $\mathcal{S}_{w_\varphi}$ is a Banach algebra which is equivalent to $\mathcal{M}$ with the norm $\|F\| = \|\sigma\|$, the total variation of σ [11].

Using the same method as used in the proof of Theorems 2.2, 2.3 and 2.4 of [3], we have the following theorems.

Theorem 2.2. *Let $1 \leq p \leq \infty$. For w_φ -a.e. $x \in C[0, t]$, let $\Psi(x) = F(x)F_r(x)$, where $F_r \in \mathcal{A}_r^{(p)}$ and $F \in \mathcal{S}_{w_\varphi}$ are given by (2.5) and (2.6), respectively. Then for $\lambda \in \mathbb{C}_+$, w_φ -a.e. $y \in C[0, t]$ and P_{X_1} -a.e. $(\xi_0, \xi_1) \in \mathbb{R}^2$,*

$$\begin{aligned} (2.7) \quad & T_\lambda[\Psi|X_1](y, \xi_0, \xi_1) \\ & = \int_{L_2[0, t]} H_1(y, \xi_0, \xi_1, v, v) A_2(f_r, 1; \lambda, (\vec{v}, y), \xi_0, \xi_1, v) d\sigma(v) \end{aligned}$$

where A_2 is given by (2.3) and H_1 is given by

$$(2.8) \quad H_1(y, \xi_0, \xi_1, v_1, v_2) = \exp \left\{ i \left[(v_1, y) + \frac{\xi_1 - \xi_0}{t} \int_0^t v_2(s) ds \right] \right\}$$

for $v_1, v_2 \in L_2[0, t]$. Moreover, as a function of y , $T_\lambda[\Psi|X_1](\cdot, \xi_0, \xi_1) \in L_p(C[0, t])$. If $p = 1$, then for nonzero real q , $T_q^{(1)}[\Psi|X_1](y, \xi_0, \xi_1)$ is given by (2.7) replacing λ by $-iq$ and $T_q^{(1)}[\Psi|X_1](\cdot, \xi_0, \xi_1) \in L_\infty(C[0, t])$.

Theorem 2.3. Let the assumptions be as given in Theorem 2.2 with one exception $1 \leq p \leq 2$ and let $\frac{1}{p} + \frac{1}{p'} = 1$. Then for $\lambda \in \mathbb{C}_+$ and P_{X_1} -a.e. $(\xi_0, \xi_1) \in \mathbb{R}^2$, $T_\lambda[\Psi|X_1](\cdot, \xi_0, \xi_1) \in L_{p'}(C[0, t])$. Furthermore, suppose that σ is concentrated on the subspace of constant functions in $L_2[0, t]$. Then for a nonzero real q , w_φ -a.e. $y \in C[0, t]$,

$$T_q^{(p)}[\Psi|X_1](y, \xi_0, \xi_1) = (f_r(T_A \cdot) * \Phi(-iq, \cdot))(((\xi_1 - \xi_0)\vec{V}_t + (\vec{v}, y))(A^T)^{-1}) \\ \times \int_{L_2[0, t]} \exp\{i(v, y) + i(\xi_1 - \xi_0)v\} d\sigma(v),$$

where for $\lambda \in \mathbb{C}_+^\sim$ and $\vec{u} \in \mathbb{R}^r$

$$(2.9) \quad \Phi(\lambda, \vec{u}) = \left(\frac{\lambda}{2\pi} \right)^{\frac{1}{2}} \exp \left\{ -\frac{\lambda}{2} \|\vec{u}\|_{\mathbb{R}^r}^2 \right\}.$$

In this case, $T_q^{(p)}[\Psi|X_1](\cdot, \xi_0, \xi_1) \in L_{p'}(C[0, t])$.

Theorem 2.4. Let $F_r \in \mathcal{A}_r^{(p_1)}$, $G_r \in \mathcal{A}_r^{(p_2)}$ and f_r, g_r be related by (2.5), respectively, where $1 \leq p_1, p_2 \leq \infty$. Let $F_1, F_2 \in \mathcal{S}_{w_\varphi}$ and $\sigma_1, \sigma_2 \in \mathcal{M}(L_2[0, t])$ be related by (2.6), respectively. Furthermore, let $\frac{1}{p_1} + \frac{1}{p'_1} = 1$, $\frac{1}{p_2} + \frac{1}{p'_2} = 1$, $\Psi_1(x) = F_r(x)F_1(x)$ and $\Psi_2(x) = G_r(x)F_2(x)$ for w_φ -a.e. $x \in C[0, t]$. Then for $\lambda \in \mathbb{C}_+$, w_φ -a.e. $y \in C[0, t]$ and P_{X_1} -a.e. $(\xi_0, \xi_1) \in \mathbb{R}^2$,

$$[(\Psi_1 * \Psi_2)_\lambda | X_1](y, \xi_0, \xi_1) \\ = \int_{L_2[0, t]} \int_{L_2[0, t]} H_1 \left(\frac{1}{\sqrt{2}}y, \frac{1}{\sqrt{2}}\xi_0, \frac{1}{\sqrt{2}}\xi_1, v_1 + v_2, v_1 - v_2 \right) A_2 \left(f_r, g_r; \right. \\ \left. 2\lambda, \left(\vec{v}, \frac{1}{\sqrt{2}}y \right), \frac{1}{\sqrt{2}}\xi_0, \frac{1}{\sqrt{2}}\xi_1, v_1 - v_2 \right) d\sigma_1(v_1) d\sigma_2(v_2)$$

where A_2 and H_1 are given by (2.3) and (2.8), respectively. Furthermore, as a function of y , $[(\Psi_1 * \Psi_2)_\lambda | X_1](\cdot, \xi_0, \xi_1) \in L_1(C[0, t])$ if either $p_2 \leq p'_1$ or $p_1 \leq p'_2$, $[(\Psi_1 * \Psi_2)_\lambda | X_1](\cdot, \xi_0, \xi_1) \in L_{p_2}(C[0, t])$ if $p_2 \geq p'_1$ and $[(\Psi_1 * \Psi_2)_\lambda | X_1](\cdot, \xi_0, \xi_1) \in L_{p_1}(C[0, t])$ if $p_1 \geq p'_2$.

3. The initial position-independent conditional transform and convolution

In this section, we evaluate the initial position-independent conditional Fourier-Feynman transform and conditional convolution product of the functions as given in the previous section.

Let $\Gamma_t = \frac{1}{1+t\|T_A^{-1}\vec{V}_t\|_{\mathbb{R}^r}^2}$. For $\lambda \in \mathbb{C}_+$, $v \in L_2[0, t]$ and $\vec{u} \in \mathbb{R}^r$, let

$$(3.1) \quad B_1(\lambda, v, \vec{u}) = \exp \left\{ \frac{t\Gamma_t}{2\lambda} \left[i \left[\frac{1}{t} \int_0^t v(s) ds - \langle \vec{c}(v), T_A^{-1}\vec{V}_t \rangle_{\mathbb{R}^r} \right] + \lambda \langle T_A^{-1}\vec{V}_t, \vec{u} \rangle_{\mathbb{R}^r} \right]^2 \right\}$$

where $\vec{c} = (c_1, \dots, c_r)$ and the c_j s are given by (2.1). Furthermore, for $\vec{z} \in \mathbb{R}^r$ let

$$(3.2) \quad B_2(f, g; \lambda, \vec{z}, v) = \Gamma_t^{\frac{1}{2}} \left(\frac{\lambda}{2\pi} \right)^{\frac{r}{2}} \int_{\mathbb{R}^r} f(\vec{z} + T_A \vec{u}) g(\vec{z} - T_A \vec{u}) A_1(\lambda, v, \vec{u}) B_1(\lambda, v, \vec{u}) d\vec{u}$$

if exists, where f and g are Borel measurable on $\mathbb{R}^r$ and A_1 is given by (2.2). Using the following well-known integration formula

$$(3.3) \quad \int_{\mathbb{R}} \exp\{-au^2 + ibu\} du = \left(\frac{\pi}{a} \right)^{\frac{1}{2}} \exp\left\{ -\frac{b^2}{4a} \right\}$$

for $a \in \mathbb{C}_+$ and $b \in \mathbb{R}$, we can prove the following lemma from Theorem 3.4 of [10].

Lemma 3.1. *Let f be Borel measurable on $\mathbb{R}^r$ and let A_2 be given by (2.3). Then for $\lambda > 0$, $\xi_0 \in \mathbb{R}$ and $v \in L_2[0, t]$*

$$\begin{aligned} & \left(\frac{\lambda}{2\pi t} \right)^{\frac{1}{2}} \int_{\mathbb{R}} \exp \left\{ \frac{i(\xi_1 - \xi_0)}{t} \int_0^t v(s) ds \right\} A_2(f, 1; \lambda, \vec{0}, \xi_0, \xi_1, v) \\ & \times \exp \left\{ -\frac{\lambda(\xi_1 - \xi_0)^2}{2t} \right\} d\xi_1 \stackrel{*}{=} B_2(f, 1; \lambda, \vec{0}, v) \end{aligned}$$

where B_2 is given by (3.2).

Theorem 3.2. *Under the assumptions as given in Theorem 2.2, we have for $\lambda \in \mathbb{C}_+$, w_φ -a.e. $y \in C[0, t]$ and P_{X_0} -a.e. $\xi_0 \in \mathbb{R}$*

$$(3.4) \quad T_\lambda[\Psi|X_0](y, \xi_0) = \int_{L_2[0, t]} B_2(f_r, 1; \lambda, (\vec{v}, y), v) \exp\{i(v, y)\} d\sigma(v)$$

where B_2 is given by (3.2). Moreover, as a function of y , $T_\lambda[\Psi|X_0](\cdot, \xi_0) \in L_p(C[0, t])$. If $p = 1$, then for nonzero real q , $T_q^{(1)}[\Psi|X_0](y, \xi_0)$ is given by (3.4) replacing λ by $-iq$ and $T_q^{(1)}[\Psi|X_0](\cdot, \xi_0) \in L_\infty(C[0, t])$.

Proof. For $\lambda > 0$, $y \in C[0, t]$ and $\xi_0 \in \mathbb{R}$,

$$K_\Psi^\lambda(y, \xi_0) = \left(\frac{\lambda}{2\pi t} \right)^{\frac{1}{2}} \int_{L_2[0, t]} \exp\{i(v, y)\} \int_{\mathbb{R}} \exp\left\{ \frac{i(\xi_1 - \xi_0)}{t} \int_0^t v(s) ds \right\} \\ \times A_2(f_r((\vec{v}, y) + \cdot), 1; \lambda, \vec{0}, \xi_0, \xi_1, v) \exp\left\{ -\frac{\lambda(\xi_1 - \xi_0)^2}{2t} \right\} d\xi_1 d\sigma(v)$$

by Theorem 2.2. By Lemma 3.1,

$$K_\Psi^\lambda(y, \xi_0) \stackrel{*}{=} \int_{L_2[0, t]} B_2(f_r, 1; \lambda, (\vec{v}, y), v) \exp\{i(v, y)\} d\sigma(v).$$

By (2.4), the Bessel's inequality and the Schwarz's inequality, we have for $\lambda \in \mathbb{C}_+^\sim$

$$(3.5) \quad |A_1(\lambda, v, \vec{u}) B_1(\lambda, v, \vec{u})| \leq \exp\left\{ -\frac{\Gamma_t \operatorname{Re} \lambda}{2} \|\vec{u}\|_{\mathbb{R}^r}^2 \right\} \leq 1$$

from the same method as used in the proof of Theorem 3.4 of [10]. Note that by the change of variable theorem

$$(3.6) \quad \|f(\vec{z} + T_A \cdot)\|_p^p = |\det(A^T)^{-1}| \|f_r\|_p^p < \infty.$$

By the Hölder's inequality, the first inequality of (3.5), the Morera's theorem and the dominated convergence theorem, we have the equality in the theorem as the analytic extension of $K_\Psi^\lambda(y, \xi_0)$ on $\mathbb{C}_+$. We also have by (3.5), (3.6), Theorem 3.5 of [11] and the Young's inequality

$$\begin{aligned} & \int_{C[0, t]} |T_\lambda[\Psi|X_0](y, \xi_0)|^p dw_\varphi(y) \\ & \leq |\det(A)| \|\sigma\|^p \left(\frac{|\lambda|}{\Gamma_t \operatorname{Re} \lambda} \right)^{\frac{pr}{2}} \|f_r(T_A \cdot) * \Phi(\Gamma_t \operatorname{Re} \lambda, \cdot)\|_p^p \\ & < \infty \end{aligned}$$

if $1 \leq p < \infty$, where Φ is given by (2.9), and $\|T_\lambda[\Psi|X_0](\cdot, \xi_0)\|_\infty \leq \|\sigma\| \|f_r\|_\infty \left(\frac{|\lambda|}{\Gamma_t \operatorname{Re} \lambda} \right)^{\frac{r}{2}}$ if $p = \infty$. If $p = 1$, the final result follows from the second inequality of (3.5) and the dominated convergence theorem. $\square$

Theorem 3.3. *Let the assumptions be as given in Theorem 3.2 with one exception $1 \leq p \leq 2$ and let $\frac{1}{p} + \frac{1}{p'} = 1$. Then for $\lambda \in \mathbb{C}_+$ and P_{X_0} -a.e. $\xi_0 \in \mathbb{R}$, $T_\lambda[\Psi|X_0](\cdot, \xi_0) \in L_{p'}(C[0, t])$. Furthermore, suppose*

that σ is concentrated on the subspace of constant functions in $L_2[0, t]$ and $\int_0^t v_l(s)ds = 0$ for $l = 1, \dots, r$. Then for a nonzero real q , w_φ -a.e. $y \in C[0, t]$ and P_{X_0} -a.e. $\xi_0 \in \mathbb{R}$,

$$T_q^{(p)}[\Psi|X_0](y, \xi_0) = (f_r * \Phi(-iq, \cdot))(\vec{v}, y) \int_{L_2[0, t]} D_{-iq}(v) \exp\{i(v, y)\} d\sigma(v),$$

where Φ is given by (2.9), $D_\lambda(v) = \exp\{-\frac{t}{2\lambda}v^2\}$ for $\lambda \in \mathbb{C}_+^\sim$. In this case, $T_q^{(p)}[\Psi|X_0](\cdot, \xi_0) \in L_{p'}(C[0, t])$.

Proof. Let $1 < p \leq 2$. Using the same process as used in the proof of Theorem 3.2, we have for $\lambda \in \mathbb{C}_+$,

$$\begin{aligned} & \int_{C[0, t]} |T_\lambda[\Psi|X_0](y, \xi_0)|^{p'} dw_\varphi(y) \\ & \leq |\det(A)| \|\sigma\|^{p'} \left(\frac{|\lambda|}{\Gamma_t \operatorname{Re} \lambda} \right)^{\frac{p'r}{2}} \|f_r(T_A \cdot) * \Phi(\Gamma_t \operatorname{Re} \lambda, \cdot)\|_{p'}^{p'} \\ & < \infty \end{aligned}$$

where the last inequality follows from Lemma 1.1 of [12]. Now, suppose that σ is concentrated on the subspace of constant functions in $L_2[0, t]$ and $\int_0^t v_l(s)ds = 0$ for $l = 1, \dots, r$. Then $\vec{V}_t = \vec{0} \in \mathbb{R}^r$, $\Gamma_t = 1$ and A is the identity matrix so that we have for σ -a.e. $v \in L_2[0, t]$, $\lambda \in \mathbb{C}_+^\sim$ and $\vec{u} \in \mathbb{R}^r$

$$(3.7) \quad \left(\frac{\lambda}{2\pi} \right)^{\frac{r}{2}} A_1(\lambda, v, \vec{u}) = \Phi(\lambda, \vec{u}) \text{ and } B_1(\lambda, v, \vec{u}) = D_\lambda(v).$$

By Theorem 3.2 and (3.7), we have for $\lambda \in \mathbb{C}_+^\sim$, w_φ -a.e. $y \in C[0, t]$ and P_{X_0} -a.e. $\xi_0 \in \mathbb{R}$

$$\begin{aligned} T_\lambda[\Psi|X_0](y, \xi_0) &= \int_{L_2[0, t]} B_2(f_r, 1; \lambda, (\vec{v}, y), v) \exp\{i(v, y)\} d\sigma(v) \\ &= (f_r * \Phi(\lambda, \cdot))(\vec{v}, y) \int_{L_2[0, t]} D_\lambda(v) \exp\{i(v, y)\} d\sigma(v). \end{aligned}$$

By Theorem 3.5 of [11] and Lemma 1.1 of [12]

$$\int_{C[0, t]} |T_q^{(p)}[\Psi|X_0](y, \xi_0)|^{p'} dw_\varphi(y) \leq \|\sigma\|^{p'} \|f_r * \Phi(-iq, \cdot)\|_{p'}^{p'} < \infty.$$

By Theorem 3.5 of [11], the Minkowski inequality and the triangle inequality, we also have for $\lambda \in \mathbb{C}_+$

$$\begin{aligned}
& \|T_\lambda[\Psi|X_0](\cdot, \xi_0) - T_q^{(p)}[\Psi|X_0](\cdot, \xi_0)\|_{p'} \\
& \leq \left[\int_{\mathbb{R}^r} \left[\int_{L_2[0,t]} |D_\lambda(v)(f_r * \Phi(\lambda, \cdot))(\vec{u}) - D_{-iq}(v)(f_r * \Phi(-iq, \cdot))(\vec{u})| \right. \right. \\
& \quad \left. \left. d|\sigma|(v) \right]^{p'} d\vec{u} \right]^{\frac{1}{p'}} \\
& \leq \left[\int_{\mathbb{R}^r} \left[|(f_r * \Phi(\lambda, \cdot))(\vec{u}) - (f_r * \Phi(-iq, \cdot))(\vec{u})| \int_{L_2[0,t]} |D_\lambda(v)| d|\sigma|(v) \right. \right. \\
& \quad \left. \left. + |(f_r * \Phi(-iq, \cdot))(\vec{u})| \int_{L_2[0,t]} |D_\lambda(v) - D_{-iq}(v)| d|\sigma|(v) \right]^{p'} d\vec{u} \right]^{\frac{1}{p'}} \\
& \leq \|\sigma\| \|f_r * \Phi(\lambda, \cdot) - f_r * \Phi(-iq, \cdot)\|_{p'} + \|f_r * \Phi(-iq, \cdot)\|_{p'} \int_{L_2[0,t]} |D_\lambda(v) - D_{-iq}(v)| d|\sigma|(v)
\end{aligned}$$

which converges to 0 as λ approaches to $-iq$ through $\mathbb{C}_+$ by the dominated convergence theorem and Lemma 1.2 of [12]. If $p = 1$, the conclusions follow from Theorem 3.2. Now, the proof is completed. $\square$

Remark 3.4. The orthonormal subset $\{v_l : l = 1, \dots, r\}$ of $L_2[0, t]$ satisfying $\int_0^t v_l(s) ds = 0$ for $l = 1, \dots, r$ exists [10].

Theorem 3.5. Under the assumptions as given in Theorem 2.4, we have for $\lambda \in \mathbb{C}_+$, w_φ -a.e. $y \in C[0, t]$ and P_{X_0} -a.e. $\xi_0 \in \mathbb{R}$

$$\begin{aligned}
[(\Psi_1 * \Psi_2)_\lambda|X_0](y, \xi_0) &= \int_{L_2[0,t]} \int_{L_2[0,t]} \exp \left\{ i \left(v_1 + v_2, \frac{1}{\sqrt{2}} y \right) \right\} \\
&\quad \times B_2 \left(f_r, g_r; 2\lambda, \left(\vec{v}, \frac{1}{\sqrt{2}} y \right), v_1 - v_2 \right) d\sigma_1(v_1) d\sigma_2(v_2)
\end{aligned}$$

where B_2 is given by (3.2). Furthermore, as a function of y , $[(\Psi_1 * \Psi_2)_\lambda|X_0](\cdot, \xi_0) \in L_1(C[0, t])$ if either $p_2 \leq p'_1$ or $p_1 \leq p'_2$, $[(\Psi_1 * \Psi_2)_\lambda|X_1](\cdot, \xi_0) \in L_{p_2}(C[0, t])$ if $p_2 \geq p'_1$ and $[(\Psi_1 * \Psi_2)_\lambda|X_0](\cdot, \xi_0) \in L_{p_1}(C[0, t])$ if $p_1 \geq p'_2$.

Proof. For $\xi_0 \in \mathbb{R}$, $\lambda > 0$ and $y \in C[0, t]$, we have by Theorem 2.4

$$\begin{aligned} & [(\Psi_1 * \Psi_2)_\lambda | X_0](y, \xi_0) \\ &= \left(\frac{\lambda}{2\pi t} \right)^{\frac{1}{2}} \int_{L_2[0, t]} \int_{L_2[0, t]} \int_{\mathbb{R}} H_1 \left(\frac{1}{\sqrt{2}} y, \frac{1}{\sqrt{2}} \xi_0, \frac{1}{\sqrt{2}} \xi_1, v_1 + v_2, v_1 - v_2 \right) \\ & \quad \times A_2 \left(f_r, g_r; 2\lambda, \left(\vec{v}, \frac{1}{\sqrt{2}} y \right), \frac{1}{\sqrt{2}} \xi_0, \frac{1}{\sqrt{2}} \xi_1, v_1 - v_2 \right) \\ & \quad \times \exp \left\{ -\frac{\lambda(\xi_1 - \xi_0)^2}{2t} \right\} d\xi_1 d\sigma_1(v_1) d\sigma_2(v_2). \end{aligned}$$

Note that by the change of variable theorem,

$$\begin{aligned} & A_2 \left(f_r, g_r; 2\lambda, \left(\vec{v}, \frac{1}{\sqrt{2}} y \right), \frac{1}{\sqrt{2}} \xi_0, \frac{1}{\sqrt{2}} \xi_1, v_1 - v_2 \right) \\ &= A_2 \left(h_r, 1; \lambda, \vec{0}, \xi_0, \xi_1, \frac{1}{\sqrt{2}}(v_1 - v_2) \right) \end{aligned}$$

where $h_r(\vec{z}) = f_r(\frac{1}{\sqrt{2}}[(\vec{v}, y) + \vec{z}])g_r(\frac{1}{\sqrt{2}}[(\vec{v}, y) - \vec{z}])$ for $\vec{z} \in \mathbb{R}^r$. Then by Lemma 3.1

$$\begin{aligned} & [(\Psi_1 * \Psi_2)_\lambda | X_0](y, \xi_0) \\ &= \left(\frac{\lambda}{2\pi t} \right)^{\frac{1}{2}} \int_{L_2[0, t]} \int_{L_2[0, t]} \exp \left\{ \frac{i}{\sqrt{2}}(v_1 + v_2, y) \right\} \int_{\mathbb{R}} \exp \left\{ \frac{i(\xi_1 - \xi_0)}{\sqrt{2}t} \right. \\ & \quad \times \left. \int_0^t (v_1(s) - v_2(s)) ds \right\} A_2 \left(h_r, 1; \lambda, \vec{0}, \xi_0, \xi_1, \frac{1}{\sqrt{2}}(v_1 - v_2) \right) \\ & \quad \times \exp \left\{ -\frac{\lambda(\xi_1 - \xi_0)^2}{2t} \right\} d\xi_1 d\sigma_1(v_1) d\sigma_2(v_2) \\ &= \int_{L_2[0, t]} \int_{L_2[0, t]} \exp \left\{ i \left(v_1 + v_2, \frac{1}{\sqrt{2}} y \right) \right\} B_2 \left(f_r, g_r; 2\lambda, \left(\vec{v}, \frac{1}{\sqrt{2}} y \right), \right. \\ & \quad \left. v_1 - v_2 \right) d\sigma_1(v_1) d\sigma_2(v_2). \end{aligned}$$

Thus for $\lambda \in \mathbb{C}_+$, we have by (3.5), Theorem 3.5 of [11] and the change of variable theorem

$$\begin{aligned} & \int_{C[0, t]} |[(\Psi_1 * \Psi_2)_\lambda | X_0](y, \xi_0)| dw_\varphi(y) \\ &\leq \|\sigma_1\| \|\sigma_2\| \left(\frac{|\lambda|}{\Gamma_t \operatorname{Re} \lambda} \right)^{\frac{r}{2}} \int_{\mathbb{R}^r} \int_{\mathbb{R}^r} \left| f_r \left(\frac{1}{\sqrt{2}}(\vec{u} + T_A \vec{z}) \right) \right| \left| g_r \left(\frac{1}{\sqrt{2}}(\vec{u} - T_A \vec{z}) \right) \right| \\ & \quad \times \Phi(\Gamma_t \operatorname{Re} \lambda, \vec{z}) d\vec{z} d\vec{u} \end{aligned}$$

where Φ is given by (2.9). By the same method as used in the proof of Theorem 3.1 of [5], we have the theorem. $\square$

4. Relationships between the conditional transforms and convolutions

In this section, we investigate the inverse transforms of the conditional Fourier-Feynman transforms of the functions as given in the previous sections. We also show that the analytic conditional Fourier-Feynman transforms of the conditional convolution product for the functions, can be expressed as the product of the analytic conditional Fourier-Feynman transforms of each function as desired.

In the following two theorems, we derive the inverse transforms of the conditional Fourier-Feynman transforms of the function Ψ as given in the previous sections.

Theorem 4.1. *For $\sigma \in \mathcal{M}$, $y \in C[0, t]$, $(\xi_0, \xi_1), (\zeta_0, \zeta_1) \in \mathbb{R}^2$ and $f_r : \mathbb{R}^r \rightarrow \mathbb{C}$, let*

$$\begin{aligned} \Psi_{\xi_0, \xi_1, \zeta_0, \zeta_1}(y) &= f_r((\vec{v}, y) + (\zeta_1 - \zeta_0 + \xi_1 - \xi_0)\vec{V}_t) \\ &\quad \times \int_{L_2[0, t]} \exp\{i(v, y) + i(\zeta_1 - \zeta_0 + \xi_1 - \xi_0)v\} d\sigma(v). \end{aligned}$$

Then, under the assumptions as given in Theorem 2.2 with one exception that σ is concentrated on the subspace of constant functions in $L_2[0, t]$, we have for P_{X_1} -a.e. $(\xi_0, \xi_1), (\zeta_0, \zeta_1) \in \mathbb{R}^2$

$$\|T_{\bar{\lambda}}[T_{\lambda}[\Psi|X_1](\cdot, \xi_0, \xi_1)|X_1](\cdot, \zeta_0, \zeta_1) - \Psi_{\xi_0, \xi_1, \zeta_0, \zeta_1}\|_p \rightarrow 0$$

for $1 \leq p < \infty$, and for $1 \leq p \leq \infty$

$$T_{\bar{\lambda}}[T_{\lambda}[\Psi|X_1](\cdot, \xi_0, \xi_1)|X_1](\cdot, \zeta_0, \zeta_1) \rightarrow \Psi_{\xi_0, \xi_1, \zeta_0, \zeta_1}(y)$$

for w_{φ} -a.e. $y \in C[0, t]$, as λ approaches to $-iq$ through $\mathbb{C}_+$.

Proof. For $\lambda_1 > 0$, $\lambda \in \mathbb{C}_+$, w_{φ} -a.e. $y \in C[0, t]$ and P_{X_1} -a.e. $(\xi_0, \xi_1), (\zeta_0, \zeta_1) \in \mathbb{R}^2$, we have by Lemma 2.1 and Theorem 2.2

$$\begin{aligned} &I_{T_{\lambda}[\Psi|X_1](\cdot, \xi_0, \xi_1)}^{\lambda_1}(y, \zeta_0, \zeta_1) \\ &= \int_{L_2[0, t]} \exp\{i(v, y) + i(\zeta_1 - \zeta_0 + \xi_1 - \xi_0)v\} \int_{C[0, t]} \exp\left\{i\lambda_1^{-\frac{1}{2}}\left(v, x - \right. \right. \\ &\quad \left. \left. x(0) - \frac{\cdot}{t}(x(t) - x(0))\right)\right\} \int_{\mathbb{R}^r} f_r\left(\lambda_1^{-\frac{1}{2}}\left(\vec{v}, x - x(0) - \frac{\cdot}{t}(x(t) - x(0))\right) \right. \\ &\quad \left. + (\vec{v}, y) + (\zeta_1 - \zeta_0 + \xi_1 - \xi_0)\vec{V}_t + T_A \vec{u}\right) \Phi(\lambda, \vec{u}) d\vec{u} dw_{\varphi}(x) d\sigma(v) \end{aligned}$$

$$= \int_{L_2[0,t]} \exp\{i(v, y) + i(\zeta_1 - \zeta_0 + \xi_1 - \xi_0)v\} \int_{\mathbb{R}^r} \int_{\mathbb{R}^r} f_r((\vec{v}, y) + (\zeta_1 - \zeta_0 + \xi_1 - \xi_0)\vec{V}_t + T_A(\vec{z} + \vec{u}))\Phi(\lambda, \vec{u})\Phi(\lambda_1, \vec{z})d\vec{z}d\vec{u}d\sigma(v)$$

where Φ is given by (2.9). By the analytic continuation, we have the existence of

$$T_{\lambda_1}[T_\lambda[\Psi|X_1](\cdot, \xi_0, \xi_1)|X_1](y, \zeta_0, \zeta_1)$$

for $\lambda_1 \in \mathbb{C}_+$. Let $\vec{l} = \vec{z} + \vec{u}$. By the change of variable theorem and (3.3)

$$\begin{aligned} & T_{\bar{\lambda}}[T_\lambda[\Psi|X_1](\cdot, \xi_0, \xi_1)|X_1](\cdot, \zeta_0, \zeta_1) \\ &= \left(\frac{|\lambda|}{2\pi}\right)^r \int_{L_2[0,t]} \exp\{i(v, y) + i(\zeta_1 - \zeta_0 + \xi_1 - \xi_0)v\} \int_{\mathbb{R}^r} \int_{\mathbb{R}^r} f_r((\vec{v}, y) \\ &+ (\zeta_1 - \zeta_0 + \xi_1 - \xi_0)\vec{V}_t + T_A\vec{l}) \exp\left\{-\operatorname{Re}\lambda\|\vec{u}\|_{\mathbb{R}^r}^2 + \bar{\lambda}\langle\vec{u}, \vec{l}\rangle_{\mathbb{R}^r} - \frac{\bar{\lambda}}{2}\|\vec{l}\|_{\mathbb{R}^r}^2\right\} \\ & d\vec{u}d\vec{l}d\sigma(v) \\ &= \left(\frac{|\lambda|^2}{4\pi\operatorname{Re}\lambda}\right)^{\frac{r}{2}} \int_{L_2[0,t]} \exp\{i(v, y) + i(\zeta_1 - \zeta_0 + \xi_1 - \xi_0)v\} \int_{\mathbb{R}^r} f_r(T_A(((\vec{v}, y) \\ &+ (\zeta_1 - \zeta_0 + \xi_1 - \xi_0)\vec{V}_t)(A^T)^{-1} - \vec{l})) \exp\left\{-\frac{|\lambda|^2}{4\operatorname{Re}\lambda}\|\vec{l}\|_{\mathbb{R}^r}^2\right\} d\vec{l}d\sigma(v) \\ &= \epsilon^{-r}(f_r(T_A\cdot) * \Phi(1, \cdot/\epsilon))(((\vec{v}, y) + (\zeta_1 - \zeta_0 + \xi_1 - \xi_0)\vec{V}_t)(A^T)^{-1}) \\ & \times \int_{L_2[0,t]} \exp\{i(v, y) + i(\zeta_1 - \zeta_0 + \xi_1 - \xi_0)v\} d\sigma(v) \end{aligned}$$

where $\epsilon = (2\operatorname{Re}\lambda/|\lambda|^2)^{1/2} > 0$. By the same method as used in the proof of Theorem 4.1 of [3], we have the theorem. $\square$

Theorem 4.2. *Under the assumptions as given in Theorem 3.2 with exceptions that σ is concentrated on the subspace of constant functions in $L_2[0, t]$ and $\int_0^t v_l(s)ds = 0$ for $l = 1, \dots, r$, we have for P_{X_0} -a.e. $\xi_0, \zeta_0 \in \mathbb{R}$*

$$\|T_{\bar{\lambda}}[T_\lambda[\Psi|X_0](\cdot, \xi_0)|X_0](\cdot, \zeta_0) - \Psi\|_p \longrightarrow 0$$

for $1 \leq p < \infty$, and for $1 \leq p \leq \infty$

$$T_{\bar{\lambda}}[T_\lambda[\Psi|X_0](\cdot, \xi_0)|X_0](y, \zeta_0) \longrightarrow \Psi(y)$$

for w_φ -a.e. $y \in C[0, t]$, as λ approaches to $-iq$ through $\mathbb{C}_+$.

Proof. Note that $\vec{V}_t = \vec{0}$, $\Gamma_t = 1$ and T_A is the identity transformation on $\mathbb{R}^r$. For $\lambda \in \mathbb{C}_+$, $\lambda_1 > 0$, $y \in C[0, t]$, $\xi_0 \in \mathbb{R}$ and $(\zeta_0, \zeta_1) \in \mathbb{R}^2$, we

have

$$\begin{aligned}
& I_{T_\lambda[\Psi|X_0](\cdot, \xi_0)}^{\lambda_1}(y, \zeta_0, \zeta_1) \\
&= \int_{L_2[0, t]} \exp\{i(v, y) + i(\zeta_1 - \zeta_0)v\} \int_{\mathbb{R}^r} \Phi(\lambda, \vec{u}) D_\lambda(v) \int_{C[0, t]} f_r \left(\lambda_1^{-\frac{1}{2}} \left(\vec{v}, x - \right. \right. \\
&\quad \left. \left. x(0) - \frac{1}{t}(x(t) - x(0)) \right) + (\vec{v}, y) + \vec{u} \right) \exp \left\{ i \lambda_1^{-\frac{1}{2}} \left(v, x - x(0) - \frac{1}{t}(x(t) - \right. \right. \right. \\
&\quad \left. \left. x(0)) \right) \right\} dw_\varphi(x) d\vec{u} d\sigma(v) \\
&= \int_{L_2[0, t]} \exp\{i(v, y) + i(\zeta_1 - \zeta_0)v\} \int_{\mathbb{R}^r} \int_{\mathbb{R}^r} \Phi(\lambda, \vec{u}) \Phi(\lambda_1, \vec{z}) D_\lambda(v) f_r((\vec{v}, y) + \\
&\quad \vec{u} + \vec{z}) d\vec{z} d\vec{u} d\sigma(v)
\end{aligned}$$

by Lemma 2.1 and Theorem 3.2, where Φ and B_2 are given by (2.9) and (3.2), respectively, and D_λ is as given in Theorem 3.3. Then we have by Lemma 3.1 and (3.3)

$$\begin{aligned}
& K_{T_\lambda[\Psi|X_0](\cdot, \xi_0)}^{\lambda_1}(y, \zeta_0) \\
&= \int_{L_2[0, t]} \exp\{i(v, y)\} \int_{\mathbb{R}^r} \int_{\mathbb{R}^r} \Phi(\lambda, \vec{u}) \Phi(\lambda_1, \vec{z}) D_\lambda(v) D_{\lambda_1}(v) f_r((\vec{v}, y) + \\
&\quad \vec{u} + \vec{z}) d\vec{z} d\vec{u} d\sigma(v)
\end{aligned}$$

which holds for $\lambda_1 \in \mathbb{C}_+$ by the analytic continuation. Let $\vec{l} = \vec{u} + \vec{z}$. Then by (3.3) and the change of variable theorem

$$\begin{aligned}
& T_\lambda[T_\lambda[\Psi|X_0](\cdot, \xi_0)|X_0](\cdot, \zeta_0) \\
&= \left(\frac{|\lambda|^2}{4\pi \operatorname{Re} \lambda} \right)^{\frac{r}{2}} \int_{L_2[0, t]} \exp\{i(v, y)\} \exp \left\{ -\frac{t \operatorname{Re} \lambda}{|\lambda|^2} v^2 \right\} \int_{\mathbb{R}^r} f_r((\vec{v}, y) - \vec{l}) \\
&\quad \times \exp \left\{ -\frac{|\lambda|^2}{4 \operatorname{Re} \lambda} \|\vec{l}\|_{\mathbb{R}^r}^2 \right\} d\vec{l} d\sigma(v) \\
&= \epsilon^{-r} (f_r * \Phi(1, \cdot/\epsilon))(\vec{v}, y) \int_{L_2[0, t]} \exp\{i(v, y)\} D_{1/\epsilon^2}(v) d\sigma(v)
\end{aligned}$$

where $\epsilon = (2 \operatorname{Re} \lambda / |\lambda|^2)^{1/2} > 0$. Let $1 \leq p < \infty$. Then we have by Theorem 3.5 of [11] and the change of variable theorem

$$\int_{C[0, t]} |T_\lambda[T_\lambda[\Psi|X_0](\cdot, \xi_0)|X_0](y, \zeta_0) - \Psi(y)|^p dw_\varphi(y)$$

$$\begin{aligned}
&\leq \int_{\mathbb{R}^r} \left[\int_{L_2[0,t]} |\epsilon^{-r}(f_r * \Phi(1, \cdot/\epsilon))(\vec{u}) D_{1/\epsilon^2}(v) - f_r(\vec{u})| d|\sigma|(v) \right]^p d\vec{u} \\
&\leq \int_{\mathbb{R}^r} \left[|\epsilon^{-r}(f_r * \Phi(1, \cdot/\epsilon))(\vec{u}) - f_r(\vec{u})| \int_{L_2[0,t]} D_{1/\epsilon^2}(v) d|\sigma|(v) + \right. \\
&\quad \left. |f_r(\vec{u})| \int_{L_2[0,t]} |D_{1/\epsilon^2}(v) - 1| d|\sigma|(v) \right]^p d\vec{u}
\end{aligned}$$

so that by the Minkowski inequality,

$$\begin{aligned}
&\|T_{\vec{\lambda}}[T_{\vec{\lambda}}[\Psi|X_0](\cdot, \xi_0)|X_0](\cdot, \zeta_0) - \Psi\|_p \\
&\leq \|\sigma\| \|\epsilon^{-r}(f_r * \Phi(1, \cdot/\epsilon)) - f_r\|_p + \|f_r\|_p \int_{L_2[0,t]} |D_{1/\epsilon^2}(v) - 1| d|\sigma|(v).
\end{aligned}$$

Letting $\lambda \rightarrow -iq$ through $\mathbb{C}_+$ which satisfies $\epsilon \rightarrow 0$, we have the first part of the theorem by the dominated convergence theorem and Theorem 1.18 of [17]. Similarly, if $1 \leq p \leq \infty$, then we have the second part of the theorem by the dominated convergence theorem and Theorem 1.25 of [17]. $\square$

Using the same method as used in the proof of Theorem 3.2 in [5] with Theorems 2.4, 3.5, (2.4) and (3.5), we can prove the following theorems.

Theorem 4.3. *Let Ψ_1 and Ψ_2 be as given in Theorem 2.4 and q be a nonzero real number. Then for $\lambda \in \mathbb{C}_+$ or $\lambda = -iq$, and P_{X_1} -a.e. $(\xi_0, \xi_1) \in \mathbb{R}^2$, we have the followings:*

1. if $F_r, G_r \in \mathcal{A}_r^{(1)}$, then $[(\Psi_1 * \Psi_2)_\lambda|X_1](\cdot, \xi_0, \xi_1) \in L_1(C[0, t])$,
2. if $F_r, G_r \in \mathcal{A}_r^{(2)}$, then $[(\Psi_1 * \Psi_2)_\lambda|X_1](\cdot, \xi_0, \xi_1) \in L_\infty(C[0, t])$,
3. if $F_r \in \mathcal{A}_r^{(1)}$ and $G_r \in \mathcal{A}_r^{(2)}$, then $[(\Psi_1 * \Psi_2)_\lambda|X_1](\cdot, \xi_0, \xi_1) \in L_2(C[0, t])$,
4. if $F_r \in \mathcal{A}_r^{(1)}$ and $G_r \in \mathcal{A}_r^{(1)} \cap \mathcal{A}_r^{(2)}$, then $[(\Psi_1 * \Psi_2)_\lambda|X_1](\cdot, \xi_0, \xi_1) \in L_1(C[0, t]) \cap L_2(C[0, t])$, and
5. if $F_r \in \mathcal{A}_r^{(1)}$ and $G_r \in \mathcal{A}_r^{(\infty)}$, then $[(\Psi_1 * \Psi_2)_\lambda|X_1](\cdot, \xi_0, \xi_1) \in L_\infty(C[0, t])$.

Theorem 4.4. *If we replace X_1 by X_0 in Theorem 4.3, then the conclusions of the theorem are still true, where $[(\Psi_1 * \Psi_2)_\lambda|X_1](\cdot, \xi_0, \xi_1)$ is replaced by $[(\Psi_1 * \Psi_2)_\lambda|X_0](\cdot, \xi_0)$.*

Theorem 4.5. *Under the assumptions as given in Theorem 2.4, we have for $\lambda \in \mathbb{C}_+$, w_φ -a.e. $y \in C[0, t]$ and P_{X_1} -a.e. $(\xi_1, \xi_2), (\zeta_0, \zeta_1) \in \mathbb{R}^2$*

$$\begin{aligned} & T_\lambda[(\Psi_1 * \Psi_2)_\lambda | X_1](\cdot, \xi_0, \xi_1) | X_1(y, \zeta_0, \zeta_1) \\ &= \left[T_\lambda[\Psi_1 | X_1] \left(\frac{1}{\sqrt{2}}y, \frac{1}{\sqrt{2}}(\zeta_0 + \xi_0), \frac{1}{\sqrt{2}}(\zeta_1 + \xi_1) \right) \right] \\ & \quad \times \left[T_\lambda[\Psi_2 | X_1] \left(\frac{1}{\sqrt{2}}y, \frac{1}{\sqrt{2}}(\zeta_0 - \xi_0), \frac{1}{\sqrt{2}}(\zeta_1 - \xi_1) \right) \right]. \end{aligned}$$

Proof. For $\lambda \in \mathbb{C}_+$ and $\lambda_1 > 0$, we have by Lemma 2.1 and Theorem 2.4

$$\begin{aligned} & I_{[(\Psi_1 * \Psi_2)_\lambda | X_1](\cdot, \xi_0, \xi_1)}^{\lambda_1}(y, \zeta_0, \zeta_1) \\ &= \int_{L_2[0, t]} \int_{L_2[0, t]} H_1 \left(\frac{1}{\sqrt{2}}y, \frac{1}{\sqrt{2}}(\zeta_0 + \xi_0), \frac{1}{\sqrt{2}}(\zeta_1 + \xi_1), v_1, v_1 \right) H_1 \left(\frac{1}{\sqrt{2}}y, \right. \\ & \quad \left. \frac{1}{\sqrt{2}}(\zeta_0 - \xi_0), \frac{1}{\sqrt{2}}(\zeta_1 - \xi_1), v_2, v_2 \right) \int_{C[0, t]} \exp \left\{ i(2\lambda_1)^{-\frac{1}{2}} \left(v_1 + v_2, x - x(0) \right. \right. \\ & \quad \left. \left. - \frac{1}{t}(x(t) - x(0)) \right) \right\} A_2 \left(f_r, g_r; 2\lambda, (2\lambda_1)^{-\frac{1}{2}} \left(\vec{v}, x - x(0) - \frac{1}{t}(x(t) - x(0)) \right) \right) \\ & \quad + \frac{1}{\sqrt{2}}(\vec{v}, y) + \frac{1}{\sqrt{2}}(\zeta_1 - \zeta_0)\vec{V}_t, \frac{1}{\sqrt{2}}\xi_0, \frac{1}{\sqrt{2}}\xi_1, v_1 - v_2 \Big) dw_\varphi(x) d\sigma_1(v_1) d\sigma_2(v_2) \\ &= \left(\frac{\lambda_1}{\pi} \right)^{\frac{r}{2}} \left(\frac{\lambda}{\pi} \right)^{\frac{r}{2}} \int_{L_2[0, t]} \int_{L_2[0, t]} H_1 \left(\frac{1}{\sqrt{2}}y, \frac{1}{\sqrt{2}}(\zeta_0 + \xi_0), \frac{1}{\sqrt{2}}(\zeta_1 + \xi_1), v_1, v_1 \right) \\ & \quad \times H_1 \left(\frac{1}{\sqrt{2}}y, \frac{1}{\sqrt{2}}(\zeta_0 - \xi_0), \frac{1}{\sqrt{2}}(\zeta_1 - \xi_1), v_2, v_2 \right) \int_{\mathbb{R}^r} \int_{\mathbb{R}^r} f_r \left(\frac{1}{\sqrt{2}}(\vec{v}, y) + \frac{1}{\sqrt{2}}(\zeta_1 + \xi_1 - (\zeta_0 + \xi_0))\vec{V}_t + T_A(\vec{z} + \vec{u}) \right) g_r \left(\frac{1}{\sqrt{2}}(\vec{v}, y) + \frac{1}{\sqrt{2}}(\zeta_1 - \xi_1 - (\zeta_0 - \xi_0))\vec{V}_t + T_A(\vec{z} - \vec{u}) \right) A_1(2\lambda, v_1 - v_2, \vec{u}) A_1(2\lambda_1, v_1 + v_2, \vec{z}) d\vec{z} d\vec{u} d\sigma_1(v_1) d\sigma_2(v_2). \end{aligned}$$

By the analytic continuation, we have the existence of $T_{\lambda_1}[(\Psi_1 * \Psi_2)_\lambda | X_1](\cdot, \xi_0, \xi_1) | X_1(y, \zeta_0, \zeta_1)$ for $\lambda_1 \in \mathbb{C}_+$. Let $\vec{w} = \vec{z} + \vec{u}$ and $\vec{l} = \vec{z} - \vec{u}$. Then for $\lambda \in \mathbb{C}_+$ we have by the change of variable theorem

$$\begin{aligned} & T_\lambda[(\Psi_1 * \Psi_2)_\lambda | X_1](\cdot, \xi_0, \xi_1) | X_1(y, \zeta_0, \zeta_1) \\ &= \left(\frac{\lambda}{2\pi} \right)^r \int_{L_2[0, t]} \int_{L_2[0, t]} H_1 \left(\frac{1}{\sqrt{2}}y, \frac{1}{\sqrt{2}}(\zeta_0 + \xi_0), \frac{1}{\sqrt{2}}(\zeta_1 + \xi_1), v_1, v_1 \right) H_1 \left(\frac{1}{\sqrt{2}}y, \right. \\ & \quad \left. \frac{1}{\sqrt{2}}(\zeta_0 - \xi_0), \frac{1}{\sqrt{2}}(\zeta_1 - \xi_1), v_2, v_2 \right) \int_{\mathbb{R}^r} \int_{\mathbb{R}^r} f_r \left(\frac{1}{\sqrt{2}}(\vec{v}, y) + \frac{1}{\sqrt{2}}(\zeta_1 + \xi_1 - (\zeta_0 + \xi_0))\vec{V}_t + T_A\vec{w} \right) g_r \left(\frac{1}{\sqrt{2}}(\vec{v}, y) + \frac{1}{\sqrt{2}}(\zeta_1 - \xi_1 - (\zeta_0 - \xi_0))\vec{V}_t + T_A\vec{l} \right) \\ & \quad \times A_1(2\lambda, v_1 - v_2, \vec{w}) A_1(2\lambda_1, v_1 + v_2, \vec{l}) d\vec{l} d\vec{w} d\sigma_1(v_1) d\sigma_2(v_2). \end{aligned}$$

$$\times A_1\left(2\lambda, v_1 - v_2, \frac{1}{2}(\vec{w} - \vec{l})\right) A_1\left(2\lambda, v_1 + v_2, \frac{1}{2}(\vec{w} + \vec{l})\right) d\vec{w} d\vec{l} d\sigma_1(v_1) d\sigma_2(v_2).$$

By a calculation,

$$\begin{aligned} & A_1\left(2\lambda, v_1 - v_2, \frac{1}{2}(\vec{w} - \vec{l})\right) A_1\left(2\lambda, v_1 + v_2, \frac{1}{2}(\vec{w} + \vec{l})\right) \\ &= A_1(\lambda, v_1, \vec{w}) A_1(\lambda, v_2, \vec{l}) \end{aligned}$$

so that we have the theorem by Theorem 2.2. $\square$

By Lemma 2.1, Theorems 2.2, 3.5 and the analytic continuation, we have the following theorem.

Theorem 4.6. *Under the assumptions as given in Theorem 2.4, we have for $\lambda, \lambda_1 \in \mathbb{C}_+$, w_φ -a.e. $y \in C[0, t]$, P_{X_0} -a.e. $\xi_0 \in \mathbb{R}$ and P_{X_1} -a.e. $(\zeta_0, \zeta_1) \in \mathbb{R}^2$*

$$\begin{aligned} & T_{\lambda_1}[(\Psi_1 * \Psi_2)_\lambda | X_0](\cdot, \xi_0) | X_1](y, \zeta_0, \zeta_1) \\ &= \Gamma_t^{\frac{1}{2}} \left(\frac{\lambda}{\pi} \right)^{\frac{r}{2}} \left(\frac{\lambda_1}{\pi} \right)^{\frac{r}{2}} \int_{L_2[0, t]} \int_{L_2[0, t]} H_1 \left(\frac{1}{\sqrt{2}} y, \frac{1}{\sqrt{2}} \zeta_0, \frac{1}{\sqrt{2}} \zeta_1, v_1 + v_2, v_1 + v_2 \right) \\ & \quad \times \int_{\mathbb{R}^r} \int_{\mathbb{R}^r} A_1(2\lambda, v_1 - v_2, \vec{u}) A_1(2\lambda_1, v_1 + v_2, \vec{z}) B_1(2\lambda, v_1 - v_2, \vec{u}) \\ & \quad f_r \left(\frac{1}{\sqrt{2}}(\vec{v}, y) + \frac{1}{\sqrt{2}}(\zeta_1 - \zeta_0) \vec{V}_t + T_A(\vec{u} + \vec{z}) \right) g_r \left(\frac{1}{\sqrt{2}}(\vec{v}, y) + \frac{1}{\sqrt{2}}(\zeta_1 - \zeta_0) \vec{V}_t - T_A(\vec{u} - \vec{z}) \right) d\vec{z} d\vec{u} d\sigma_1(v_1) d\sigma_2(v_2) \end{aligned}$$

where A_1 , H_1 and B_1 are given by (2.2), (2.8) and (3.1), respectively.

Theorem 4.7. *Under the assumptions as given in Theorem 3.5, we have for $\lambda \in \mathbb{C}_+$, w_φ -a.e. $y \in C[0, t]$ and P_{X_0} -a.e. $\xi_0, \zeta_0 \in \mathbb{R}$*

$$\begin{aligned} & T_\lambda[(\Psi_1 * \Psi_2)_\lambda | X_0](\cdot, \xi_0) | X_0](y, \zeta_0) \\ &= \left[T_\lambda[\Psi_1 | X_0] \left(\frac{1}{\sqrt{2}} y, 0 \right) \right] \left[T_\lambda[\Psi_2 | X_0] \left(\frac{1}{\sqrt{2}} y, 0 \right) \right]. \end{aligned}$$

Proof. Note that $B_1(2\lambda_1, v_1 + v_2, \vec{z}) = B_1(\lambda_1, \frac{1}{\sqrt{2}}(v_1 + v_2), \vec{\alpha})$ and $A_1(2\lambda_1, v_1 + v_2, \vec{z}) = A_1(\lambda_1, \frac{1}{\sqrt{2}}(v_1 + v_2), \vec{\alpha})$ if $\vec{\alpha} = \sqrt{2}\vec{z}$, where A_1 and B_1 are given by (2.2) and (3.1), respectively. For $\lambda \in \mathbb{C}_+$ and $\lambda_1 > 0$, we

have by Lemma 3.1, Theorem 4.6 and the change of variable theorem

$$\begin{aligned}
& K_{[(\Psi_1 * \Psi_2)_\lambda | X_0](\cdot, \xi_0)}^{\lambda_1}(y, \zeta_0) \\
= & \Gamma_t^{\frac{1}{2}} \left(\frac{\lambda}{\pi} \right)^{\frac{r}{2}} \left(\frac{\lambda_1}{2\pi} \right)^{\frac{r}{2}} \left(\frac{\lambda_1}{2\pi t} \right)^{\frac{1}{2}} \int_{L_2[0, t]} \int_{L_2[0, t]} \exp \left\{ \frac{i}{\sqrt{2}} (v_1 + v_2, y) \right\} \int_{\mathbb{R}^r} A_1(\\
& 2\lambda, v_1 - v_2, \vec{u}) B_1(2\lambda, v_1 - v_2, \vec{u}) \int_{\mathbb{R}} \exp \left\{ \frac{i}{\sqrt{2}} \frac{\zeta_1 - \zeta_0}{t} \int_0^t (v_1(s) + v_2(s)) ds \right\} \\
& \times \int_{\mathbb{R}^r} A_1 \left(\lambda_1, \frac{v_1 + v_2}{2}, \vec{\alpha} \right) f_r \left(\frac{1}{\sqrt{2}} [(\vec{v}, y) + (\zeta_1 - \zeta_0) \vec{V}_t + T_A \vec{\alpha}] + T_A \vec{u} \right) \\
& g_r \left(\frac{1}{\sqrt{2}} [(\vec{v}, y) + (\zeta_1 - \zeta_0) \vec{V}_t + T_A \vec{\alpha}] - T_A \vec{u} \right) \exp \left\{ -\frac{\lambda_1}{2} \frac{(\zeta_1 - \zeta_0)^2}{t} \right\} \\
& d\vec{\alpha} d\zeta_1 d\vec{u} d\sigma_1(v_1) d\sigma_2(v_2) \\
= & \Gamma_t \left(\frac{\lambda}{\pi} \right)^{\frac{r}{2}} \left(\frac{\lambda_1}{\pi} \right)^{\frac{r}{2}} \int_{L_2[0, t]} \int_{L_2[0, t]} \exp \left\{ \frac{i}{\sqrt{2}} (v_1 + v_2, y) \right\} \int_{\mathbb{R}^r} \int_{\mathbb{R}^r} A_1(2\lambda, v_1 - \\
& v_2, \vec{u}) B_1(2\lambda, v_1 - v_2, \vec{u}) A_1(2\lambda_1, v_1 + v_2, \vec{z}) B_1(2\lambda_1, v_1 + v_2, \vec{z}) f_r \left(\frac{1}{\sqrt{2}} (\vec{v}, y) \right. \\
& \left. + T_A(\vec{u} + \vec{z}) \right) g_r \left(\frac{1}{\sqrt{2}} (\vec{v}, y) + T_A(\vec{z} - \vec{u}) \right) d\vec{z} d\vec{u} d\sigma_1(v_1) d\sigma_2(v_2).
\end{aligned}$$

By the analytic continuation, we have the existence of $T_{\lambda_1}[(\Psi_1 * \Psi_2)_\lambda | X_0](\cdot, \xi_0) | X_0](y, \zeta_0)$ for $\lambda_1 \in \mathbb{C}_+$. Let $\vec{w} = \vec{z} + \vec{u}$ and $\vec{l} = \vec{z} - \vec{u}$. By a calculation, we can prove that for $\lambda \in \mathbb{C}_+$, $B_1(2\lambda, v_1 - v_2, \frac{1}{2}(\vec{w} - \vec{l})) B_1(2\lambda, v_1 + v_2, \frac{1}{2}(\vec{w} + \vec{l})) = B_1(\lambda, v_1, \vec{w}) B_1(\lambda, v_2, \vec{l})$ so that by the change of variable theorem

$$\begin{aligned}
& T_{\lambda}[(\Psi_1 * \Psi_2)_\lambda | X_0](\cdot, \xi_0) | X_0](y, \zeta_0) \\
= & \Gamma_t \left(\frac{\lambda}{2\pi} \right)^r \int_{L_2[0, t]} \int_{L_2[0, t]} \exp \left\{ \frac{i}{\sqrt{2}} (v_1, y) \right\} \exp \left\{ \frac{i}{\sqrt{2}} (v_2, y) \right\} \int_{\mathbb{R}^r} \int_{\mathbb{R}^r} A_1(\lambda, \\
& v_1, \vec{w}) A_1(\lambda, v_2, \vec{l}) B_1(\lambda, v_1, \vec{w}) B_1(\lambda, v_2, \vec{l}) f_r \left(\frac{1}{\sqrt{2}} (\vec{v}, y) + T_A \vec{w} \right) g_r \left(\frac{1}{\sqrt{2}} (\vec{v}, \\
& y) + T_A \vec{l} \right) d\vec{w} d\vec{l} d\sigma_1(v_1) d\sigma_2(v_2)
\end{aligned}$$

which completes the proof by Theorem 3.2. $\square$

Finally, we have the following relationships from Theorems 2.2, 2.4, 3.2, 3.5, 4.3, 4.4, 4.5 and 4.7.

Theorem 4.8. *Let Ψ_1 and Ψ_2 be as given in Theorem 2.4 and q be a nonzero real number. Then we have the followings:*

1. if $F_r, G_r \in \mathcal{A}_r^{(1)}$, then we have for w_φ -a.e. $y \in C[0, t]$ and P_{X_1} -a.e. $(\xi_0, \xi_1), (\zeta_0, \zeta_1) \in \mathbb{R}^2$

$$\begin{aligned} & T_q^{(1)}[(\Psi_1 * \Psi_2)_q | X_1](\cdot, \xi_0, \xi_1) | X_1(y, \zeta_0, \zeta_1) \\ &= \left[T_q^{(1)}[\Psi_1 | X_1] \left(\frac{1}{\sqrt{2}}y, \frac{1}{\sqrt{2}}(\zeta_0 + \xi_0), \frac{1}{\sqrt{2}}(\zeta_1 + \xi_1) \right) \right] \\ & \quad \times \left[T_q^{(1)}[\Psi_2 | X_1] \left(\frac{1}{\sqrt{2}}y, \frac{1}{\sqrt{2}}(\zeta_0 - \xi_0), \frac{1}{\sqrt{2}}(\zeta_1 - \xi_1) \right) \right], \end{aligned}$$

and for P_{X_0} -a.e. $\xi_0, \zeta_0 \in \mathbb{R}$

$$\begin{aligned} & T_q^{(1)}[(\Psi_1 * \Psi_2)_q | X_0](\cdot, \xi_0) | X_0(y, \zeta_0) \\ &= \left[T_q^{(1)}[\Psi_1 | X_0] \left(\frac{1}{\sqrt{2}}y, 0 \right) \right] \left[T_q^{(1)}[\Psi_2 | X_0] \left(\frac{1}{\sqrt{2}}y, 0 \right) \right], \end{aligned}$$

2. if $F_r \in \mathcal{A}_r^{(1)}$ and $G_r \in \mathcal{A}_r^{(2)}$, then we have for w_φ -a.e. $y \in C[0, t]$ and P_{X_1} -a.e. $(\xi_0, \xi_1), (\zeta_0, \zeta_1) \in \mathbb{R}^2$

$$\begin{aligned} & T_q^{(2)}[(\Psi_1 * \Psi_2)_q | X_1](\cdot, \xi_0, \xi_1) | X_1(y, \zeta_0, \zeta_1) \\ &= \left[T_q^{(1)}[\Psi_1 | X_1] \left(\frac{1}{\sqrt{2}}y, \frac{1}{\sqrt{2}}(\zeta_0 + \xi_0), \frac{1}{\sqrt{2}}(\zeta_1 + \xi_1) \right) \right] \\ & \quad \times \left[T_q^{(2)}[\Psi_2 | X_1] \left(\frac{1}{\sqrt{2}}y, \frac{1}{\sqrt{2}}(\zeta_0 - \xi_0), \frac{1}{\sqrt{2}}(\zeta_1 - \xi_1) \right) \right] \end{aligned}$$

and for P_{X_0} -a.e. $\xi_0, \zeta_0 \in \mathbb{R}$

$$\begin{aligned} & T_q^{(2)}[(\Psi_1 * \Psi_2)_q | X_0](\cdot, \xi_0) | X_0(y, \zeta_0) \\ &= \left[T_q^{(1)}[\Psi_1 | X_0] \left(\frac{1}{\sqrt{2}}y, 0 \right) \right] \left[T_q^{(2)}[\Psi_2 | X_0] \left(\frac{1}{\sqrt{2}}y, 0 \right) \right]. \end{aligned}$$

References

- [1] Cameron R.H., Storvick D.A., Some Banach algebras of analytic Feynman integrable functionals, Lecture Notes in Mathematics, 798, Springer, Berlin-New York, 1980
- [2] Chang S.J., Skoug D., The effect of drift on conditional Fourier-Feynman transforms and conditional convolution products, Int. J. Appl. Math., 2000, 2(4), 505–527
- [3] Cho D.H., Conditional Fourier-Feynman transforms and convolutions of unbounded functions on a generalized Wiener space, preprint
- [4] Cho D.H., A time-independent conditional Fourier-Feynman transform and convolution product on an analogue of Wiener space, Bull. Korean Math. Soc., (in Press, in Korea)

- [5] Cho D.H., A time-dependent conditional Fourier-Feynman transform and convolution product on an analogue of Wiener space, preprint
- [6] Cho D.H., Conditional integral transforms and convolutions of bounded functions on an analogue of Wiener space, preprint
- [7] Cho D.H., Conditional integral transforms and conditional convolution products on a function space, *Integral Transforms Spec. Funct.*, 2012, 23(6), 405–420
- [8] Cho D.H., A simple formula for an analogue of conditional Wiener integrals and its applications II, *Czechoslovak Math. J.*, 2009, 59(2), 431–452
- [9] Cho D.H., A simple formula for an analogue of conditional Wiener integrals and its applications, *Trans. Amer. Math. Soc.*, 2008, 360(7), 3795–3811
- [10] Cho D.H., Kim B.J., Yoo I., Analogues of conditional Wiener integrals and their change of scale transformations on a function space, *J. Math. Anal. Appl.*, 2009, 359(2), 421–438
- [11] Im M.K., Ryu K.S., An analogue of Wiener measure and its applications, *J. Korean Math. Soc.*, 2002, 39(5), 801–819
- [12] Johnson G.W., Skoug D.L., The Cameron-Storvick function space integral: an $\mathcal{L}(L_p, L_{p'})$ theory, *Nagoya Math. J.*, 1976, 60, 93–137
- [13] Kim M.J., Conditional Fourier-Feynman transform and convolution product on a function space, *Int. J. Math. Anal.*, 2009, 3(10), 457–471
- [14] Park S.B., Cho D.H., Choi Y.H., Conditional Fourier-Feynman transforms and convolutions over continuous paths, *Int. Math. Forum.* 2013, 8(9), 443–456
- [15] Ryu K.S., Im M.K., A measure-valued analogue of Wiener measure and the measure-valued Feynman-Kac formula, *Trans. Amer. Math. Soc.*, 2002, 354(12), 4921–4951
- [16] Ryu K.S., Im M.K., Choi K.S., Survey of the theories for analogue of Wiener measure space, *Interdiscip. Inform. Sci.*, 2009, 15(3), 319–337
- [17] Stein E.M., Weiss G., *Introduction to Fourier Analysis on Euclidean Spaces*, Princeton Univ. Press, Princeton, 1971.

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