STABILITY OF A QUADRATIC-ADDITIVE TYPE FUNCTIONAL EQUATION IN NON-ARCHIMEDEAN NORMED SPACES

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Abstract. In this paper, we investigate the stability for the functional equation

$$2f(x+y) + f(x-y) + f(y-x) - f(2x) - f(2y) = 0$$

in non-Archimedean normed spaces.

1. Introduction

We introduce some terminologies and notations used in the theory of non-Archimedean spaces (see [3]).

Definition 1.1. A field $\mathbb{K}$ equipped with a function (valuation) $|\cdot|$ from $\mathbb{K}$ into $[0, \infty)$ is called a *non-Archimedean field* if the function $|\cdot| : \mathbb{K} \rightarrow [0, \infty)$ satisfies the following conditions:

- (i) $|r| = 0$ if and only if $r = 0$;
- (ii) $|rs| = |r||s|$;
- (iii) $|r+s| \leq \max\{|r|, |s|\}$ for all $r, s \in \mathbb{K}$.

Clearly, $|1| = |-1|$ and $|n| \leq 1$ for all $n \in \mathbb{N}$.

Definition 1.2. Let X be a vector space over a scalar field $\mathbb{K}$ with a non-Archimedean nontrivial valuation $|\cdot|$. A function $\|\cdot\| : X \rightarrow \mathbb{R}$ is a *non-Archimedean norm* if it satisfies the following conditions:

- (i) $\|x\| = 0$ if and only if $x = 0$;
- (ii) $\|rx\| = |r|\|x\|$ ($r \in \mathbb{K}, x \in X$);

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(iii) the strong triangle inequality, namely,

$$\|x + y\| \leq \max\{\|x\|, \|y\|\}$$

for all $x, y \in X$. In this case, the pair $(X, \|\cdot\|)$ is called a *non-Archimedean space*. Due to the fact that

$$\|x_n - x_m\| \leq \max\{\|x_{j+1} - x_j\| : m \leq j \leq n-1\} \quad (n > m)$$

a sequence $\{x_n\}$ is Cauchy if and only if $\{x_{n+1} - x_n\}$ converges to zero in a non-Archimedean space. By a *complete non-Archimedean* we mean one in which every Cauchy sequence is convergent.

The stability problem of the functional equation, was formulated by S. M. Ulam [5] in 1940. In the next year, D. H. Hyers [2] gave a partial solution of Ulam's problem for the case of an approximate additive mapping. Subsequently, Hyers' result was generalized by T. Aoki [1] for an additive mapping and by Th. M. Rassias [4] for a linear mapping with unbounded Cauchy differences.

Recently M. S. Moslehian and Th. M. Rassias [3] discussed the Hyers-Ulam stability of the Cauchy functional equation

$$(1.1) \quad f(x + y) = f(x) + f(y)$$

and the quadratic functional equation

$$(1.2) \quad f(x + y) + f(x - y) - 2f(x) - 2f(y) = 0$$

in non-Archimedean normed spaces.

Now we consider a *quadratic-additive type functional equation*

$$(1.3) \quad 2f(x + y) + f(x - y) + f(y - x) - f(2x) - f(2y) = 0$$

which solution is called a *quadratic-additive mapping*. We remark that the general solution of the equation (1.3) is of the form $f(x) = Q(x) + A(x)$, where Q is a quadratic mapping and A is an additive mapping.

In this paper, we get a general stability result of the quadratic-additive type functional equation (1.3) in non-Archimedean normed spaces.

2. Stability of a quadratic-additive type functional equation

Throughout this section, we assume that X is a real linear space and Y is a complete non-Archimedean space with $|2| < 1$.

For a given mapping $f : X \rightarrow Y$, we use the abbreviation

$$Df(x, y) := 2f(x + y) + f(x - y) + f(y - x) - f(2x) - f(2y)$$

for all $x, y \in X$. Now, we will prove the stability of the quadratic-additive type functional equation (1.3).

Theorem 2.1. *Let $\varphi : X^2 \rightarrow [0, \infty)$ be a function such that*

$$(2.1) \quad \lim_{n \rightarrow \infty} \frac{\varphi(2^n x, 2^n y)}{|4|^n} = 0, \quad (x, y \in X)$$

and let for each $x \in X$ the limit

$$(2.2) \quad \lim_{n \rightarrow \infty} \max \left\{ \frac{\varphi(2^j x, 0)}{|2| \cdot |4|^{j+1}}, \frac{\varphi(-2^j x, 0)}{|2| \cdot |4|^{j+1}} : 0 \leq j < n \right\}$$

denoted by $\tilde{\varphi}(x)$, exists. Suppose that $f : X \rightarrow Y$ is a mapping satisfying

$$(2.3) \quad \|Df(x, y)\| \leq \varphi(x, y), \quad (x, y \in X).$$

Then there exists a unique quadratic-additive mapping $T : X \rightarrow Y$ such that

$$(2.4) \quad \|f(x) - T(x)\| \leq \tilde{\varphi}(x), \quad (x \in X).$$

In particular, T is given by

$$T(x) = \lim_{n \rightarrow \infty} \frac{f(2^n x) + f(-2^n x)}{2 \cdot 4^n} + \frac{f(2^n x) - f(-2^n x)}{2^{n+1}}$$

for all $x \in X$.

Proof. It follows from $|2| < 1$ and (2.1) that

$$\varphi(0, 0) \leq \lim_{n \rightarrow \infty} \frac{\varphi(0, 0)}{|4|^n} = 0.$$

So we have

$$\|2f(0)\| = \|Df(0, 0)\| \leq \varphi(0, 0) = 0,$$

i.e., $f(0) = 0$. Let $J_n f : X \rightarrow Y$ be a function defined by

$$J_n f(x) = \frac{f(2^n x) + f(-2^n x)}{2 \cdot 4^n} + \frac{f(2^n x) - f(-2^n x)}{2^{n+1}}$$

for all $x \in X$ and $n \in \mathbb{N}$. Notice that $J_0 f(x) = f(x)$ and

$$\begin{aligned} & \|J_j f(x) - J_{j+1} f(x)\| \\ &= \left\| \frac{Df(2^j x, 0)}{2 \cdot 4^{j+1}} + \frac{Df(-2^j x, 0)}{2 \cdot 4^{j+1}} + \frac{Df(2^j x, 0)}{2^{j+2}} - \frac{Df(-2^j x, 0)}{2^{j+2}} \right\| \\ &\leq \max \left\{ \frac{\|Df(2^j x, 0)\|}{|2| \cdot |4|^{j+1}}, \frac{\|Df(-2^j x, 0)\|}{|2| \cdot |4|^{j+1}}, \frac{\|Df(2^j x, 0)\|}{|2|^{j+2}}, \frac{\|Df(-2^j x, 0)\|}{|2|^{j+2}} \right\} \\ (2.5) \quad &\leq \max \left\{ \frac{\varphi(2^j x, 0)}{|2| \cdot |4|^{j+1}}, \frac{\varphi(-2^j x, 0)}{|2| \cdot |4|^{j+1}} \right\} \end{aligned}$$

for all $x \in X$ and $j \geq 0$. It follows from (2.1) and (2.5) that the sequence $\{J_n f(x)\}$ is Cauchy. Since Y is complete, we conclude that $\{J_n f(x)\}$ is convergent. Set

$$T(x) := \lim_{n \rightarrow \infty} J_n f(x).$$

Using induction one can show that

$$(2.6) \quad \|J_n f(x) - f(x)\| \leq \max \left\{ \frac{\varphi(2^j x, 0)}{|2|^{2j+3}}, \frac{\varphi(-2^j x, 0)}{|2|^{2j+3}} : 0 \leq j < n \right\}$$

for all $x \in X$ and $n \in \mathbb{N}$. By taking n to approach infinity in (2.6) and using (2.2), one obtains (2.4). Replacing x and y by $2^n x$ and $2^n y$, respectively, in (2.3), we get

$$\begin{aligned} \|DJ_n f(x, y)\| &= \left\| \frac{Df(2^n x, 2^n y) - Df(-2^n x, -2^n y)}{2^{n+1}} \right. \\ &\quad \left. + \frac{Df(2^n x, 2^n y) + Df(-2^n x, -2^n y)}{2^{2n+3}} \right\| \\ &\leq \max \left\{ \frac{\varphi(2^n x, 2^n y)}{|2|^{n+1}}, \frac{\varphi(-2^n x, -2^n y)}{|2|^{n+1}}, \right. \\ &\quad \left. \frac{\varphi(2^n x, 2^n y)}{|2| \cdot |4|^n}, \frac{\varphi(-2^n x, -2^n y)}{|2| \cdot |4|^n} \right\} \end{aligned}$$

for all $x, y \in X$ and $n \in \mathbb{N}$. Taking the limit as $n \rightarrow \infty$ and using (2.1), we get $DT(x, y) = 0$. Thus, the mapping T is a quadratic-additive mapping satisfying (2.4). Now, to prove that the quadratic-additive mapping T is unique. If T' is another quadratic-additive mapping satisfying (2.4), then

$$\begin{aligned} &T'(x) - J_k T'(x) \\ &= \sum_{j=0}^{k-1} \left(\frac{(1 + 2^{j+1})DT'(2^j x, 0)}{2 \cdot 4^{j+1}} + \frac{(1 - 2^{j+1})DT'(-2^j x, 0)}{2 \cdot 4^{j+1}} \right) \\ &= 0 \end{aligned}$$

for any $k \in \mathbb{N}$ and so

$$\begin{aligned}
& \|T(x) - T'(x)\| \\
&= \lim_{k \rightarrow \infty} \|J_k T(x) - J_k T'(x)\| \\
&\leq \lim_{k \rightarrow \infty} \max\{\|J_k T(x) - J_k f(x)\|, \|J_k f(x) - J_k T'(x)\|\} \\
&\leq \lim_{k \rightarrow \infty} |2|^{-2k-1} \max\{\|T(2^k x) - f(2^k x)\|, \|T(-2^k x) - f(-2^k x)\|, \\
&\quad \|f(2^k x) - T'(2^k x)\|, \|f(-2^k x) - T'(-2^k x)\|\} \\
&\leq \lim_{k \rightarrow \infty} \lim_{n \rightarrow \infty} \max\left\{\frac{\varphi(2^j x, 0)}{|4|^{j+2}}, \frac{\varphi(-2^j x, 0)}{|4|^{j+2}} : k \leq j < n+k\right\} \\
&= 0 \quad (x \in X).
\end{aligned}$$

Therefore $T = T'$. This completes the proof of the uniqueness of T . $\square$

Corollary 2.2. *Let X and Y be non-Archimedean normed spaces over $\mathbb{K}$ with $|2| < 1$. If Y is complete and for some $2 < r$, $f : X \rightarrow Y$ satisfies the condition*

$$\|Df(x, y)\| \leq \theta(\|x\|^r + \|y\|^r)$$

for all $x, y \in X$. Then there exists a unique quadratic-additive mapping $T : X \rightarrow Y$ such that

$$(2.7) \quad \|f(x) - T(x)\| \leq |2|^{-3}\theta\|x\|^r.$$

Proof. Let $\varphi(x, y) = \theta(\|x\|^r + \|y\|^r)$. Since $|2| < 1$ and $r - 2 > 0$,

$$\lim_{n \rightarrow \infty} |4|^{-n} \varphi(2^n x, 2^n y) = \lim_{n \rightarrow \infty} |2|^{n(r-2)} \varphi(x, y) = 0$$

for all $x, y \in X$. Therefore f and φ satisfy the conditions of Theorem 2.1. It is easy to see that $\tilde{\varphi}(x) = |2|^{-3}\theta\|x\|^r$. By Theorem 2.1 there is a unique quadratic-additive mapping $T : X \rightarrow Y$ satisfying (2.7). $\square$

Theorem 2.3. *Let $\varphi : X^2 \rightarrow [0, \infty)$ be a function such that*

$$(2.8) \quad \lim_{n \rightarrow \infty} |2|^n \varphi(2^{-n}x, 2^{-n}y) = 0, \quad (x, y \in X)$$

and let for each $x \in X$ the limit

$$(2.9) \quad \lim_{n \rightarrow \infty} \max\left\{|2|^{j-1} \varphi\left(\frac{x}{2^{j+1}}, 0\right), |2|^{j-1} \varphi\left(\frac{-x}{2^{j+1}}, 0\right) : 0 \leq j < n\right\}.$$

denoted by $\tilde{\varphi}(x)$, exists. Suppose that $f : X \rightarrow Y$ is a mapping satisfying

$$(2.10) \quad \|Df(x, y)\| \leq \varphi(x, y), \quad (x, y \in X)$$

with $f(0) = 0$. Then there exists a unique quadratic-additive mapping $T : X \rightarrow Y$ such that

$$(2.11) \quad \|f(x) - T(x)\| \leq \tilde{\varphi}(x), \quad (x \in X).$$

In particular, T is given by

$$T(x) = \lim_{n \rightarrow \infty} \left(2^{2n-1} \left(f\left(\frac{x}{2^n}\right) + f\left(\frac{-x}{2^n}\right) \right) + 2^{n-1} \left(f\left(\frac{x}{2^n}\right) - f\left(\frac{-x}{2^n}\right) \right) \right)$$

for all $x \in X$.

Proof. Let $J_n f : X \rightarrow Y$ be a function defined by

$$J_n f(x) = 2^{2n-1} \left(f\left(\frac{x}{2^n}\right) + f\left(\frac{-x}{2^n}\right) \right) + 2^{n-1} \left(f\left(\frac{x}{2^n}\right) - f\left(\frac{-x}{2^n}\right) \right)$$

for all $x \in X$ and $n \in \mathbb{N}$. Notice that $J_0 f(x) = f(x)$ and

$$\begin{aligned} & \|J_j f(x) - J_{j+1} f(x)\| \\ &= \left\| -\frac{4^j + 2^j}{2} Df\left(\frac{x}{2^{j+1}}, 0\right) - \frac{4^j - 2^j}{2} Df\left(\frac{-x}{2^{j+1}}, 0\right) \right\| \\ (2.12) \quad & \leq \max \left\{ |2|^{j-1} \varphi\left(\frac{x}{2^{j+1}}, 0\right), |2|^{j-1} \varphi\left(\frac{-x}{2^{j+1}}, 0\right) \right\} \end{aligned}$$

for all $x \in X$ and $j \geq 0$. It follows from (2.8) and (2.12) that the sequence $\{J_n f(x)\}$ is Cauchy. Since Y is complete, we conclude that $\{J_n f(x)\}$ is convergent. Set

$$T(x) := \lim_{n \rightarrow \infty} J_n f(x).$$

Using induction one can show that

$$\begin{aligned} \|J_n f(x) - f(x)\| &\leq \max \left\{ |2|^{j-1} \varphi\left(\frac{x}{2^{j+1}}, 0\right), \right. \\ (2.13) \quad & \left. |2|^{j-1} \varphi\left(\frac{-x}{2^{j+1}}, 0\right) : 0 \leq j < n \right\} \end{aligned}$$

for all $x \in X$ and $n \in \mathbb{N}$. By taking n to approach infinity in (2.13) and using (2.9), one obtains (2.11). Replacing x and y by $2^{-n}x$ and $2^{-n}y$, respectively, in (2.10), we get

$$\begin{aligned} \|DJ_n f(x, y)\| &= \left\| 2^{n-1} Df\left(\frac{x}{2^n}, \frac{y}{2^n}\right) - 2^{n-1} Df\left(\frac{-x}{2^n}, \frac{-y}{2^n}\right) \right. \\ &\quad \left. + 2^{2n-1} Df\left(\frac{x}{2^n}, \frac{y}{2^n}\right) + 2^{2n-1} Df\left(\frac{-x}{2^n}, \frac{-y}{2^n}\right) \right\| \\ &\leq \max \left\{ |2|^{n-1} \varphi\left(\frac{x}{2^n}, \frac{y}{2^n}\right), |2|^{n-1} \varphi\left(\frac{-x}{2^n}, \frac{-y}{2^n}\right) \right\} \end{aligned}$$

for all $x, y \in X$ and $n \in \mathbb{N}$. Taking the limit as $n \rightarrow \infty$ and using (2.8), we get $DT(x, y) = 0$. Thus, the mapping T is a quadratic-additive mapping satisfying (2.11).

If T' is another quadratic-additive mapping satisfying (2.11), then

$$T'(x) - J_k T'(x) = \sum_{j=0}^{k-1} \left(-\frac{4^j + 2^j}{2} Df\left(\frac{x}{2^{j+1}}, 0\right) - \frac{4^j - 2^j}{2} Df\left(\frac{-x}{2^{j+1}}, 0\right) \right) = 0$$

for any $k \in \mathbb{N}$ and so

$$\begin{aligned} & \|T(x) - T'(x)\| \\ &= \lim_{k \rightarrow \infty} \|J_k T(x) - J_k T'(x)\| \\ &\leq \lim_{k \rightarrow \infty} \max\{\|J_k T(x) - J_k f(x)\|, \|J_k f(x) - J_k T'(x)\|\} \\ &\leq \lim_{k \rightarrow \infty} |2|^{k-1} \max\left\{\left\|T\left(\frac{x}{2^k}\right) - f\left(\frac{x}{2^k}\right)\right\|, \left\|T\left(-\frac{x}{2^k}\right) - f\left(-\frac{x}{2^k}\right)\right\|, \right. \\ &\quad \left.\left\|f\left(\frac{x}{2^k}\right) - T'\left(\frac{x}{2^k}\right)\right\|, \left\|f\left(-\frac{x}{2^k}\right) - T'\left(-\frac{x}{2^k}\right)\right\|\right\} \\ &\leq \lim_{k \rightarrow \infty} |2|^{k-1} \tilde{\varphi}\left(\frac{x}{2^k}\right) \\ &= 0 \quad (x \in X). \end{aligned}$$

Therefore $T = T'$. This completes the proof of the uniqueness of T . $\square$

Corollary 2.4. *Let X and Y be non-Archimedean normed spaces over $\mathbb{K}$ with $|2| < 1$. If Y is complete and for some $0 \leq r < 1$, $f : X \rightarrow Y$ satisfies the condition*

$$\|Df(x, y)\| \leq \theta(\|x\|^r + \|y\|^r)$$

for all $x, y \in X$. Then there exists a unique quadratic-additive mapping $T : X \rightarrow Y$ such that

$$(2.14) \quad \|f(x) - T(x)\| \leq \begin{cases} |2|^{-1-r} \theta \|x\|^r & \text{if } 0 < r < 1, \\ 2|2|^{-1} \theta & \text{if } r = 0. \end{cases}$$

Proof. Let $\varphi(x, y) = \theta(\|x\|^r + \|y\|^r)$. Since $|2| < 1$ and $1 - r > 0$,

$$\lim_{n \rightarrow \infty} |2|^n \varphi(2^{-n}x, 2^{-n}y) = \lim_{n \rightarrow \infty} |2|^{n(1-r)} \varphi(x, y) = 0$$

for all $x, y \in X$. Therefore f and φ satisfy the conditions of Theorem 2.3. It is easy to see that $\tilde{\varphi}(x) = |2|^{-1-r} \theta \|x\|^r$ if $0 < r < 1$ and $\tilde{\varphi}(x) = 2|2|^{-1} \theta$ if $r = 0$. By Theorem 2.3 there is a unique quadratic-additive mapping $T : X \rightarrow Y$ satisfying (2.14). $\square$

References

- [1] T. Aoki, *On the stability of the linear transformation in Banach spaces*, J. Math. Soc. Japan **2** (1950), 64-66.
- [2] D. H. Hyers, *On the stability of the linear functional equation*, Proc. Natl. Acad. Sci. USA **27** (1941), 222-224.
- [3] M.S. Moslehian and Th. M. Rassias, *Stability of functional equations in non-Archimedean spaces*, Appl. Anal. Discrete Math. **1** (2007), 325-334.
- [4] Th. M. Rassias, *On the stability of the linear mapping in Banach spaces*, Proc. Amer. Math. Soc. **72** (1978), 297-300.
- [5] S.M. Ulam, *A Collection of Mathematical Problems*, Interscience, New York, 1960.

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