THE BOREL-CANTELLI LEMMA UNDER PAIRWISE EXTENDED NEGATIVE QUADRANT DEPENDENCE

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Abstract. By extending the negatively quadrant dependence, the paper puts forth the concept of extended negative quadrant dependence. A generalization of the second Borel-Cantelli lemma is obtained under extended negative quadrant dependence. Some applications are also introduced.

1. Introduction

The definitions, properties and applications regarding negative quadrant dependence (NQD) can be found in Lehmann (1966) and Block et al. (1982). We call two events A and B are extended negatively quadrant dependent (ENQD) if there exists a constant $M > 0$ such that

$$(1.1) \quad P(A \cap B) \leq MP(A)P(B).$$

Recall that two events A and B are called negatively quadrant dependent (NQD) if (1.1) holds when $M = 1$; they are called positively quadrant dependent (PQD) if the inequality (1.1) holds in the reverse direction when $M = 1$. Obviously, NQD events must be ENQD events. On the other hand, for some PQD events, it is possible to find a corresponding positive constant M such that (1.1) holds. Therefore the ENQD events in that it can reflect not only a negative quadrant dependence but also a positive one, to some extent. (See Ko and Tang (2008) and Liu (2009).)

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We call random variables X and Y extended negatively quadrant dependent if there exists a constant $M > 0$ such that

$$(1.2) \quad \begin{aligned} P(X \leq x, Y \leq y) &\leq MP(X \leq x)P(Y \leq y) \\ (\text{or } P(X > x, Y > y) &\leq MP(X > x)P(Y > y)) \end{aligned}$$

holds for all x, y .

In the following example, the extended negatively quadrant dependent random variables X and Y can be taken as negatively quadrant dependent or positive quadrant dependent since there are no restrictions on the dependence structure between X and Y .

Example. If X is possibly valued at $x_1 \leq y_1$. Then, X and Y are extended negatively quadrant dependent. In fact, for any x and y such that

$$P(X \leq x)P(Y \leq y) = 0,$$

$$(1.3) \quad P(X \leq x, Y \leq y) \leq MP(X \leq x)P(Y \leq y) \text{ for a constant } M \geq 0$$

hold. Additionally, for any x and y such that $P(X \leq x)P(Y \leq y) \neq 0$, take $M = 1/\min\{P(X = x_1), P(Y = y_1)\}$ then (1.3) still holds.

The following Borel-Cantelli lemma plays an exceptionally important role in probability theory: if $A_1, A_2, \dots$ is a sequence of events on common probability space $(\Omega, \mathcal{F}, \mathcal{P})$ and if $\sum_{n=1}^{\infty} P(A_n) < \infty$, then $P(\limsup A_n) = 0$; if $A_1, A_2, \dots$ is a sequence of independent events and if $\sum_{n=1}^{\infty} P(A_n) = \infty$, then $P(\limsup A_n) = 1$, where $\limsup A_n = \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} A_k$.

Erdős and Renyi(1959) investigated that the independence condition in the second part of the Borel-Cantelli lemma can be replaced by the weaker condition of pairwise independence of events $A_1, A_2, \dots$. They also proved that the condition of pairwise independence of events $A_1, A_2, \dots$ can be replaced by the weaker condition $P(A_i A_j) \leq P(A_i)P(A_j)$ for every i and j such that $i \neq j$. Recently, Petrov(2002, 2004) and Chandra, T.K.(2008) studied a generalization of the Borel-Cantelli lemma. Especially, note that Petrov(2002) investigated the following Theorem 2.1 only for the case $M \geq 1$.

In this paper we consider the Borel-Cantelli lemma under extended negative quadrant dependence. As applications we also obtain sufficient condition of strong law of large numbers for extended negatively quadrant dependent random variables.

2. Results

Theorem 2.1. Let $A_1, A_2, \dots$ be a sequence of pairwise ENQD events i.e. there exists a constant $M > 0$ such that

$$(2.1) \quad P(A_i A_j) \leq M P(A_i) P(A_j)$$

for all i and j ($i \neq j$). If

$$(2.2) \quad \sum_{n=1}^{\infty} P(A_n) = \infty,$$

then

$$(2.3) \quad P(\limsup A_n) \geq 1/M \text{ for } M > 1,$$

$$(2.4) \quad P(\limsup A_n) = 1 \text{ for } 0 < M \leq 1.$$

Proof. As the proof of Theorem 2.1 in Petrov(2002) the proof is based on the inequality of Chung and Erdős(1952). For arbitrary events $A_1, A_2, \dots, A_n$, we have

$$(2.5) \quad P(\cup_{k=1}^n A_k) \geq \left(\sum_{k=1}^n P(A_k) \right)^2 / \sum_{k,j=1}^n P(A_k A_j).$$

Let

$$(2.6) \quad S = \sum_{k,j=n, k \neq j}^N P(A_k) P(A_j) \text{ and } T = \sum_{k=n}^N P(A_k).$$

It follows from condition (2.1) that

$$(2.7) \quad \sum_{k,j=n}^N P(A_k A_j) \leq MS + T.$$

Petrov(2002) proved (2.3). For completeness we repeat it here.

If $M > 1$, (2.7) implies the inequality

$$(2.8) \quad \sum_{k,j=n}^N P(A_k A_j) \leq MS + T \leq M(S + T).$$

We also have

$$(2.9) \quad S = \left(\sum_{k=n}^N P(A_k) \right)^2 - \sum_{k=n}^N (P(A_k))^2.$$

Therefore,

$$\begin{aligned}
(2.10) \quad S + T &= \left(\sum_{k=n}^N P(A_k) \right)^2 - \sum_{k=n}^N (P(A_k))^2 + \sum_{k=n}^N P(A_k) \\
&\leq \left(\sum_{k=n}^N P(A_k) \right)^2 + \sum_{k=n}^N P(A_k) \\
&= T^2 + T.
\end{aligned}$$

By (2.5)-(2.10), we obtain

$$\begin{aligned}
(2.11) \quad P(\cup_{k=n}^N A_k) &\geq M^{-1} T^2 (T^2 + T)^{-1} \\
&\geq M^{-1} (1 + T^{-1})^{-1} \\
&= M^{-1} \{1 + (\sum_{k=n}^N P(A_k))^{-1}\}^{-1}.
\end{aligned}$$

It follows from condition (2.2) that for a fixed n

$$(2.12) \quad 1 + \left(\sum_{k=n}^N P(A_k) \right)^{-1} \rightarrow 1 \text{ as } N \rightarrow \infty.$$

Thus,

$$\liminf_{N \rightarrow \infty} P(\cup_{k=n}^N A_k) \geq M^{-1}$$

and

$$(2.13) \quad P(\cup_{k=n}^{\infty} A_k) \geq M^{-1}.$$

Let $B_n = \cup_{k=n}^{\infty} A_k$, Then, we observe that $B_1 \supset B_2 \supset \dots$ and

$$(2.14) \quad \cap_{n=1}^{\infty} B_n = \limsup A_n.$$

From (2.13) and (2.14) we obtain

$$(2.15) \quad \lim P(B_n) = P(\cap_{n=1}^{\infty} B_n) = P(\limsup A_n)$$

and the inequality

$$P(\limsup A_n) \geq M^{-1}$$

by (2.13)-(2.15).

If $0 < M \leq 1$, (2.7) implies the inequality

$$\begin{aligned}
(2.16) \quad \sum_{k,j=n}^N P(A_k A_j) &\leq MS + T \\
&\leq S + T.
\end{aligned}$$

It follows from (2.5), (2.7), (2.9), (2.10) and (2.16) that

$$\begin{aligned}
 P(\cup_{k=n}^N A_k) &\geq (\sum_{k=n}^N P(A_k))^2 / \sum_{k,j=n}^N P(A_k A_j) \\
 &\geq T^2 (S + T)^{-1} \\
 (2.17) \quad &\geq T^2 (T^2 + T)^{-1} \\
 &\geq (1 + T^{-1})^{-1} \\
 &= \{1 + (\sum_{k=n}^N P(A_k))^{-1}\}^{-1}.
 \end{aligned}$$

It follows from (2.12) and (2.17) that

$$(2.18) \quad P(\cup_{k=n}^\infty A_k) \geq 1.$$

Hence by (2.13), (2.14), (2.15) and (2.18) we obtain

$$P(\limsup A_n) \geq 1.$$

The proof is complete.

Remark. Recently Petrov(2002, 2004) found that (2.3) is satisfied and $P(A_i A_k) \leq H P(A_i) P(A_k)$ for all $i, k > N$ such that $i \neq k$ and for some constants only $H \geq 1$ and N , then $P(\limsup A_n) \geq H$. He did not considered for $0 < H < 1$.

3. Applications

Theorem 3.1. Let $\{X_n, n \geq 1\}$ be a sequence of pairwise extended negatively quadrant dependent random variables i.e., there exists a constant $M > 0$ such that

$$(3.1) \quad P(X_i \leq x_i, X_j \leq x_j) \leq M P(X_i \leq x_i) P(X_j \leq x_j)$$

for all i and $j(i \neq j)$ and all x_i, x_j . If

$$(3.2) \quad \sum_{n=1}^\infty P(X_n \leq x_n) = \infty,$$

then

$$(3.3) \quad P(\limsup\{X_n \leq x_n\}) \geq 1/M \text{ for } M > 1,$$

$$(3.4) \quad P(\limsup\{X_n \leq x_n\}) = 1 \text{ for } 0 < M \leq 1.$$

Proof. Let $A_k = \{X_k \leq x_k\}$. Then A_k 's are pairwise extended negatively quadrant dependent. Hence, by Theorem 2.1 the desired results (3.3) and (3.4) follow.

Theorem 3.2. Let $\{X_n, n \geq 1\}$ be a sequence of pairwise extended negatively quadrant dependent random variables and $\{a_n, n \geq 1\}$ a sequence of positive numbers. If

$$S_n/a_n \rightarrow 0 \text{ a.s. as } n \rightarrow \infty$$

and

$$\sup_{n \geq 1} a_{n-1}/a_n \leq M < \infty$$

for some positive M then

$$\sum_{n=1}^{\infty} P\{|X_n| > a_n\} < \infty,$$

where $S_n = \sum_{i=1}^n X_i$.

Proof. If $S_n/a_n \rightarrow 0$ a.s., then $X_n/a_n \rightarrow 0$ a.s. as $n \rightarrow \infty$. Thus putting

$$X_n^+ = \max\{0, X_n\},$$

$$X_n^- = \max\{0, -X_n\},$$

we also have

$$X_n^+/a_n \rightarrow 0 \text{ a.s.}$$

and

$$X_n^-/a_n \rightarrow 0 \text{ a.s.}$$

Define

$$A_n = [X_n^+ > \frac{1}{3}a_n] \text{ and } B_n = [X_n^- > \frac{1}{3}a_n], \quad n \geq 1.$$

Note that A_i 's and B_i 's are pairwise extended negatively quadrant dependent events. Hence, for some positive constant $M > 0$ we have

$$P(A_k \cap A_j) \leq MP(A_k)P(A_j) \text{ for } k \neq j,$$

$$P(B_k \cap B_j) \leq MP(B_k)P(B_j) \text{ for } k \neq j.$$

By Theorem 2.1 if $\sum_{n=1}^{\infty} P[X_n^+ > \frac{1}{3}a_n] = \infty$, then $P[\limsup A_k] \geq \frac{1}{M}$ contrary to the almost sure convergence of $X_n^+/a_n \rightarrow 0$. Hence,

$$\sum_{n=1}^{\infty} P[X_n^+ > \frac{1}{3}a_n] < \infty.$$

By the similar consideration we have

$$\sum_{n=1}^{\infty} P[X_n^- > \frac{1}{3}a_n] < \infty.$$

Thus

$$\begin{aligned} & \sum_{n=1}^{\infty} P[|X_n| > a_n] \\ &= \sum_{n=1}^{\infty} P[X_n^+ + X_n^- > a_n] \\ &\leq \sum_{n=1}^{\infty} P[X_n^+ > \frac{1}{3}a_n] + \sum_{n=1}^{\infty} P[X_n^- > \frac{1}{3}a_n] \\ &< \infty, \end{aligned}$$

which completes the proof.

Theorem 3.3. Let $\{X_n, n \geq 1\}$ be a sequence of pairwise extended negatively quadrant dependent random variables with the same distribution $F(x)$. If $S_n/n \rightarrow 0$ a.s., then $E|X_1| < \infty$.

Proof. In Theorem 3.2 set $a_n = n$ and take into account that X 's are equidistributed. Then by Theorem 3.2 we get $\sum_{n=1}^{\infty} P[|X_n| > n] > \infty$. Thus $E|X_1| < \infty$.

In the appendix we introduced the properties of extended negative quadrant dependence needed to prove main results.

Appendix

Theorem A.1. Two random variables X and Y satisfy that there exists a positive constant $M > 0$ such that

$$(A.1) \quad P(X \leq x, Y \leq y) \leq MP(X \leq x)P(Y \leq y),$$

for all x and y .

Then there is a positive constant $M' > 0$ such that

$$(A.2) \quad P(X > x, Y > y) \leq M'P(X > x)P(Y > y),$$

for all x and y .

Proof. When $P(X > x, Y > y) = 0$ obviously (A.2) holds. Now assume $P(X > x, Y > y) \neq 0$.

$$\begin{aligned}
 P(X > x, Y > y) &= P(X > x) - P(X > x, Y \leq y) \\
 &= P(X > x) - P(Y \leq y) + P(X \leq x, Y \leq y) \\
 &\leq P(X > x) + P(Y > y) - 1 + MP(X \leq x)P(Y \leq y) \\
 &= P(X > x) + P(Y > y) - 1 \\
 &\quad + M(1 - P(X > x))(1 - P(Y > y)) \\
 &\leq (M - 1) + MP(X > x)P(Y > y).
 \end{aligned}$$

If $0 < M \leq 1$, then $P(X > x, Y > y) \leq MP(X > x)P(Y > y)$.

If $M > 1$, then

$$\begin{aligned}
 P(X > x, Y > y) &\leq \left(M + \frac{M - 1}{P(X > x)P(Y > y)}\right)P(X > x)P(Y > y) \\
 &= M'P(X > x)P(Y > y),
 \end{aligned}$$

where $M' = M + \frac{M-1}{P(X>x)P(Y>y)} \geq M > 1$.

Theorem A.2. If (X, Y) is extended negatively quadrant dependent, then for nondecreasing functions f and g $(f(X), g(Y))$ is also extended negatively quadrant dependent.

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