MINIMUM PERMANENTS ON DOUBLY STOCHASTIC MATRICES WITH PRESCRIBED ZEROS

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Abstract. We consider permanent function on the faces of the polytope of certain doubly stochastic matrices, whose nonzero entries coincide with those of fully indecomposable square $(0, 1)$ -matrices containing identity submatrix. We determine the minimum permanents and minimizing matrices on the given faces of the polytope using the contraction method.

1. Introduction and preliminaries

Let Ω_n be the polyhedron of $n \times n$ doubly stochastic matrices, that is, the $n \times n$ nonnegative matrices whose row and column sums are all equal to 1. The *permanent* of an $n \times n$ matrix $A = [a_{ij}]$ is defined by

$$\text{per}(A) = \sum_{\sigma} a_{1\sigma(1)} \cdots a_{n\sigma(n)}$$

where σ runs over all permutations of $\{1, 2, \dots, n\}$.

Let $J_{r,s}$ denote the $r \times s$ matrix all of whose entries are 1. In 1981 Egorycev [3] and Falikman [4] proved the van der Waerden permanent conjecture: If $A \in \Omega_n$, then

$$\text{per}(A) \geq \text{per}\left(\frac{1}{n}J_{n,n}\right).$$

The techniques of Egorycev have been used, with some success, for determination of minimum permanents in various faces of Ω_n (See [2], [5]–[9]). The key technique is replacing rows (or columns) of a matrix with

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minimum permanent by their average without altering its permanent. Unfortunately, the presence of fixed zeros restricts the use of this technique. Indeed this tool is not available at all in the case of faces which consist of matrices with at least one fixed zero in each row and column. In this paper we use this technique in some parts of proofs.

Let $D = [d_{ij}]$ be an n -square nonnegative matrix, and let

$$\Omega(D) = \{X = [x_{ij}] \in \Omega_n | x_{ij} = 0 \text{ whenever } d_{ij} = 0\}.$$

Then $\Omega(D)$ is a face of Ω_n , and since it is compact, $\Omega(D)$ contains a minimizing matrix A such that $\text{per}(A) \leq \text{per}(X)$ for all $X \in \Omega(D)$.

Brualdi [1] defined an n -square $(0,1)$ matrix D to be *cohesive* if there is a matrix Z in the interior of $\Omega(D)$ for which

$$\text{per}(Z) = \min\{\text{per}(X) | X \in \Omega(D)\}.$$

And he defined an n -square $(0,1)$ matrix D to be *barycentric* if

$$\text{per}(b(D)) = \min\{\text{per}(X) | X \in \Omega(D)\},$$

where the *barycenter* $b(D)$ of $\Omega(D)$ is given by

$$b(D) = \frac{1}{\text{per}(D)} \sum_{P \leq D} P,$$

where the summation extends over the set of all permutation matrices P with $P \leq D$ and $\text{per}(D)$ is their number.

Let I_n denote the identity matrix of order n and 0_k the $k \times k$ zero matrix.

In [9], Song considered a matrix $V_{m,n} = \begin{bmatrix} J_{m,m} & J_{m,n} \\ J_{n,m} & I_n \end{bmatrix}$ and suggested determining the minimum permanents and minimizing matrices on $\Omega(V_{m,n})$ for $m \geq 2$, and $n \geq 3$. This face $\Omega(V_{m,n})$ is extended one of $\Omega(W_n)$ in Theorem 5 in [1].

Brualdi ([1]) determined the minimum permanent and minimizing matrix on $\Omega(V_{1,n-1})$. Song determined the minimum permanents on $\Omega(V_{3,n})$ in [8] and on $\Omega(V_{m,3})$ in [7], respectively. In [6] the authors determined the minimum permanents for various cases and gave a conjecture.

In this paper, we consider the faces $\Omega(V_{4,n})$ and $\Omega(U_{4,n})$, where $U_{4,n} = \begin{bmatrix} 0_{4,4} & J_{4,n} \\ J_{n,4} & I_n \end{bmatrix}$. We show that $U_{4,n}$ is both cohesive and barycentric and determine the minimum permanents and minimizing matrices on the face $\Omega(V_{4,n})$ of Ω_{4+n} for $n \geq 6$ using the contraction method, which was obtained under certain assumption in [6]. We show that the Conjecture in [6] holds in some part. Our results contribute toward characterizations of cohesive and barycentric matrices respectively, which were asked for in [1].

Recall that an n -square nonnegative matrix is said to be *fully indecomposable* if it contains no $k \times (n-k)$ zero submatrix for $k = 1, \dots, n-1$.

For a matrix A , let $A(i, j, \dots, k \mid l, m, \dots, n)$ denote the submatrix obtained from A by deleting rows $i, j, \dots, k$, and columns $l, m, \dots, n$. In particular, we simplify the notation $A(i, j, \dots, k \mid i, j, \dots, k)$ to $A(i, j, \dots, k)$.

We use the following well-known Lemma ([5]).

Lemma 1.1. ([5]) *Let $D = [d_{ij}]$ be an n -square fully indecomposable $(0, 1)$ matrix, and $A = [a_{ij}]$ be a minimizing matrix on $\Omega(D)$. Then A is fully indecomposable, and for (i, j) such that $d_{ij} = 1$,*

$$\text{per}(A(i \mid j)) = \text{per}(A) \quad \text{if } a_{ij} > 0,$$

$$\text{per}(A(i \mid j)) \geq \text{per}(A) \quad \text{if } a_{ij} = 0.$$

2. The Cohesiveness of $U_{4,n}$

In this section, we consider the $(4+n)$ -square $(0, 1)$ matrix $U_{4,n}$ and we show that this matrix is cohesive for $n \geq 6$.

If a column j of an $n \times n$ matrix A contains exactly k nonzero entries ($2 \leq k \leq n$), say in rows $r_1, \dots, r_k$, then the $(n-1)$ square matrix $GC(A)$ obtained from A by replacing rows $r_2, \dots, r_k$ with $\frac{1}{k-1}(r_1 + r_2 + \dots + r_k)$ and deleting row r_1 and column j is called a *generalized contraction* of A .

Lemma 2.1. ([8]) *Suppose $D \in \Omega_n$ is fully indecomposable and has a column (row) with exactly k positive entries and those k rows (columns) have the same zero pattern. Let A be a minimizing matrix on $\Omega(D)$.*

Then $GC(A)$ is fully indecomposable, and there is a minimizing matrix $\overline{GC(A)} \in \Omega(GC(A))$ satisfying

$$\text{per}(A) = \left(\frac{k-1}{k}\right)^{k-1} \text{per}(GC(A)) \geq \left(\frac{k-1}{k}\right)^{k-1} \text{per}(\overline{GC(A)}).$$

Lemma 2.2. ([8]) For $n \geq 5$, $U_{3,n} = \begin{bmatrix} 0_3 & J_{3,n} \\ J_{n,3} & I_n \end{bmatrix}$ is barycentric and the minimum permanent on $\Omega(U_{3,n})$ is $\frac{6(n-1)(n-2)(n-3)^{n-3}}{n^{n+2}}$.

Theorem 2.3. For $n \geq 5$, we show that $U_{4,n}$ is a cohesive matrix.

Proof. Let $B_{4,n}$ be a minimizing matrix on $\Omega(U_{4,n})$. Since the first 4 columns and first 4 rows of $U_{4,n}$ are the same, we can use the averaging method (see [3] or [6]) on those columns and rows of $B_{4,n}$, respectively. Thus we can write $B_{4,n}$ as follows:

$$(2.1) \quad B_{4,n} = \left[\begin{array}{c|cccc} 0_4 & \bar{\mathbf{b}}_1 & \bar{\mathbf{b}}_2 & \dots & \bar{\mathbf{b}}_n \\ \hline \bar{\mathbf{b}}_1^t & x_1 & & & 0 \\ \bar{\mathbf{b}}_2^t & & x_2 & & \\ \vdots & & & \ddots & \\ \bar{\mathbf{b}}_n^t & & & & x_n \end{array} \right],$$

where $\bar{\mathbf{b}}_i(\bar{\mathbf{b}}_i^t)$ is a column (row) vector with b_i as all its entries for $i = 1, 2, \dots, n$.

Since the permanent value is invariant under the interchange of rows (and columns, respectively), we may assume that

$$(2.2) \quad b_{i+1} \leq b_i \quad (\text{i.e. } x_{i+1} \geq x_i)$$

for $i = 1, 2, \dots, n-1$, without loss of generality. Since $U_{3,n}$ is fully indecomposable, each

$$(2.3) \quad b_i \neq 0$$

for $i = 1, \dots, n$.

Suppose $x_1 = 0$. Then the fourth row and column of $B_{4,n}$ have exactly 4 nonzero entries. Thus we can obtain a generalized contraction $GC(B_{4,n})$ of $B_{4,n}$. Since the third row of $GC(B_{4,n})$ has exactly 3 nonzero

entries, we can obtain its generalized contraction $GC(GC(B_{4,n}))$, which is contained in $\Omega(U_{3,n-1})$. Using Lemmas 2.1 and 2.2, we have

$$\begin{aligned}
 \text{per}(B_{4,n}) &= \left(\frac{3}{4}\right)^3 \text{per}(GC(B_{4,n})) \\
 (2.4) \quad &\geq \left(\frac{3}{4}\right)^3 \left\{ \left(\frac{3}{4}\right)^3 b(U_{3,n-1}) \right\} \\
 &= \left(\frac{3}{4}\right)^6 \cdot \frac{6(n-2)(n-3)(n-4)^{n-4}}{(n-1)^{n+1}}.
 \end{aligned}$$

But $\text{per}(B_{4,n})$ is less than or equal to the permanent of the barycenter $b(U_{4,n})$. That is,

$$\begin{aligned}
 \text{per}(B_{4,n}) &\leq \text{per}(b(U_{4,n})) \\
 (2.5) \quad &= 4 \cdot \frac{1}{n} [(n-1) \cdot \frac{1}{n} \cdot \frac{3}{n} \cdot (n-2) \cdot \frac{1}{n} \cdot \frac{2}{n} \cdot (n-3) \cdot \frac{1}{n} \cdot \frac{1}{n}] \left(\frac{n-4}{n}\right)^{n-4} \\
 &= \frac{4!(n-1)(n-2)(n-3)(n-4)^{n-4}}{n^{n+3}}.
 \end{aligned}$$

If we divide the last value in (2.5) by the last value in (2.4), then we can show that the result is less than 1 by a direct calculation using Mathematica Program for $n \geq 5$. Thus we have a contradiction from the inequalities in (2.4) and (2.5). Hence x_1 is not zero. By (2.2), each

$$(2.6) \quad x_i \neq 0$$

for $i = 1, \dots, n$. Hence $U_{4,n}$ is cohesive by (2.3) and (2.6). $\square$

Remark 2.4. In [6], the cohesiveness of $U_{m,n}$ was not proved. And the fact that $x_i > 0$ was assumed in the proof of the Theorem 2.1 in [6]. But this fact that each $x_i > 0$ is essential for the determining of the minimum permanents on $\Omega(U_{4,n})$.

3. Minimum permanents on $\Omega(U_{4,n})$

In this section, we show that $U_{4,n}$ is a barycentric matrix and hence we determine the minimum permanents on $\Omega(U_{4,n})$ for $n \geq 5$ using the results in Theorem 2.3.

Theorem 3.1. For $n \geq 5$, the minimum permanent on $\Omega(U_{4,n})$ is

$$(3.1) \quad \frac{4!(n-1)(n-2)(n-3)(n-4)^{n-4}}{n^{n+3}}.$$

which occurs at the barycenter $b(U_{4,n})$.

Proof. Let $B_{4,n}$ be a minimizing matrix on $\Omega(U_{4,n})$. Then $B_{4,n}$ has the form of $B_{4,n}$ in (2.1) as the proof of Theorem 2.3. Without loss of generality, we also assume

$$(3.2) \quad b_{i+1} \leq b_i \text{ (i.e. } x_{i+1} \geq x_i \text{)}$$

for $i = 1, 2, \dots, n-1$. Since $U_{4,n}$ is a cohesive matrix by Theorem 2.3, each b_i and x_i are positive for $i = 1, 2, \dots, n$.

Now, we will show that all b_i are the same. Suppose to the contrary that $b_1 > b_n$. Then we have $\text{per}B_{4,n} = \text{per}B_{4,n}(1 \mid 5) = \text{per}B_{4,n}(1 \mid n+4)$, from Lemma 1.1 since all $b_i > 0$. But

$$\text{per}B_{4,n}(1 \mid 5) = (4b_1)\{(3b_n)^2\text{per}B_{4,n}(1, 2, 5, n) + x_n\text{per}B_{4,n}(1, 5, n)\},$$

and

$$\text{per}B_{4,n}(1 \mid n) = (4b_n)\{(3b_1)^2\text{per}B_{4,n}(1, 2, 5, n) + x_1\text{per}B_{4,n}(1, 5, n)\}.$$

From these values, using $x_i = 1 - 4b_i$ for $i = 1$ and n , we have

$$(3.3) \quad 9b_1b_n\text{per}B_{4,n}(1, 2, 5, n) = \text{per}B_{4,n}(1, 5, n).$$

Since all $x_i > 0$, we have

$$\text{per}B_{4,n} = \text{per}B_{4,n}(5 \mid 5) = \text{per}B_{4,n}(n \mid n),$$

from Lemma 1.1.

But

$$\text{per}B_{4,n}(5 \mid 5) = (4b_n)^2\text{per}B_{4,n}(1, 5, n) + x_n\text{per}B_{4,n}(5, n),$$

and

$$\text{per}B_{4,n}(n \mid n) = (4b_1)^2\text{per}B_{4,n}(1, 5, n) + x_1\text{per}B_{4,n}(5, n).$$

From these values, using $x_i = 1 - 4b_i$ for $i = 1$ and n , we have

$$(3.4) \quad 4(b_1 + b_n)\text{per}B_{4,n}(1, 5, n) = \text{per}B_{4,n}(5, n).$$

Let us calculate the $\text{per}B_{4,n}$ by expanding along the n -th and 5 -th columns. Then we have

$$\begin{aligned} \text{per}B_{4,n} &= (4b_n)^2\text{per}B_{4,n}(1, n) + x_n\text{per}B_{4,n}(n \mid n) \\ &= (4b_n)^2\{(3b_1)^2\text{per}B_{4,n}(1, 2, 5, n) + x_1\text{per}B_{4,n}(1, 5, n)\} \\ (3.5) \quad &+ x_n\{(4b_1)^2\text{per}B_{4,n}(1, 5, n) + x_1\text{per}B_{4,n}(5, n)\} \\ &= 4^2 \cdot 3^2 \cdot b_1^2 b_n^2 \text{per}B_{4,n}(1, 2, 5, n) + 16b_n^2(1 - 4b_1)\text{per}B_{4,n}(1, 5, n) \\ &+ 16b_1^2(1 - 4b_n)\text{per}B_{4,n}(1, 5, n) + (1 - 4b_n)(1 - 4b_1)\text{per}B_{4,n}(5, n). \end{aligned}$$

Now applying (3.3) and (3.4) to (3.5), we have

$$\begin{aligned}
 \text{per} B_{4,n} &= 16(b_1 b_n + b_1^2 + b_n^2) \text{per} B_{4,n}(1, 5, n) \\
 &\quad + (1 - 4(b_1 + b_n)) \text{per} B_{4,n}(5, n) \\
 &= 16(b_1 + b_n)^2 \text{per} B_{4,n}(1, 5, n) - 16b_1 b_n \text{per} B_{4,n}(1, 5, n) \\
 &\quad + \text{per} B_{4,n}(5, n) - 4(b_1 + b_n) \text{per} B_{4,n}(5, n) \\
 &= -16b_1 b_n \text{per} B_{4,n}(1, 5, n) + \text{per} B_{4,n}(5, n) \\
 &= 4(b_1 + b_n - 4b_1 b_n) \text{per} B_{4,n}(1, 5, n).
 \end{aligned}$$

Thus if we increase b_1 and decrease b_n with fixed $b_1 + b_n$ in the above equation, then we can decrease $\text{per} B_{4,n}$. This contradicts to the fact that $B_{4,n}$ is a minimizing matrix. Hence we must have $b_1 = b_n$. Using (3, 2), we have that all b_i and x_i are the same, respectively. Thus $B_{4,n}$ with each $b_i = \frac{1}{n}$ and $x_i = \frac{n-4}{n}$ is a minimizing matrix on $\Omega(U_{4,n})$, and it is the barycenter of $\Omega(U_{4,n})$. Moreover, the minimum permanent is $\text{per} b(\Omega(U_{4,n})) = \text{per} B_{4,n}(1 | 5) = \frac{4!(n-1)(n-2)(n-3)(n-4)^{n-4}}{n^{n+3}}$ as required in (3.1). $\square$

4. Minimum permanents on $\Omega(V_{4,n})$

In this section, we consider the $(n+4)$ -square $(0, 1)$ -matrix $V_{4,n}$ and determine the minimizing matrices and minimum permanents on the faces $\Omega(V_{4,n})$ for $n \geq 6$. In [6], the authors gave the following conjecture:

Conjecture 4.1. ([6], Conjecture 2.1) $V_{m,n}$ is cohesive but not barycentric for $1 < n < m + \sqrt{m}$, while for $n \geq m + \sqrt{m}$, $V_{m,n}$ is not cohesive and $b(U_{m,n})$ is a minimizing matrix.

We show that this Conjecture 4.1 holds for the case $n \geq m + \sqrt{m}$, with $n = 4$ and use contraction method to determine the minimum permanents on $\Omega(V_{4,n})$.

Lemma 4.2. ([7]) For $n \geq 5$, the minimum permanent on $\Omega(V_{3,n})$ is $\frac{6(n-1)(n-2)(n-3)^{n-3}}{n^{n+2}}$ and the unique minimizing matrix is the barycenter $b(U_{3,n})$ of $\Omega(U_{3,n})$.

Lemma 4.3. For $n \geq 5$, the minimizing matrix on $\Omega(V_{4,n})$ is the form

$$(4.1) \quad A_{4,n} = \begin{bmatrix} aJ_{4,4} & dJ_{4,n} \\ dJ_{n,4} & cI_n \end{bmatrix}$$

with $4a + nd = 1 = 4d + c$.

Proof. Let $A_{4,n}$ be a minimizing matrix on $\Omega(V_{4,n})$. Since the first 4 columns and first 4 rows of $V_{4,n}$ are the same, we can use the averaging method on those columns and rows of $A_{4,n}$, respectively. Thus we can write $A_{4,n}$ as follows:

$$(4.2) \quad A_{4,n} = \left[\begin{array}{c|cccc} aJ_{4,4} & \bar{\mathbf{d}}_1 & \bar{\mathbf{d}}_2 & \dots & \bar{\mathbf{d}}_n \\ \hline \bar{\mathbf{d}}_1^t & c_1 & & & 0 \\ \bar{\mathbf{d}}_2^t & & c_2 & & \\ \vdots & 0 & & \ddots & \\ \bar{\mathbf{d}}_n^t & & & & c_n \end{array} \right],$$

where $\bar{\mathbf{d}}_i$ ($\bar{\mathbf{d}}_i^t$) is a column (row) vector with d_i as all its entries for $i = 1, 2, \dots, n$. Since the permanent value is invariant under the interchange of rows (and columns), we may assume that

$$(4.3) \quad d_{i+1} \leq d_i \text{ (i.e. } c_{i+1} \geq c_i)$$

for $i = 1, \dots, n-1$ without loss of generality. Since $V_{4,n}$ is fully indecomposable, we can have that

$$(4.4) \quad d_i > 0$$

for $i = 1, \dots, n$ by Lemma 1.1.

Suppose $c_1 = 0$. Then the fourth row and column of $A_{4,n}$ have exactly 4 nonzero entries. Thus we can obtain a generalized contraction $GC(A_{4,n})$ of $A_{4,n}$. Since the third row of $GC(A_{4,n})$ has exactly 3 nonzero entries, we can obtain its generalized contraction $GC(GC(A_{4,n}))$, which is contained in $\Omega(V_{3,n-1})$. Using Lemmas 2.1 and 2.2, we have

$$(4.5) \quad \begin{aligned} \text{per}(A_{4,n}) &= \left(\frac{3}{4}\right)^3 \text{per}(GC(A_{4,n})) \\ &\geq \left(\frac{3}{4}\right)^3 \left\{ \left(\frac{3}{4}\right)^3 b(U_{3,n-1}) \right\} \\ &= \left(\frac{3}{4}\right)^6 \cdot \frac{6(n-2)(n-3)(n-4)^{n-4}}{(n-1)^{n+1}}. \end{aligned}$$

But $\text{per}(A_{4,n})$ is less than or equal to the permanent of the barycenter $b(U_{4,n})$. That is,

$$\begin{aligned}
 (4.6) \quad & \text{per}(A_{4,n}) \leq \text{per}(b(U_{4,n})) \\
 & = 4 \cdot \frac{1}{n} [(n-1) \cdot \frac{1}{n} \cdot \frac{3}{n} \cdot (n-2) \cdot \frac{1}{n} \cdot \frac{2}{n} \cdot (n-3) \cdot \frac{1}{n} \cdot \frac{1}{n}] \left(\frac{n-4}{n}\right)^{n-4} \\
 & = \frac{4!(n-1)(n-2)(n-3)(n-4)^{n-4}}{n^{n+3}}.
 \end{aligned}$$

If we divide the last value in (4.6) by the last value in (4.5), then we can show that the result is less than 1 by a direct calculation using Mathematica Program for $n \geq 5$. Thus we have a contradiction from the inequalities in (4.5) and (4.6). Hence c_1 is not zero. Hence all $c_i > 0$ by (4.3) for $i = 1, \dots, n$.

Suppose to the contrary that $b_1 > b_n$. Let

$$\begin{aligned}
 Q_1 &= \text{per}A_{4,n}(5, n+4), \\
 Q_2 &= \text{per}A_{4,n}(1, 5, n+4), \\
 Q_3 &= \text{per}A_{4,n}(1, 2, 5, n+4).
 \end{aligned}$$

From $\text{per}A_{4,n}(1 \mid 5) = \text{per}A_{4,n} = \text{per}A_{4,n}(1 \mid n+4)$, we have

$$4d_1\{(3d_n)^2Q_3 + c_nQ_2\} = 4d_n\{(3d_1)^2Q_3 + c_1Q_2\},$$

and hence

$$4 \cdot 9 \cdot d_1d_n(d_n - d_1)Q_3 = 4(c_1d_n - d_1c_n)Q_2.$$

Using $c_i = 1 - 4d_i$, we obtain

$$(4.7) \quad 9b_1b_nQ_3 = Q_2.$$

On the other hand, from

$$\text{per}A_{4,n}(5 \mid 5) = \text{per}A_{4,n} = \text{per}A_{4,n}(n+4 \mid n+4),$$

we have

$$c_1Q_1 + (4d_1)^2Q_2 = c_nQ_1 + (4d_n)^2Q_2.$$

Using $c_i = 1 - 4d_i$, we obtain

$$(4.8) \quad Q_1 = 4(d_1 + d_n)Q_2.$$

Let us calculate $\text{per}A_{4,n}$ by expanding along the 5th and $(4+n)$ th columns. Then we have

$$\begin{aligned}
 \text{per}A_{4,n} &= (4d_1)^2\text{per}A_{4,n}(1, 5) + c_1\text{per}A_{4,n}(5 \mid 5) \\
 &= (4d_1)^2\{(3d_n)^2Q_3 + c_nQ_2\} + c_1\{(4d_1)^2Q_2 + c_nQ_1\}.
 \end{aligned}$$

Using $c_i = 1 - 4d_i$ and applying (4.7), (4.8) to this equation, we have

$$\begin{aligned} \text{per} A_{4,n} &= 16(d_1 d_n + d_1^2 + d_n^2) \text{per} A_{4,n}(1, 5, n+4) \\ &\quad + \{1 - 4(d_1 + d_n)\} \text{per} A_{4,n}(5, n+4) \\ &= 4(d_1 + d_n - 4d_1 d_n) Q_2. \end{aligned}$$

Thus by increasing d_1 and decreasing d_n , with fixed the sum $d_1 + d_n$, we can decrease $\text{per} A_{4,n}$. This is impossible since $A_{4,n}$ is a minimizing matrix. So we must have $d_1 = d_n$, and hence all $d_i = d_1$ and all $c_i = c_1$ for $i = 1, \dots, n$.

Therefore we have shown the required form (4.1) with $4a + nd = 1 = 4d + c$ from (4.2) as the minimizing matrix on $\Omega(V_{4,n})$ with $c_i = c$ and $d_i = d$ for $i = 1, \dots, n$. $\square$

Remark 4.4. In [6], the fact that $c_i > 0$ was assumed in the proof of the Theorem 2.1 in [6]. But in order to determine the minimum permanents on $\Omega(V_{4,n})$ in the following Theorem 4.5, we need to show that each $c_i > 0$. We proved this fact in the above Lemma 4.3.

Theorem 4.5. For $n \geq 7$, the minimum permanent on $\Omega(V_{4,n})$ is $\frac{4!(n-1)(n-2)(n-3)(n-4)^{n-4}}{n^{n+3}}$, which occurs at the barycenter on $\Omega(U_{4,n})$.

Proof. Let $A_{4,n} = \begin{bmatrix} aJ_{4,4} & dJ_{4,n} \\ dJ_{n,4} & cI_n \end{bmatrix}$ be a minimizing matrix on $\Omega(V_{4,n})$ with $4a + nd = 1 = 4d + c$ from Lemma 4.3. Since $nd \leq 1$, we have $d \leq \frac{1}{n}$. From $4d + c = 1$, we have

$$(4.9) \quad c = 1 - 4d \geq 1 - 4 \cdot \frac{1}{n} = \frac{n-4}{n} \geq (n-4)d.$$

Now, suppose to the contrary that $a \neq 0$. Then we have

$$\begin{aligned} (4.10) \quad 0 &= \text{per} A_{4,n}(1 \mid 1) - \text{per} A_{4,n}(1 \mid n+4) \\ &= (3d)^2 \text{per} A_{4,n}(1, 2, n+4) + (c) \text{per} A_{4,n}(1, n+4) \\ &\quad - (4d) \text{per} A_{4,n}(1, n+4) \\ &= (9d^2) \text{per} A_{4,n}(1, 2, n+4) + (c - 4d) \text{per} A_{4,n}(1, n+4). \end{aligned}$$

Case 1) $n \geq 8$. Since $\text{per} A_{4,n}(1, 2, n+4)$ and $\text{per} A_{4,n}(1, n+4)$ are positive, equation (4.10) implies that $(c - 4d)$ must be negative. Then $c < 4d$. But $c \geq (n-4)d$ from (4.9). Thus $(n-8)d < 0$ and $n < 8$, which is a contradiction. Thus a must be zero for $n \geq 8$.

Case 2) $n = 7$. Consider

$$\begin{aligned}
 0 &= \text{per} A_{4,7}(1 \mid 1) - \text{per} A_{4,7}(1 \mid 11) \\
 (4.11) \quad &= (3d)^2 \text{per} A_{4,7}(1, 2, 11) + (c) \text{per} A_{4,7}(1, 11) \\
 &\quad - (4d) \text{per} A_{4,7}(1, 11).
 \end{aligned}$$

If $c \geq 4d$, we have a contradiction that the right of (4.11) is positive. Hence

$$(4.12) \quad 3d \leq c < 4d$$

from (4.9). And hence

$$(4.13) \quad \frac{1}{8} < d \leq \frac{1}{7}, \quad 0 < a < \frac{1}{32}, \quad \frac{3}{8} < c < \frac{4}{7}$$

from doubly stochastic property of $A_{4,7}$. From (4.11), we have

$$\begin{aligned}
 0 &= 9d^2 \text{per} A_{4,7}(1, 2, 11) - d \text{per} A_{4,7}(1, 11) \\
 &\quad + (c - 3d) \text{per} A_{4,7}(1, 11) \\
 &= 9d^2 \text{per} A_{4,7}(1, 2, 11) - d \{ 3a \text{per} A_{4,7}(1, 2, 11) \\
 &\quad + 18d^2 \text{per} A_{4,7}(1, 2, 10, 11) \} + (c - 3d) \text{per} A_{4,7}(1, 11) \\
 (4.14) \quad &= (d^2 - 3ad) \text{per} A_{4,7}(1, 2, 11) + \{ 8d^2 \text{per} A_{4,7}(1, 2, 11) \\
 &\quad - 18d^3 \text{per} A_{4,7}(1, 2, 10, 11) \} + (c - 3d) \text{per} A_{4,7}(1, 11) \\
 &= d(d - 3a) \text{per} A_{4,7}(1, 2, 11) + [8d^2 \{ 4d^2 \text{per} A_{4,7}(1, 2, 3, 10, 11) \\
 &\quad + c \text{per} A_{4,7}(1, 2, 10, 11) \} - 18d^3 \text{per} A_{4,7}(1, 2, 10, 11)] \\
 &\quad + (c - 3d) \text{per} A_{4,7}(1, 11).
 \end{aligned}$$

Now consider

$$\begin{aligned}
 d(d - 3a) &> d\left(\frac{1}{8} - 3 \cdot \frac{1}{32}\right) > 0, \\
 8d^2c - 18d^3 &= d^2(8c - 18d) > d^2\left(8 \cdot \frac{3}{8} - 18 \cdot \frac{1}{7}\right) = \frac{3}{7}d^2 > 0,
 \end{aligned}$$

and

$$c - 3d \geq 0$$

from (4.12) and (4.13). These values show that the last equation of (4.14) is positive, which is impossible. Hence a must be zero.

Thus the minimizing matrices on $\Omega(V_{4,n})$ have the form $A_{4,n}$ in (4.1) with $a = 0$ for $n \geq 7$. Therefore we have obtained that the minimum permanent on $\Omega(V_{4,n})$ is $\frac{4!(n-1)(n-2)(n-3)(n-4)^{n-4}}{n^{n+3}}$, which occurs at the barycenter $b(U_{4,n})$ of $\Omega(U_{4,n})$ by Theorem 3.1 for $n \geq 7$. $\square$

Theorem 4.6. *The minimum permanent on $\Omega(V_{4,6})$ is*

$$\frac{4!5 \cdot 4 \cdot 3 \cdot 2^2}{6^9} \approx 0.00057155921353452217650,$$

which occurs at the barycenter $b(U_{4,6})$ of $\Omega(U_{4,6})$.

Proof. Let $A_{4,6}$ be a minimizing matrix on $\Omega(V_{4,6})$. Then we have $A_{4,6} =$

$$\begin{bmatrix} aJ_{4,4} & dJ_{4,6} \\ dJ_{6,4} & cI_6 \end{bmatrix} \text{ with } 4a + 6d = 1 = 4d + c \text{ from Lemma 4.3.}$$

Assume that $a \neq 0$.

Then we have $\text{per}A_{4,6}(1 \mid 1) = \text{per}A_{4,6}(1 \mid 10)$ from Lemma 1.1.

Now, consider

$$\begin{aligned} \text{per}A_{4,6}(1 \mid 1) &= 9d^2 \text{per}A_{4,6}(1, 2, 10) + c \text{per}A_{4,6}(1, 10) \\ (4.15) \quad &= 9d^2(2a^2c^5 + 20ad^2c^4 + 40d^4c^3) \\ &\quad + c(6a^3c^5 + 90a^2d^2c^4 + 360ad^4c^3 + 360d^6c^2), \end{aligned}$$

and

$$\begin{aligned} \text{per}A_{4,6}(1 \mid 10) &= 4d \text{per}A_{4,6}(1, 10) \\ (4.16) \quad &= 4d(6a^3c^5 + 90a^2d^2c^4 + 360ad^4c^3 + 360d^6c^2). \end{aligned}$$

If $c \geq 4d$, then $\text{per}A_{4,6}(1 \mid 1) > \text{per}A_{4,6}(1 \mid 10)$ from (4.15) and (4.16). This is a contraction. Thus $c < 4d$. Since $a \neq 0$, we have $6d < 1$, and hence $d < \frac{1}{6}$. From $4d + c = 1$, we have $1 - 4d = c < 4d$. Hence $\frac{1}{8} < d < \frac{1}{6}$. By substituting $c = 1 - 4d$, $a = \frac{1-6d}{4}$ to (4.15) and (4.16), we get two equal equations with one variable d . By using the Mathematica Program, we get

$$\begin{aligned} 0 &= \text{per}A_{4,6}(1 \mid 1) - \text{per}A_{4,6}(1 \mid 10) \\ &= 9d^2 \text{per}A_{4,6}(1, 2, 10) + c \text{per}A_{4,6}(1, 10) - 4d \text{per}A_{4,6}(1, 10) \\ &= 9d^2(2a^2c^5 + 20ad^2c^4 + 40d^4c^3) \\ &\quad + c(6a^3c^5 + 90a^2d^2c^4 + 360ad^4c^3 + 360d^6c^2) \\ &\quad - 4d(6a^3c^5 + 90a^2d^2c^4 + 360ad^4c^3 + 360d^6c^2) \\ &= \frac{3}{32} - \frac{69}{16}d + \frac{753}{8}d^2 - \frac{5049}{4}d^3 + 11355d^4 - 70536d^5 + 300576d^6 \\ &\quad - 841824d^7 + 1396224d^8 - 1036800d^9 \\ &> 0 \end{aligned}$$

for $\frac{1}{8} < d < \frac{1}{6}$.

This contradiction shows that $a = 0$.

Thus the minimizing matrix on $\Omega(V_{4,6})$ has the form $A_{4,6}$ with $a = 0$. Therefore we have obtained that the minimum permanent on $\Omega(V_{4,6})$ is $\frac{4!5 \cdot 4 \cdot 3 \cdot 2^2}{6^9}$, which occurs at the barycenter $b(U_{4,6})$ of $\Omega(U_{4,6})$ by Theorem 3.1 for $n = 6$. Using the Mathematica Program, we get the approximate value 0.00057155921353452217650. $\square$

Hence we have determined minimizing matrices and minimum permanents on $\Omega(V_{4,n})$ for $n \geq 6$. Also this results show that the Conjecture 4.1 is true for the case $V_{4,n}$ with $n \geq 6$.

References

- [1] R. A. Brualdi, *An interesting face of the polytope of doubly stochastic matrices*, Linear and Multilinear Algebra, **17** (1985), 5-18.
- [2] R. A. Brualdi and B. L. Shader, *Minimum permanents on special faces of the polytope of doubly stochastic matrices*, Linear Algebra Appl. **201** (1994), 103-111.
- [3] G. P. Egorycev, *The solution of the van der Waerden problem for permanents*, Dokl. Akad. Nauk. SSSR **258** (1981), 1041-1044.
- [4] D. I. Falikman, *A proof of vander Waerden's conjecture on the permanent of a doubly stochastic matrix*, Mat. Zametki **29** (1981), 31-38, Translated in Math. Notes **29** (1981), 475-479.
- [5] T. H. Foregger, *On the minimum value of the permanent of a nearly decomposable doubly stochastic matrix*, Linear Algebra Appl. **32** (1980), 75-85.
- [6] K. Pula, S. Z. Song and I. M. Wanless, *Minimum permanents on two faces of the polytope of doubly stochastic matrices*, Linear Algebra Appl. **434** (2011), 232-238.
- [7] S. Z. Song, *Minimum permanents on certain faces of matrices containing an identity submatrix*, Linear Algebra Appl. **108** (1988), 263-280.
- [8] S. Z. Song, *Minimum permanents on a face of the polytope of doubly stochastic matrices*, Linear Algebra Appl. **259** (1997), 169-182.
- [9] S. Z. Song, *Minimum permanents and minimizing matrices on certain doubly stochastic matrices*, Linear Algebra Appl. **162-164** (1992), 786-790.

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