COMMUTATIVE DIAGRAMS WITH FIBER PRODUCTS OF THREE MODULES

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Abstract. In this paper, we investigate some properties in commutative diagrams with fiber products of three modules.

1. Introduction

We assume that every ring R is a commutative ring with identity and every module is unitary.

In algebraic K-theory, we can find the concept of fiber products of two R -modules([1], [5]). The notion of a fiber product is one that comes up very often in mathematics and logic. We knew various properties of fiber products of two R -modules ([1], [3], [4], [5]). In general the definition of fiber products is useful for two modules. In [2], Chung, Kim and Seo gave a generalization of fiber products of modules, that was, we defined the fiber product of three modules and studied some criteria of the fiber product of three R -modules(cf. Lemma 2.1, Lemma 2.2, Lemma 2.3).

We introduce the usual definition([1], [5]) of the fiber product of R -modules M_1 and M_2 over N . Let $f_1 : M_1 \rightarrow N$ and $f_2 : M_2 \rightarrow N$ be homomorphisms of R -modules. The *fiber product* of M_1 and M_2 is a triple (M, g_1, g_2) where $g_1 : M \rightarrow N$ and $g_2 : M \rightarrow M_2$ are R -linear maps such that $f_1 g_1 = f_2 g_2$, and the triple is universal in the sense that given any other triple (M', g'_1, g'_2) of this kind with $f_1 g'_1 = f_2 g'_2$. There is a unique homomorphism $h : M' \rightarrow M$ such that $g_1 h = g'_1$ and $g_2 h = g'_2$.

The purpose of this paper is to investigate some properties in commutative diagrams with fiber products of three R -modules in Theorem 2.4.

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2. Commutative diagrams with fiber Products

Firstly, we define the fiber product of three modules. For R -modules M, M_1, M_2, M_3 and N , let $f_1 : M_1 \rightarrow N$, $f_2 : M_2 \rightarrow N$ and $f_3 : M_3 \rightarrow N$ be R -homomorphisms. The *fiber product* of M_1, M_2 and M_3 over N is a quadruple (M, g_1, g_2, g_3) where $g_1 : M \rightarrow M_1$, $g_2 : M \rightarrow M_2$ and $g_3 : M \rightarrow M_3$ are R -homomorphisms such that $f_1 g_1 = f_2 g_2 = f_3 g_3$, and the quadruple is universal in the sense that given any other quadruple (M', g'_1, g'_2, g'_3) of this kind with $f_1 g'_1 = f_2 g'_2 = f_3 g'_3$, there is a unique homomorphism $h : M' \rightarrow M$ such that $g_1 h = g'_1$, $g_2 h = g'_2$ and $g_3 h = g'_3$.

Let $f_i : M_i \rightarrow N$ for $i = 1, 2, 3$ be R -homomorphisms and let $p_i : M_1 \oplus M_2 \oplus M_3 \rightarrow M_i$ for $i = 1, 2, 3$ be natural projections. Consider the map

$$(f_1 p_1 - f_3 p_3, f_2 p_2 - f_3 p_3) : M_1 \oplus M_2 \oplus M_3 \rightarrow N \oplus N.$$

Then we have

$$\begin{aligned} & \text{Im}(f_1 p_1 - f_3 p_3, f_2 p_2 - f_3 p_3) \\ &= \{(f_1(m_1) - f_3(m_3), f_2(m_2) - f_3(m_3)) \in N \oplus N : \\ & \quad (m_1, m_2, m_3) \in M_1 \oplus M_2 \oplus M_3\} \end{aligned}$$

and

$$\begin{aligned} & \text{Ker}(f_1 p_1 - f_3 p_3, f_2 p_2 - f_3 p_3) \\ &= \{(m_1, m_2, m_3) \in M_1 \oplus M_2 \oplus M_3 : f_1(m_1) = f_2(m_2) = f_3(m_3)\}. \end{aligned}$$

To obtain our main Theorem, we need the following three Lemmas in [2].

Lemma 2.1. ([2] Lemma 3.1) Let R be a commutative ring with identity, and let M, M_1, M_2, M_3 , and N be R -modules. Let $f_i : M_i \rightarrow N$ for $i = 1, 2, 3$ be R -homomorphisms and let $p_i : M_1 \oplus M_2 \oplus M_3 \rightarrow M_i$ for $i = 1, 2, 3$ be natural projections. Let

$$M_{123} = \{(m_1, m_2, m_3) \in M_1 \oplus M_2 \oplus M_3 : f_1(m_1) = f_2(m_2) = f_3(m_3)\}.$$

Then we have the following.

- (1) $(M_{123}, p_1|_{M_{123}}, p_2|_{M_{123}}, p_3|_{M_{123}})$ is a fiber product of M_1, M_2 and M_3 over N .
- (2) Consider the following commutative diagram.

If (M, g_1, g_2, g_3) is a fiber product of M_1, M_2 and M_3 over N , then we have

- (i) there is a unique isomorphism $h : M \rightarrow M_{123}$ such that $(p_1|_{M_{123}})h = g_1$, $(p_2|_{M_{123}})h = g_2$ and $(p_3|_{M_{123}})h = g_3$.
- (ii) $\text{Ker}(g_1) \cap \text{Ker}(g_2) \cap \text{Ker}(g_3) = \{0\}$.

Lemma 2.2. ([2] Lemma 3.3) Let R be a commutative ring and

be a commutative diagram of R -modules. Then this diagram is a fiber product diagram if and only if for each pair of elements $m_1 \in M_1$, $m_2 \in M_2$, $m_3 \in M_3$ with $f_1(m_1) = f_2(m_2) = f_3(m_3)$ there is a unique element $m \in M$ with $g_1(m) = m_1$, $g_2(m) = m_2$ and $g_3(m) = m_3$.

Lemma 2.3. ([2] Theorem 3.4) Let R be a commutative ring with identity and M, M_1, M_2, M_3, N be R -modules. Let $g_i : M \rightarrow M_i$ for $i = 1, 2, 3$ and $f_i : M_i \rightarrow N$ for $i = 1, 2, 3$ be R -homomorphisms. Let

$$p_i : M_1 \oplus M_2 \oplus M_3 \rightarrow M_i \text{ for } i = 1, 2, 3$$

be natural projections. Consider the map

$$(g_1, g_2, g_3) : M \rightarrow M_1 \oplus M_2 \oplus M_3$$

and

$$(f_1p_1 - f_2p_2, f_1p_1 - f_3p_3) : M_1 \oplus M_2 \oplus M_3 \rightarrow N \oplus N.$$

Then the following are equivalent.

- (1) The diagram

is a fiber product diagram of M_1 , M_2 and M_3 over N .

- (2) The following sequence

$$0 \longrightarrow M \xrightarrow{g} M_1 \oplus M_2 \oplus M_3 \xrightarrow{f} \text{Im}(f) \longrightarrow 0$$

is exact, where $g = (g_1, g_2, g_3)$ and $f = (f_1p_1 - f_2p_2, f_1p_1 - f_3p_3)$.

- (3) For the diagram in (1), we have

- (i) (g_1, g_2, g_3) is a monomorphism.
- (ii) $\{(m_1, m_2, m_3) \in M_1 \oplus M_2 \oplus M_3 : f_1(m_1) = f_2(m_2) = f_3(m_3)\} \subset \text{Im}(g_1, g_2, g_3)$.

The following Theorem 2.4 is the our main result.

Theorem 2.4. Suppose that each square in the following diagram is a fiber product (or commutative).

Then we have the following.

- (1) The quadruple (M_1, a_1, a_2, a_3) is a fiber product of M_2, M_4, M_6 over M_8 if and only if c_3 is a monomorphism.
- (2) The quadruple $(M_1, b_1, a_1, a_2, a_3)$ is a fiber product of M_3, M_4, M_6 over M_8 , when a_2 (or a_3), c_3 are monomorphisms and b_1 is an epimorphism.
- (3) (A commutative case) The quadruple $(M_1, b_1, a_1, b_4, a_2, a_3)$ is a fiber product of M_3, M_7, M_6 over M_8 , when c_1, c_3 are monomorphisms and a_3 is an isomorphism.

- (4) The quadruple $(M_1, b_1a_1, b_2a_1, a_3)$ is a fiber product of M_3, M_5, M_6 over M_8 , when b_4 is an isomorphism.
- (5) The quadruple $(M_1, b_1a_1, b_1a_2, b_3a_1)$ is a fiber product of M_3, M_5, M_7 over M_8 if and only if a_1 is an isomorphism.

PROOF: (1) Let

$$p_i : M_2 \oplus M_4 \oplus M_6 \rightarrow M_i \text{ for } i = 2, 4, 6.$$

be natural maps.

($\implies$) Let $m_7 \in \text{Ker}(c_3)$. Then we have

$$c_3(m_7) = 0 = c_1(0) = c_2(0).$$

By Lemma 2.2, there is an element $m_2 \in M_2$ such that

$$b_1(m_2) = 0, \quad b_2(m_2) = 0 \text{ and } b_3(m_2) = m_7.$$

Then we have

$$c_3(b_3(m_2)) = c_3(m_7) = 0 = c_3(b_4(0)) = c_3(b_5(0)).$$

By the hypothesis and Lemma 2.2, there is an element $m \in M$ such that

$$a_1(m) = m_2, \quad a_2(m) = 0 \text{ and } a_3(m) = 0.$$

Then we have

$$m_7 = b_3(m_2) = b_3(a_1(m)) = b_4(a_2(m)) = b_4(0) = 0.$$

Therefore c_3 is a monomorphism.

($\impliedby$) Let the map

$$(a_1, a_2, a_3) : M_1 \rightarrow M_2 \oplus M_4 \oplus M_6$$

where $(a_1, a_2, a_3)(m_1) = (a_1(m_1), a_2(m_1), a_3(m_1))$ for all $m_1 \in M_1$.

Consider a map

$$(c_3b_3p_2 - c_3b_4p_4, c_3b_3p_2 - c_3b_5p_6) : M_2 \oplus M_4 \oplus M_6 \rightarrow M_8 \oplus M_8.$$

Since (M_1, a_1, a_2, a_3) is a fiber product of M_2, M_4, M_6 over M_7 , by Lemma 2.3 the map (a_1, a_2, a_3) is a monomorphism.

(Method I) Hence it suffices to show that by Lemma 2.3

$$\text{Ker}(c_3b_3p_2 - c_3b_4p_4, c_3b_3p_2 - c_3b_5p_6) \subset \text{Im}(a_1, a_2, a_3).$$

Let $(m_2, m_4, m_6) \in \text{Ker}(c_3b_3p_2 - c_3b_4p_4, c_3b_3p_2 - c_3b_5p_6)$. Then we have

$$c_3(b_3(m_2)) = c_3(b_4(m_4)) = c_3(b_5(m_6)).$$

Since c_3 is a monomorphism, we have

$$b_3(m_2) = b_4(m_4) = b_5(m_6).$$

By Lemma 2.2, there is an element $m_1 \in M_1$ such that

$$a_1(m_1) = m_2, \quad a_2(m_1) = m_4 \quad \text{and} \quad a_3(m_1) = m_6.$$

Therefore

$$(m_2, m_4, m_6) = (a_1(m_1), a_2(m_1), a_3(m_1))$$

and hence

$$\text{Ker}(c_3b_3p_2 - c_3b_4p_4, c_3b_3p_2 - c_3b_5p_6) = \text{Im}(a_1, a_2, a_3).$$

(Method II) Consider a map

$$(b_3p_2 - b_4p_4, b_3p_2 - b_5p_6) : M_2 \oplus M_4 \oplus M_6 \rightarrow M_7 \oplus M_7.$$

Then since c_3 is a monomorphism,

$$\text{Ker}(b_3p_2 - b_4p_4, b_3p_2 - b_5p_6) = \text{Ker}(c_3b_3p_2 - c_3b_4p_4, c_3b_3p_2 - c_3b_5p_6).$$

Hence by the hypothesis we have

$$\begin{aligned} M_1 &\cong \text{Im}(a_1, a_2, a_3) \\ &= \text{Ker}(b_3p_2 - b_4p_4, b_3p_2 - b_5p_6) \\ &= \text{Ker}(c_3b_3p_2 - c_3b_4p_4, c_3b_3p_2 - c_3b_5p_6). \end{aligned}$$

The result follows from Lemma 2.3.

(2) For natural maps

$$p_i : M_3 \oplus M_4 \oplus M_6 \rightarrow M_i \quad \text{for } i = 3, 4, 6$$

consider a map

$$(c_1p_3 - c_3b_4p_4, c_1p_3 - c_3b_5p_6) : M_3 \oplus M_4 \oplus M_6 \rightarrow M_8 \oplus M_8.$$

Since a_2 (or a_3) is a monomorphism, (b_1a_1, a_2, a_3) is a monomorphism. Next we show that

$$\text{Ker}(c_1p_3 - c_3b_4p_4, c_1p_3 - c_3b_5p_6) \subset \text{Im}(b_1a_1, a_2, a_3).$$

Let $(m_3, m_4, m_6) \in \text{Ker}(c_1p_3 - c_3b_4p_4, c_1p_3 - c_3b_5p_6)$. Then we have

$$c_1(m_3) = c_3(b_4(m_4)) = c_3(b_5(m_6)).$$

Since b_1 is surjective, there is an element $m_2 \in M_2$ such that

$$b_1(m_2) = m_3.$$

Then we have

$$c_3(b_3(m_2)) = c_1(b_1(m_2)) = c_1(m_3) = c_3(b_4(m_4)) = c_3(b_5(m_6)).$$

Since c_3 is a monomorphism, we have

$$b_3(m_2) = b_4(m_4) = b_5(m_6).$$

By Lemma 2.2, there is an element $m_1 \in M_1$ such that

$$a_1(m_1) = m_2, a_2(m_1) = m_4 \text{ and } a_3(m_1) = m_6.$$

Then we have $b_1(a_1(m_1)) = b_1(m_2) = m_3$. Hence

$$(m_3, m_4, m_6) = (b_1(a_1(m_1)), a_2(m_1), a_3(m_1)).$$

Thus we have

$$Im(b_1a_1, a_2, a_3) = Ker(c_1p_3 - c_3b_4p_4, c_1p_3 - c_3b_5p_6).$$

(3) For natural maps

$$p_i : M_3 \oplus M_7 \oplus M_6 \rightarrow M_i \text{ for } i = 3, 7, 6$$

consider a map

$$(c_1p_3 - c_3p_7, c_1p_3 - c_3b_5p_6) : M_3 \oplus M_7 \oplus M_6 \rightarrow M_8 \oplus M_8.$$

Since a_3 is a monomorphism, (b_1a_1, b_4a_2, a_3) is a monomorphism. Next we show that

$$Ker(c_1p_3 - c_3p_7, c_1p_3 - c_3b_5p_6) \subset Im(b_1a_1, b_4a_2, a_3).$$

Let $(m_3, m_7, m_6) \in Ker(c_1p_3 - c_3p_7, c_1p_3 - c_3b_5p_6)$. Then we have

$$c_1(m_3) = c_3(m_7) = c_3(b_5(m_6)).$$

Since c_3 is a monomorphism, we have $m_7 = b_5(m_6)$. Since a_3 is an epimorphism, there is an element $m_1 \in M_1$ such that

$$a_3(m_1) = m_6.$$

Then we have

$$m_7 = b_5(m_6) = b_5(a_3(m_1)) = b_4(a_2(m_1)) = b_3(a_1(m_1)).$$

Hence we have

$$c_1(m_3) = c_3(m_7) = c_3(b_3(a_1(m_1))) = c_1(b_1(a_1(m_1))).$$

Since c_1 is a monomorphism, we have

$$m_3 = b_1(a_1(m_1)).$$

Hence we have

$$(m_3, m_7, m_6) = (b_1(a_1(m_1)), b_4(a_2(m_1)), a_3(m_1)).$$

Thus we have

$$Im(b_1a_1, b_4a_2, a_3) = Ker(c_1p_3 - c_3p_7, c_1p_3 - c_3b_5p_6).$$

(4) For natural maps

$$p_i : M_3 \oplus M_5 \oplus M_6 \rightarrow M_i \text{ for } i = 3, 5, 6$$

consider a map

$$(c_1p_3 - c_2p_5, c_1p_3 - c_3b_5p_6) : M_3 \oplus M_5 \oplus M_6 \rightarrow M_8 \oplus M_8.$$

Let $m_1 \in \text{Ker}(b_1a_1, b_2a_1, a_3)$. Then we have

$$b_1(a_1(m_1)) = 0, \quad b_2(a_1(m_1)) = 0, \quad a_3(m_1) = 0.$$

Thus

$$0 = b_5(a_3(m_1)) = b_4(a_2(m_1)) = b_3(a_1(m_1)).$$

Therefore we have

$$a_1(m_1) \in \text{Ker}(b_1) \cap \text{Ker}(b_2) \cap \text{Ker}(b_3).$$

By Lemma 2.1

$$a_1(m_1) = 0.$$

Since b_4 is a monomorphism, we have $a_2(m_1) = 0$ and then

$$m_1 \in \text{Ker}(a_1) \cap \text{Ker}(a_2) \cap \text{Ker}(a_3).$$

By Lemma 2.1

$$m_1 = 0.$$

Hence (b_1a_1, b_2a_1, a_3) is a monomorphism.

Next we show that

$$\text{Ker}(c_1p_3 - c_2p_5, c_1p_3 - c_3b_5p_6) \subset \text{Im}(b_1a_1, b_2a_1, a_3).$$

Since $(c_1p_3 - c_2p_5, c_1p_3 - c_3b_5p_6)(b_1a_1, b_2a_1, a_3) = (0, 0)$, we have

$$\text{Im}(b_1a_1, b_2a_1, a_3) \subset \text{Ker}(c_1p_3 - c_2p_5, c_1p_3 - c_3b_5p_6).$$

Let $(m_3, m_5, m_6) \in \text{Ker}(c_1p_3 - c_2p_5, c_1p_3 - c_3b_5p_6)$. Then we have

$$c_1(m_3) = c_2(m_5) = c_3(b_5(m_6)).$$

By Lemma 2.2, there is an element $m_2 \in M_2$ such that

$$b_1(m_2) = m_3, \quad b_2(m_2) = m_5 \quad \text{and} \quad b_3(m_2) = b_5(m_6).$$

Since b_4 is an epimorphism, there is an element $m_4 \in M_4$ such that

$$b_4(m_4) = b_3(m_2).$$

Then we have

$$b_3(m_2) = b_4(m_4) = b_5(m_6).$$

By Lemma 2.2, there is an element $m_1 \in M_1$ such that

$$a_1(m_1) = m_2, \quad a_2(m_1) = m_4 \quad \text{and} \quad a_3(m_1) = m_6.$$

Then we have

$$b_1(a_1(m_1)) = b_1(m_2) = m_3, \quad b_2(a_1(m_1)) = b_2(m_2) = m_5 \quad \text{and} \quad a_1(m_1) = m_6.$$

Hence

$$(m_3, m_5, m_6) = (b_1(a_1(m_1)), b_2(a_1(m_1)), a_3(m_1)).$$

Thus we have

$$Im(b_1a_1, b_2a_1, a_3) = Ker(c_1p_3 - c_2p_5, c_1p_3 - c_3b_5p_6).$$

(5) It follows from Lemma 2.1. □

References

- [1] H. Bass, *Algebraic K-Theory*, W. A. Benjamin, Inc. 1968.
- [2] S. C. Chung, S. A. Kim, and G. S. Seo, *Some properties of fiber products of three modules*, Honam Mathematical J. **32(2)** (2010), 239-253.
- [3] E. Kunz, *Introduction to Commutative Algebra and Algebraic Geometry*, Birkhäuser Boston, 1985.
- [4] S. Mandal, *Projective Modules and Complete Intersections*, Lecture Notes in Mathematics 1672, Springer-Verlag Berlin Heidelberg, 1997.
- [5] J. Milnor, *Introduction to Algebraic K-Theory*, Princeton University Press, 1971.

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