THE REPRESENTATION OF THE GOLDEN RATIO BY THE CONTINUED FRACTION

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Abstract. There are several theories to say that ‘Mathematics is beautiful’, but the typical one of them is a theory about the golden ratio. Often the golden ratio apt to be considered only as the geometric shapes or the simple number of ratio used in buildings and arts. However in this paper, we studied to consider the mathematical theories which are contained in their inside. In particular, we investigate the various expressions of the continued fraction which are represented by the golden ratio.

1. Introduction

First, we introduce the well known expression of the continued fraction from the definition of golden ratio as follows.

$$(1.1) \quad \phi = 1 + \frac{1}{1 + \frac{1}{1 + \frac{1}{\ddots}}}.$$

As is well known, we say that two quantities are in the **golden ratio** if the ratio of the sum of the quantities to the larger quantity is equal to the ratio of the larger quantity to the smaller one.

Let a and b be the two quantities satisfying the golden ratio with $a > b$. By definition of the golden ratio, two ratio $\frac{a+b}{a}$ and $\frac{a}{b}$ are equal to the golden ratio ϕ . That is,

$$(1.2) \quad \frac{a+b}{a} = \frac{a}{b} = \phi.$$

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Hence the golden ratio ϕ satisfies the equation

$$(1.3) \quad \phi = \frac{a+b}{a} = 1 + \frac{b}{a} = 1 + \frac{1}{\phi}.$$

By multiplying ϕ on the both sides of equation (1.3), we give the equation

$$(1.4) \quad \phi^2 = \phi + 1.$$

Using this quadratic formula (1.4), we obtain the only positive solution as

$$(1.5) \quad \phi = \frac{1 + \sqrt{5}}{2} = 1.6180339887 \dots$$

From the equation (1.3), the golden ratio ϕ also satisfies $\phi - 1 = \frac{1}{\phi}$.

The golden ratio can be expressed by continued fraction. From the equation $\phi = 1 + \frac{1}{\phi}$ and once again substituting this expression into the ϕ of right-hand side, then we have $\phi = 1 + \frac{1}{1 + \frac{1}{\phi}}$. If you continue to repeat, it can be expressed continued fraction as the equation (1.1).

However, it is not clear mathematically. We looked for the mathematical theory to clearly explain the mathematical meaning of '...' in the above representation. For this purpose, we construct the sequence which converges to the golden ratio by fixed point iteration method.

This paper is consists as following: Chapter 2 explain about definition of the golden ratio and the expression of the continued fraction of the various representations. We also introduce about the fixed point theorem that the functions which satisfy some conditions have only one fixed point. In chapter 3, we introduce about the fixed point iteration that the sequence founded from the function that satisfy the condition of the fixed point theorem converges to fixed point for the appropriate initial value. We find some example sequences through fixed point iteration converges to fixed point by apply the expression of continued fraction of the golden ratio. Finally in chapter 4, we calculated by the MS-EXCEL program for functions that intuitively and theoretically converges to golden ratio. And we reported the practical values of the sequence and the rate of convergence through the fixed point iteration.

2. Preliminaries

In this section, we introduce some definitions and the basic facts about the golden ratio and the fixed point iteration method.

Now let's look at the method of another way to find the expression of the continued fraction. The golden ratio is the root of the quadratic equation

$$x^2 - x - 1 = 0.$$

To obtain the root $x = c$ of the equation $f(x) = 0$, which can be transformed into $g(x) - x = 0$, define the sequence $\{x_n\}$ by $x_{n+1} = g(x_n)$, $n \geq 0$ with initial guess x_0 . If this sequence is a convergent sequence, then a limit value of this convergence sequence $c = \lim_{n \rightarrow \infty} x_n$ is the fixed point of the function g .

If a function g is continuous and the limit value $\lim_{n \rightarrow \infty} x_n = c$, then

$$g(c) = g(\lim_{n \rightarrow \infty} x_n) = \lim_{n \rightarrow \infty} g(x_n) = \lim_{n \rightarrow \infty} x_{n+1} = c.$$

Hence, the limit value c is a fixed point of the function g . This scheme is called as a fixed point iteration method. We only state the theorem without proof. Here, the equation $x_{n+1} = g(x_n)$ is called a iteration equation and the function $g(x)$ is called a iteration function. Also the constant K is the degree of the convergence.

Theorem 2.1 (Fixed Point Iteration). Let $I = [a, b]$ be an interval and let $g(x)$ be a function from I to I . Assume that the function $g(x)$ is continuous on I and is differentiable on (a, b) . If there exists a constant $0 \leq K < 1$ satisfying

$$|g'(x)| \leq K < 1, \quad x \in (a, b).$$

Then the sequence $\{x_n\}$ generated by iteration equation

$$x_{n+1} = g(x_n), \quad n = 0, 1, 2, \dots$$

with for any initial guess $x_0 \in [a, b]$ converges to the fixed point c of the function g on $I = [a, b]$.

In previous theorem, we see that the fixed point iteration is used to way to look for the fixed point of the function $g(x)$, i.e., the roots of the equation $g(x) = x$. In order to find the roots of the equation $f(x) = 0$, at first, transformed in the form of $f(x) = 0$ to $x = g(x)$. At this $g(x)$ is a function that must be satisfied with condition of the fixed point theorem. That is, if $g(x)$ is a function satisfying $|g'(x)| \leq K < 1$ on $I = [a, b]$ for some constant K , after we choose an arbitrary initial value x_0 in the interval I , find sequence $\{x_n\}$ to apply $x_{n+1} = g(x_n)$. This sequence converges and limit value of $\{x_n\}$ is fixed point of $g(x)$ i.e. That is a solution of the equation $f(x) = 0$.

Now, let's apply the equation $x^2 - x - 1 = 0$ to obtain the golden ratio. As seen earlier, we know that one of those roots of this equation is the golden ratio ϕ . Expressed in the form $x = g(x)$ of the equation $x^2 - x - 1 = 0$ to obtain this value. Next, make sure that the function $g(x)$ should satisfy the condition of the fixed point iteration in the interval $[a, b]$, where $a > 0$. If it satisfies the conditions, by choosing an appropriate initial value x_0 , find a sequence $\{x_n\}$ by applying iteration $x_{n+1} = f(x_n)$. The limit value of the sequence $\{x_n\}$ is a solution of the equation $x^2 - x - 1 = 0$. This is the golden ratio ϕ . Now, let's find out the function $g(x)$ by transforming into continued fraction the equation $x^2 - x - 1 = 0$.

Proposition 2.2. The golden ratio can be represented as following:

$$\begin{aligned}
 \text{(a) } \phi &= 1 + \frac{1}{1 + \frac{1}{1 + \frac{1}{1 + \frac{1}{\ddots}}}} \\
 \text{(b) } \phi &= 2 - \frac{1}{3 - \frac{1}{3 - \frac{1}{3 - \frac{1}{\ddots}}}} \\
 \text{(c) } \phi &= 3 - \frac{5}{5 - \frac{5}{5 - \frac{5}{5 - \frac{5}{\ddots}}}} \\
 \text{(d) } \phi &= 2 - \frac{1}{2 + \frac{1}{2 - \frac{1}{2 + \frac{1}{\ddots}}}}
 \end{aligned}$$

Moreover, there may be many other representation of the continued fraction for golden ratio.

Proof. (a) From the equation $x^2 - x - 1 = 0$, we have $x^2 = x + 1$. Dividing both sides by x , then we obtain $x = 1 + \frac{1}{x}$. Hence we take $g_1(x) = 1 + \frac{1}{x}$. For all $x \in [1.5, 2]$,

$$1.5 = 1 + \frac{1}{2} \leq g_1(x) = 1 + \frac{1}{x} \leq 1 + \frac{1}{1.5} < 2.$$

Hence $g_1 : [1.5, 2] \rightarrow [1.5, 2]$ is continuous. Also, g_1 satisfies $|g'_1(x)| = |-\frac{1}{x^2}| \leq \frac{4}{9} < 1$. Therefore by the Theorem 2.1, the sequence constructed by the fixed point iteration $x_{n+1} = g_1(x_n)$ converges to the fixed point of $g_1(x)$ for any initial $x_0 \in [1.5, 2]$. Let $x_0 = 2$ be the initial value, then

the sequence $\{x_n\}$ is as follows:

$$\begin{aligned} x_1 &= 1 + \frac{1}{2} = \frac{3}{2} \\ x_2 &= 1 + \frac{1}{x_1} = 1 + \frac{1}{1 + \frac{1}{2}} = \frac{5}{3} \\ x_3 &= 1 + \frac{1}{x_2} = 1 + \frac{1}{1 + \frac{1}{\frac{5}{3}}} = \frac{8}{5} \\ &\vdots \end{aligned}$$

Hence this sequence $\{x_n\}$ converges to the fixed point of the function $g_1(x)$ i.e. the solution of $x^2 - x - 1 = 0$. Since the limit value exists in the interval $[1.5, 2]$,

$$\phi = \lim_{n \rightarrow \infty} x_n = 1 + \frac{1}{1 + \frac{1}{1 + \frac{1}{\ddots}}}.$$

(b) From the equation $x^2 - x - 1 = 0$, we have $x^2 - x - 2 = -1$.

$$(x - 2)(x + 1) = -1 \implies x - 2 = -\frac{1}{x + 1} \implies x = 2 - \frac{1}{x + 1}.$$

Hence we take $g_2(x) = 2 - \frac{1}{x+1}$. For all $x \in [1, 2]$,

$$1 < \frac{3}{2} \leq g_2(x) = 2 - \frac{1}{x+1} \leq \frac{5}{3} < 2.$$

Hence $g_2 : [1, 2] \rightarrow [1, 2]$ is continuous. Also, g_2 satisfies $|g_2'(x)| = \left| \frac{1}{(x+1)^2} \right| \leq \frac{1}{4} < 1$. Therefore by the Theorem 2.1, the sequence constructed by the fixed point iteration $x_{n+1} = g_2(x_n)$ converges to the fixed point of $g_2(x)$ for any initial $x_0 \in [1, 2]$. If initial value is chosen $x_0 = 1$, then the sequence $\{x_n\}$ is as follows:

$$\begin{aligned} x_1 &= 2 - \frac{1}{1+1} = \frac{3}{2} \\ x_2 &= 2 - \frac{1}{1+x_1} = 2 - \frac{1}{1+2-\frac{1}{1+1}} = \frac{8}{5} \\ x_3 &= 2 - \frac{1}{1+x_2} = 2 - \frac{1}{1+2-\frac{1}{1+2-\frac{1}{1+1}}} = \frac{21}{13} \\ &\vdots \end{aligned}$$

Hence this sequence $\{x_n\}$ converges to the fixed point of the function $g_2(x)$ i.e. the solution of the equation $x^2 - x - 1 = 0$. Since the limit value exists in the interval $[1, 2]$,

$$\phi = \lim_{n \rightarrow \infty} x_n = 2 - \frac{1}{3 - \frac{1}{3 - \frac{1}{3 - \frac{1}{\ddots}}}}.$$

(c) From the equation $x^2 - x - 1 = 0$, we also have $x^2 - x - 6 = -5$.

$$(x - 3)(x + 2) = -5 \implies x - 3 = -\frac{5}{x + 2} \implies x = 3 - \frac{5}{x + 2}.$$

Hence we take $g_3(x) = 3 - \frac{5}{2+x}$. For all $x \in [1, 2]$,

$$1 < \frac{4}{3} \leq g_3(x) = 3 - \frac{5}{2+x} \leq \frac{7}{4} < 2.$$

Hence $g_3 : [1, 2] \rightarrow [1, 2]$ is continuous. Also, g_3 satisfies $|g'_3(x)| = \left| \frac{5}{(2+x)^2} \right| \leq \frac{5}{9} < 1$. Therefore by the Theorem 2.1, the sequence constructed by the fixed point iteration $x_{n+1} = g_3(x_n)$ converges to the fixed point of $g_3(x)$ for any initial $x_0 \in [1, 2]$. If initial value is chosen $x_0 = 1$, then the sequence $\{x_n\}$ is as follows:

$$\begin{aligned} x_1 &= 3 - \frac{5}{2+1} = \frac{4}{3} \\ x_2 &= 3 - \frac{5}{2+x_1} = 3 - \frac{5}{2+3-\frac{5}{2+1}} = \frac{3}{2} \\ x_3 &= 3 - \frac{5}{2+x_2} = 3 - \frac{5}{2+3-\frac{5}{2+3-\frac{5}{2+1}}} = \frac{11}{7} \\ &\vdots \end{aligned}$$

Hence this sequence $\{x_n\}$ converges to the fixed point of the function $g_3(x)$ i.e. the solution of the equation $x^2 - x - 1 = 0$. Since the limit value exists in the interval $[1, 2]$,

$$\phi = \lim_{n \rightarrow \infty} x_n = 3 - \frac{5}{5 - \frac{5}{5 - \frac{5}{5 - \frac{5}{\ddots}}}}.$$

(d) From the equation $x^2 - x - 1 = 0$, we also have $2x^2 - 2x - 2 = 0$ and $2x^2 - 3x - 2 = -x$.

$$(x - 2)(2x + 1) = -x \implies x - 2 = -\frac{x}{2x + 1} \implies x = 2 - \frac{x}{2x + 1}.$$

Hence we take $g_4(x) = 2 - \frac{x}{2x+1} = 2 - \frac{1}{2+\frac{1}{x}}$. For all $x \in [1, 2]$,

$$1 < 2 - \frac{2}{5} \leq g_4(x) = 2 - \frac{x}{2x+1} \leq 2 - \frac{1}{3} < 2.$$

Hence $g_4 : [1, 2] \rightarrow [1, 2]$ is continuous. Also, g_4 satisfies $|g'_4(x)| = \left| \frac{1}{(2x+1)^2} \right| \leq \frac{1}{9} < 1$. Therefore by the Theorem 2.1, the sequence constructed by the fixed point iteration $x_{n+1} = g_4(x_n)$ converges to the fixed point of $g_4(x)$ for any initial $x_0 \in [1, 2]$. If initial value is chosen $x_0 = 1$, then the sequence $\{x_n\}$ is as follows:

$$\begin{aligned} x_1 &= 2 - \frac{1}{2+1} = \frac{5}{3} \\ x_2 &= 2 - \frac{1}{2+\frac{1}{x_1}} = 2 - \frac{1}{2+\frac{3}{5}} = \frac{21}{13} \\ x_3 &= 2 - \frac{1}{2+\frac{1}{x_2}} = 2 - \frac{1}{2+\frac{13}{21}} = \frac{89}{55} \\ &\vdots \end{aligned}$$

Hence this sequence $\{x_n\}$ converges to the fixed point of the function $g_4(x)$ i.e. the solution of the equation $x^2 - x - 1 = 0$. Since the limit value exists in the interval $[1, 2]$,

$$\phi = \lim_{n \rightarrow \infty} x_n = 2 - \frac{1}{2 + \frac{1}{2 + \frac{1}{2 + \frac{1}{\ddots}}}}.$$

□

3. Numerical Computation

As can be seen in the previous examples, we were able to get the golden ratio represented by continued fraction from functions. By searching for another functions in $g(x)$, can be obtained new representation. The values of these sequences in fact, can be calculated. Let's look at the values of sequence and the number of iteration through numerical calculations. For this computation, we used the MS-EXCEL program.

From the Table 3.2., we know that the less degree K of the convergence makes the faster convergence. Hence we will show the next theorem can be reduce the degree K .

n	$g_1(x)$	$g_2(x)$	$g_3(x)$	$g_4(x)$
0	2	1	1	1
1	1.5	1.5	1.333333333	1.666666667
2	1.666666667	1.6	1.5	1.615384615
3	1.6	1.615384615	1.571428571	1.618181818
4	1.625	1.617647059	1.6	1.618025751
5	1.615384615	1.617977528	1.611111111	1.618034448
6	1.619047619	1.618025751	1.615384615	1.618033963
7	1.617647059	1.618032787	1.617021277	1.618033990
8	1.618181818	1.618033813	1.617647059	1.618033989
9	1.617977528	1.618033963	1.617886179	1.618033989
10	1.618055556	1.618033985	1.617977528	1.618033989
11	1.618025751	1.618033988	1.618012422	1.618033989
12	1.618037135	1.618033989	1.618025751	1.618033989
13	1.618032787	1.618033989	1.618030842	1.618033989
14	1.618034448	1.618033989	1.618032787	1.618033989
15	1.618033813	1.618033989	1.618033530	1.618033989
16	1.618034056	1.618033989	1.618033813	1.618033989
17	1.618033963	1.618033989	1.618033922	1.618033989
18	1.618033999	1.618033989	1.618033963	1.618033989
19	1.618033985	1.618033989	1.618033979	1.618033989
20	1.618033990	1.618033989	1.618033985	1.618033989
21	1.618033988	1.618033989	1.618033987	1.618033989
22	1.618033989	1.618033989	1.618033988	1.618033989
23	1.618033989	1.618033989	1.618033989	1.618033989
24	1.618033989	1.618033989	1.618033989	1.618033989
25	1.618033989	1.618033989	1.618033989	1.618033989

TABLE 3.1. The values of sequences for each iteration function

n	$g_1(x)$	$g_2(x)$	$g_3(x)$	$g_4(x)$
the number of iteration	22	12	23	8
degree of the convergence	$\frac{4}{9}$	$\frac{1}{4}$	$\frac{5}{9}$	$\frac{1}{9}$

TABLE 3.2. The comparison of the number of iteration with the degree of the convergence K

Theorem 3.1. Assume that the functions g_1 and g_2 are the iteration function in the Theorem 2.1 with the degree of the convergence K_1 and K_2 and that two functions have the same fixed point c . Then we have

another iteration function $g_1 \circ g_2$ which satisfies the condition in Theorem 2.1 with the degree of the convergence $K_1 K_2$ which is less than K_1 and K_2 . In other words, the equation $\{x_n\}$ generated by iteration equation

$$x_{n+1} = g_1 \circ g_2(x_n) = g_1(g_2(x_n)), \quad n = 0, 1, 2, \dots$$

with for any initial guess x_0 converges to the same fixed point c .

Proof. Since g_1 and g_2 are the iteration function with the degree K_1 and K_2 , respectively, i.e.,

$$|g'_1(x)| \leq K_1, \quad |g'_2(x)| \leq K_2, \quad x \in I,$$

then

$$|(g_1 \circ g_2)'(x)| = |g'_1(g_2(x)) \cdot g'_2(x)| = |g'_1(g_2(x))| \cdot |g'_2(x)| \leq K_1 K_2, \quad x \in I.$$

Hence $g_1 \circ g_2$ is iteration function with the degree $K_1 K_2$. By Theorem 2.1., the sequence $\{x_n\}$ generated by the iteration function $g_1 \circ g_2$ converges to the fixed point of $g_1 \circ g_2$.

Since g_1 and g_2 have the same fixed point c , i.e., $g_1(c) = c$ and $g_2(c) = c$, then

$$(g_1 \circ g_2)(c) = g_1(g_2(c)) = g_1(c) = c. \quad \square$$

Corollary 3.2. If $g(x)$ is a iteration function, then the composite function $g \circ \dots \circ g$ is also iteration function.

By Proposition 2.2. (a), $g_1(x) = 1 + \frac{1}{x}$ is the iteration function with $K_1 = \frac{4}{9} < 1$. By Corollary 3.2, $g_1 \circ g_1$ is also the iteration function for the same fixed point. It is easy to check that the function $g_1 \circ g_1$ is equal to the function g_2 in Proposition 2.2. (b).

By Corollary 3.2., $g_1 \circ g_1 \circ \dots \circ g_1 = 1 + \frac{1}{1 + \frac{1}{1 + \frac{1}{\ddots + \frac{1}{x}}}}$ is the iteration

function with $K = (\frac{4}{9})^n$ as n tends to ∞ , the limit function $g = \lim(g_1)^n$ is the iteration function and satisfies

$$|g'(x)| = \lim |(g'_1)^n| \leq \lim K_1^n = 0.$$

Hence $g(x)$ is a constant function with fixed point c . Therefore

$$\lim_{n \rightarrow \infty} g_1 \circ g_1 \circ \dots \circ g_1 = 1 + \frac{1}{1 + \frac{1}{1 + \frac{1}{\ddots}}} = \phi.$$

4. Concluding Remarks

We investigated the definition and expression of the golden ratio, and inquired about the required courses for the golden ratio is represented by continued fraction, fixed point, sequence, fixed point iteration, by considering the mathematical theory for only known contents as the numerical value. Through this process, we could understand for relationship between the golden ratio and continued fraction. Also we were able to visualize the result through numerical calculations using the MS-EXCEL program.

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