VARIATIONS OF THE LENGTH INTEGRAL

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Abstract. In this paper, we obtain a necessary and sufficient condition for the second variation of an arbitrarily given smooth variation of a geodesic on a Riemannian manifold to be 0.

1. Introduction

Let $\tau_s : [0, 1] \rightarrow M$ ($-\varepsilon < s < \varepsilon$) be a smooth variation of a geodesic $\tau_0 = \tau = x_t$ ($0 \leq t \leq 1$) on a Riemannian manifold (M, g) such that $\tau_s(0) = x_0$ and $\tau_s(1) = x_1$ for every $s \in (-\varepsilon, \varepsilon)$. We put

$$L(s) := L(\tau_s) := \int_0^1 \left\| \frac{d\tau_s}{dt} \right\| dt,$$

and calculate the second variation $(d^2L(s)/ds^2)_{s=0}$. The second variation is expressed in terms of the sectional curvature ([1, 2, 3, 4]) and the variation vector field along the curve $\tau = x_t$ ($0 \leq t \leq 1$). And then, we get a necessary and sufficient condition for the second variation $(d^2L(s)/ds^2)_{s=0}$ to be 0.

These calculus methods in variation problems are very useful in the study on natural sciences. But, in most of references, the calculus of variations is insufficient and omitted. In this paper, we make a minute and detailed proof of the calculus parts which are insufficient and omitted in variation problems of the length integral.

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2. The first and second variations of the length integral

Let (M, g) be a complete Riemannian manifold and ∇ the Levi-Civita connection for the Riemannian metric g . For a C^∞ -curve $\tau = x_t$ ($0 \leq t \leq 1$) on M , let

$$\phi : (t, s) \in [0, 1] \times (-\varepsilon, \varepsilon) \mapsto \phi(t, s) \in M$$

be a C^∞ -map which satisfies

$$\phi(t, 0) = x_t.$$

Such a map ϕ is called a *variation* of $\tau = x_t$ ($0 \leq t \leq 1$). From now on, we assume that $\tau = x_t$ ($0 \leq t \leq 1$) is parametrized by its arc length. Let $\tau_s : [0, 1] \rightarrow M$ ($-\varepsilon < s < \varepsilon$) be a C^∞ -map which is defined by $\tau_s(t) := \phi(t, s)$. The length $L(s)$ of τ_s is given by

$$(2.1) \quad L(s) := L(\tau_s) = \int_0^1 \sqrt{g(\phi_*(\partial/\partial t)_{(t,s)}, \phi_*(\partial/\partial t)_{(t,s)})} dt.$$

Let $\phi_*(\partial/\partial t)_{(t,s)}$ and $\phi_*(\partial/\partial s)_{(t,s)}$ are defined on $[0, 1] \times (-\varepsilon, \varepsilon)$. Moreover, we define a vector field X_t along the curve $\tau = x_t$ ($0 \leq t \leq 1$) by

$$(2.2) \quad X_t := \phi_*(\partial/\partial s)_{(t,0)} \quad (0 \leq t \leq 1).$$

Such a vector field X_t ($0 \leq t \leq 1$) along the curve $\tau = x_t$ ($0 \leq t \leq 1$) is called the *variation vector field* along ϕ . In this paper, we assume that $\phi(0, s) = x_0$ and $\phi(1, s) = x_1$, $s \in (-\varepsilon, \varepsilon)$, for any variation ϕ of the curve $\tau = x_t$ ($0 \leq t \leq 1$). Then we have

$$(2.3) \quad X_0 = 0 \in T_{x_0}(M), \quad X_1 = 0 \in T_{x_1}(M).$$

Now we calculate the first variation $(dL/ds)_{s=0}$ of $L(s)$. First of all, we get

$$(2.4) \quad \begin{aligned} & \frac{\partial}{\partial s} \left(\sqrt{g(\phi_*(\partial/\partial t)_{(t,s)}, \phi_*(\partial/\partial t)_{(t,s)})} \right) \\ &= \frac{g(\nabla_s \phi_*(\partial/\partial t)_{(t,s)}, \phi_*(\partial/\partial t)_{(t,s)})}{\sqrt{g(\phi_*(\partial/\partial t)_{(t,s)}, \phi_*(\partial/\partial t)_{(t,s)})}}. \end{aligned}$$

Since

$$\begin{aligned} & \nabla_s \phi_*(\partial/\partial t)_{(t,s)} - \nabla_t \phi_*(\partial/\partial s)_{(t,s)} - \phi_*[(\partial/\partial s)_{(t,s)}, (\partial/\partial t)_{(t,s)}] \\ &= T^\nabla(\phi_*(\partial/\partial s)_{(t,s)}, \phi_*(\partial/\partial t)_{(t,s)}) \end{aligned}$$

and $T^\nabla = 0$ (cf, [3, Lemma 1.16, p. 151]), we obtain on $[0, 1] \times (-\varepsilon, \varepsilon)$

$$(2.5) \quad \nabla_s \phi_*(\partial/\partial t)_{(t,s)} = \nabla_t \phi_*(\partial/\partial s)_{(t,s)}.$$

Here, T^∇ is the torsion of ∇ . We get from (2.2) and (2.5)

$$(2.6) \quad \phi_*(\partial/\partial t)_{(t,0)} = x'_t, \quad (\nabla_s \phi_*(\partial/\partial t)_{(t,s)})_{s=0} = \nabla_t X_t.$$

From (2.1), (2.4) and (2.6), we obtain

$$(2.7) \quad \left(\frac{dL(s)}{ds} \right)_{s=0} = \int_0^1 g(\nabla_t X_t, x'_t) dt.$$

Moreover, we get

$$(2.8) \quad g(\nabla_t X_t, x'_t) = \frac{d}{dt}(g(X_t, x'_t)) - g(X_t, \nabla_t x'_t).$$

By the help of (2.7) and (2.8), we have

$$(2.9) \quad \begin{aligned} \left(\frac{dL(s)}{ds} \right)_{s=0} &= \int_0^1 \left\{ \frac{d}{dt}(g(X_t, x'_t)) - g(X_t, \nabla_t x'_t) \right\} dt \\ &= [g(X_t, x'_t)]_0^1 - \int_0^1 g(X_t, \nabla_t x'_t) dt. \end{aligned}$$

Since $X_0 = 0 \in T_{x_0}(M)$ and $X_1 = 0 \in T_{x_1}(M)$ from (2.3), we get from (2.9)

$$\left(\frac{dL(s)}{ds} \right)_{s=0} = - \int_0^1 g(X_t, \nabla_t x'_t) dt.$$

Hence we have the following theorem.

Theorem 2.1. *Let $\tau_s : [0, 1] \rightarrow M$ ($-\varepsilon < s < \varepsilon$) be an arbitrarily given smooth variation of $\tau = x_t$ ($0 \leq t \leq 1$) such that $\tau_s(0) = x_0$ and $\tau_s(1) = x_1$ for every $s \in (-\varepsilon, \varepsilon)$. Then, $(dL(s)/ds)_{s=0} = 0$ if and only if $\nabla_t x'_t = 0$ for every $t \in (0, 1)$, that is, $\tau = x_t$ ($0 \leq t \leq 1$) is a geodesic in the Riemannian manifold (M, g) .*

Next, we calculate the second variation $(d^2L(s)/ds^2)_{s=0}$ of the geodesic $\tau = x_t$ ($0 \leq t \leq 1$). From (2.4) and (2.5), we get

$$(2.10) \quad \begin{aligned} &\frac{\partial^2}{\partial s^2} \left(\sqrt{g(\phi_*(\partial/\partial t)_{(t,s)}, \phi_*(\partial/\partial t)_{(t,s)})} \right) \\ &= \frac{g(\nabla_s \nabla_t \phi_*(\partial/\partial s)_{(t,s)}, \phi_*(\partial/\partial t)_{(t,s)}) + \|\nabla_t \phi_*(\partial/\partial s)_{(t,s)}\|^2}{\{g(\phi_*(\partial/\partial t)_{(t,s)}, \phi_*(\partial/\partial t)_{(t,s)})\}^{\frac{1}{2}}} \\ &\quad - \frac{\{g(\nabla_t \phi_*(\partial/\partial s)_{(t,s)}, \phi_*(\partial/\partial t)_{(t,s)})\}^2}{\{g(\phi_*(\partial/\partial t)_{(t,s)}, \phi_*(\partial/\partial t)_{(t,s)})\}^{\frac{3}{2}}}. \end{aligned}$$

Moreover, the following is well known ([2, Proposition 10, p. 261]):

$$(2.11) \quad \begin{aligned} & \nabla_s \nabla_t \phi_*(\partial/\partial s)_{(t,s)} - \nabla_t \nabla_s \phi_*(\partial/\partial s)_{(t,s)} \\ &= R(\phi_*(\partial/\partial s)_{(t,s)}, \phi_*(\partial/\partial t)_{(t,s)}) \phi_*(\partial/\partial s)_{(t,s)}. \end{aligned}$$

Here, R is the curvature tensor field on (M, g) . Furthermore,

$$(2.12) \quad \begin{aligned} & g(\nabla_t \nabla_s \phi_*(\partial/\partial s)_{(t,s)}, \phi_*(\partial/\partial t)_{(t,s)}) \\ &= \frac{d}{dt} \{g(\nabla_s \phi_*(\partial/\partial s)_{(t,s)}, \phi_*(\partial/\partial t)_{(t,s)})\} \\ &\quad - g(\nabla_s \phi_*(\partial/\partial s)_{(t,s)}, \nabla_t \phi_*(\partial/\partial t)_{(t,s)}). \end{aligned}$$

By the help of (2.6), (2.10), (2.11) and (2.12), we obtain

$$(2.13) \quad \begin{aligned} \left(\frac{d^2 L(s)}{ds^2} \right)_{s=0} &= [(g(\nabla_s \phi_*(\partial/\partial s)_{(t,s)}, \phi_*(\partial/\partial t)_{(t,s)}))_{s=0}]_{t=0}^1 \\ &\quad + \int_0^1 \{g(R(X_t, x'_t)X_t, x'_t) + \|\nabla_t X_t\|^2 \\ &\quad - (g(\nabla_t X_t, x'_t))^2\} dt. \end{aligned}$$

Let $X_t^\perp$ ($0 \leq t \leq 1$) be the component of X_t perpendicular to the geodesic $\tau = x_t$ ($0 \leq t \leq 1$), that is,

$$(2.14) \quad X_t^\perp = X_t - g(X_t, x'_t) x'_t.$$

Since $\tau_s(0) = \phi(0, s) = x_0$ and $\tau_s(1) = \phi(1, s) = x_1$ for every $s \in (-\varepsilon, \varepsilon)$, we get

$$(2.15) \quad \nabla_s \phi_*(\partial/\partial s)_{(0,s)} = 0, \quad \nabla_s \phi_*(\partial/\partial s)_{(1,s)} = 0.$$

From (2.14), we obtain

$$(2.16) \quad \begin{aligned} \nabla_t X_t &= \nabla_t X_t^\perp + \frac{d(g(X_t, x'_t))}{dt} x'_t, \\ 0 &= \frac{d(g(X_t^\perp, x'_t))}{dt} = g(\nabla_t X_t^\perp, x'_t), \end{aligned}$$

because x_t ($0 \leq t \leq 1$) is a geodesic in the Riemannian manifold (M, g) .

From (2.16), we get

$$(2.17) \quad \|\nabla_t X_t\|^2 - (g(\nabla_t X_t, x'_t))^2 = \|\nabla_t X_t^\perp\|^2.$$

We obtain from (2.13), (2.14), (2.15) and (2.17)

$$\left(\frac{d^2 L(s)}{ds^2} \right)_{s=0} = \int_0^1 \{\|\nabla_t X_t^\perp\|^2 - g(R(X_t^\perp, x'_t)x'_t, X_t^\perp)\} dt.$$

Thus, we get the following theorem.

Theorem 2.2. Let $\tau_s : [0, 1] \rightarrow M$ ($-\varepsilon < s < \varepsilon$) be an arbitrarily given variation of a geodesic $\tau_0 = \tau = x_t$ ($0 \leq t \leq 1$) on (M, g) such that $\tau_s(0) = x_0$ and $\tau_s(1) = x_1$ for every $s \in (-\varepsilon, \varepsilon)$. Then the second variation $(d^2L/ds^2)_{s=0}$ is given as follows:

$$\begin{aligned} \left(\frac{d^2L(s)}{ds^2} \right)_{s=0} &= \int_0^1 \{ \|\nabla_t X_t^\perp\|^2 - g(R(X_t^\perp, x'_t)x'_t, X_t^\perp) \} dt \\ &= \int_0^1 \{ \|\nabla_t X_t^\perp\|^2 - \|X_t^\perp\|^2 \sigma(X_t^\perp, x'_t) \} dt, \end{aligned}$$

where $\sigma(X_t^\perp, x'_t)$ is the sectional curvature determined by $\{X_t^\perp, x'_t\}$.

By virtue of Theorem 2.2, we obtain the following corollary.

Corollary 2.3. Assume that (M, g) is a space of constant negative. For an arbitrarily given variation $\tau_s : [0, 1] \rightarrow M$ ($-\varepsilon < s < \varepsilon$) of a geodesic $\tau_0 = \tau = x_t$ ($0 \leq t \leq 1$) on (M, g) such that $\tau_s(0) = x_0$ and $\tau_s(1) = x_1$ for every $s \in (-\varepsilon, \varepsilon)$, a necessary and sufficient condition for the second variation $(d^2L/ds^2)_{s=0}$ to be 0 is $X_t^\perp = 0$ ($0 \leq t \leq 1$).

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