ON DERIVATIONS OF BE-ALGEBRAS

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Abstract. In this paper, we introduce the notion of derivation in a BE-algebra, and consider the properties of derivations. Also, we characterize the fixed set $Fix_d(X)$ and $Kerd$ by derivations. Moreover, we prove that if d is a derivation of BE-algebra, every filter F is a d -invariant.

1. Introduction

Y. Imai and K. Iséki introduced two classes of abstract algebras: BCK-algebras and BCI-algebras ([3, 4]). It is known that the class of BCK-algebras is a proper subclass of the class of BCI-algebras. In [1, 2], Q. P. Hu and X. Li introduced a wide class of abstracts: BCH-algebras. They have shown that the class of BCI-algebras is a proper subclass of the class of BCH-algebras. In [5], H. S. Kim and Y. H. Kim introduced the notion of a BE-algebra as a dualization of a generation of a BCK-algebras. In this paper, we introduced the notion of derivation in a BE- algebra, and considered the properties of derivations. Also, we characterized the fixed set $Fix_d(X)$ and $Kerd$ by derivations. Moreover, we prove that if d is a derivation of BE-algebra, every filter F is a d -invariant.

2. Preliminaries

In what follows, let X denote an BE-algebra unless otherwise specified.

Received December 26, 2013. Accepted January 9, 2014.

2010 Mathematics Subject Classification. 06F35, 03G25, 08A30.

Key words and phrases. BE-algebra, derivation, self-distributive, filter, normal filter, isotone.

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By a *BE-algebra*, we mean an algebra $(X; *, 1)$ of type $(2, 0)$ with a single binary operation “ $*$ ” that satisfies the following identities, for any $x, y, z \in X$,

- (BE1) $x * x = 1$ for all $x \in X$.
- (BE2) $x * 1 = 1$ for all $x \in X$.
- (BE3) $1 * x = x$ for all $x \in X$.
- (BE4) $x * (y * z) = y * (x * z)$.

A BE-algebra X is said to be *self-distributive* if $x * (y * z) = (x * y) * (x * z)$ for all $x, y, z \in X$. A non-empty subset S of a BE-algebra X is said to be a *subalgebra* of X if $x * y \in S$ whenever $x, y \in S$. For any x, y in a BE-algebra X , we define $x \vee y = (y * x) * x$. A map d is said to be *isotone* if $x \leq y$, then $d(x) \leq d(y)$ for all $x, y \in X$.

Let X be a BE-algebra. We define the binary operation “ $\leq$ ” as the following,

$$x \leq y \Leftrightarrow x * y = 1$$

for all $x, y \in X$.

Let X be a BE-algebra. Then the following identities are true for all $x, y, z \in X$.

- (p1) $x * (y * x) = 1$.
- (p2) $x * ((x * y) * y) = 1$.
- (p3) Let X be a self-distributive BE-algebra. If $x \leq y$, then $z * x \leq z * y$ and $y * z \leq x * z$.

Let X be a BE-algebra and let F be a non-empty subset of X . Then F is called a *filter* of X if

- (F1) $1 \in F$,
- (F2) If $x \in F$ and $x * y \in F$, then $y \in F$.

3. Derivations of BE-algebras

Definition 3.1. A self-map d on a BE-algebra X is called a *derivation* if

$$d(x * y) = (x * d(y)) \vee (d(x) * y)$$

for every $x, y \in X$.

Example 3.2. Let $X = \{1, a, b\}$ be a set in which “ $*$ ” is defined by

$*$	1	a	b
1	1	a	b
a	1	1	1
b	1	1	1

Then X is a BE-algebra. Define a map $d : X \rightarrow X$ by

$$d(x) = \begin{cases} 1 & \text{if } x = 1 \\ b & \text{if } x = a \\ a & \text{if } x = b \end{cases}$$

Then it is easy to check that d is a derivation of X .

Example 3.3. Let $X = \{1, a, b, c\}$ be a set in which “ $*$ ” is defined by

$*$	1	a	b	c
1	1	a	b	c
a	1	1	a	a
b	1	1	1	a
c	1	1	a	1

Then X is a BE-algebra. Define a map $d : X \rightarrow X$ by

$$d(x) = \begin{cases} 1 & \text{if } x = 1, a \\ a & \text{if } x = b, c \end{cases}$$

Then it is easy to check that d is a derivation of a BE-algebra X .

Example 3.4. Let $X = \{1, a, b, c\}$ be a set in which “ $*$ ” is defined by

$*$	1	a	b	c
1	1	a	b	c
a	1	1	b	1
b	1	c	1	c
c	1	1	b	1

Then X is a BE-algebra. Define a map $d : X \rightarrow X$ by

$$d(x) = \begin{cases} 1 & \text{if } x = 1, b \\ c & \text{if } x = a, c. \end{cases}$$

Then it is easy to check that d is a derivation of X .

Example 3.5. Let $X = \{1, a, b, c\}$ be a set in which “ $*$ ” is defined by

$*$	1	a	b	c
1	1	a	b	c
a	1	1	b	c
b	1	a	1	c
c	1	1	1	1

Then X is a BE-algebra. Define a map $d : X \rightarrow X$ by

$$d(x) = \begin{cases} 1 & \text{if } x = 1, b, c \\ a & \text{if } x = a \end{cases}$$

Then it is easy to check that d is a derivation of X .

Proposition 3.6. *Let d be a derivation of X . Then we have*

- (1) $d(1) = 1$,
- (2) $d(x) = d(x) \vee x$ for all $x \in X$.

Proof. (1) Let d be a derivation of X . Then we have

$$\begin{aligned} d(1) &= d(1 * 1) = (1 * d(1)) \vee (d(1) * 1) \\ &= d(1) \vee 1 = (1 * d(1)) * d(1) = d(1) * d(1) = 1. \end{aligned}$$

(2) For all $x \in X$, we have

$$\begin{aligned} d(x) &= d(1 * x) = (1 * d(x)) \vee (d(1) * x) \\ &= d(x) \vee (1 * x) = d(x) \vee x. \end{aligned}$$

□

Proposition 3.7. *Let d be a derivation of X . Then the following identities hold:*

- (1) $x \leq d(x)$ for all $x \in X$,
- (2) $d(x) * y \leq x * y \leq x * d(y)$ for all $x, y \in X$.

Proof. (1) By Proposition 3.6(2) and (BE4), we have for all $x \in X$,

$$\begin{aligned} x * d(x) &= x * (d(x) \vee x) = x * ((x * d(x)) * d(x)) \\ &= (x * d(x)) * (x * d(x)) = 1 \end{aligned}$$

which implies $x \leq d(x)$.

(2) By (1) and (p3), we have $d(x) * y \leq x * y \leq x * d(y)$ for all $x, y \in X$. □

Theorem 3.8. *Let X be a self-distributive BE-algebra and let d be a derivation of X . Then we have $d(x * y) = x * d(y)$ for all $x, y \in X$.*

Proof. Let d be a derivation on X and $x, y \in X$. Then we have $d(x) * y \leq x * d(y)$ from Proposition 3.7 (2). Hence we get

$$\begin{aligned} d(x * y) &= (x * d(y)) \vee (d(x) * y) \\ &= ((d(x) * y) * (x * d(y))) * (x * d(y)) \\ &= 1 * (x * d(y)) = x * d(y). \end{aligned} \quad \square$$

Proposition 3.9. *If d is a derivation of X , then we have $d(x * d(x)) = d(d(x) * x) = 1$ for all $x \in X$.*

Proof. Let d be a derivation of X . Then we have

$$\begin{aligned} d(x * d(x)) &= (x * d(d(x))) \vee (d(x) * d(x)) \\ &= (x * d(d(x))) \vee 1 = 1 \end{aligned}$$

for all $x \in X$. Similarly, we get

$$\begin{aligned} d(d(x) * x) &= (d(x) * d(x)) \vee (d(d(x))) * x \\ &= 1 \vee (d(d(x))) * x = 1 \end{aligned}$$

for all $x \in X$. $\square$

Proposition 3.10. *Let X be a BE-algebra. Then*

$$x \leq d_n(d_{n-1}(\dots(d_2(d_1(x)))\dots))$$

for $n \in \mathbb{N}$, where $d_1, d_2, \dots, d_n$ are derivations of X .

Proof. For $n = 1$,

$$\begin{aligned} d_1(x) &= d_1(1 * x) = (1 * d_1(x)) \vee (d_1(1) * x) = d_1(x) \vee (1 * x) \\ &= d_1(x) \vee x = (x * d_1(x)) * d_1(x). \end{aligned}$$

Hence we have

$$x * d_1(x) = x * ((x * d_1(x)) * d_1(x)) = (x * d_1(x)) * (x * d_1(x)) = 1$$

which implies $x * d_1(x) = 1$. Thus $x \leq d_1(x)$.

Let $n \in \mathbb{N}$ and $x \leq d_n(d_{n-1}(\dots(d_2(d_1(x)))\dots))$. For simplicity, let

$$D_n = d_n(d_{n-1}(\dots(d_2(d_1(x)))\dots)).$$

Then

$$\begin{aligned} d_{n+1}(D_n) &= d_{n+1}(1 * D_n) = (1 * d_{n+1}(D_n)) \wedge (d_{n+1}(1) * D_n) \\ &= d_{n+1}(D_n) \wedge D_n = (D_n * d_{n+1}(D_n)) * d_{n+1}(D_n). \end{aligned}$$

Hence $D_n * D_{n+1} = 1$, which implies $D_n \leq D_{n+1}$. By assumption, $x \leq D_n \leq D_{n+1}$. $\square$

Let d be a derivation of X . Define a set $Fix_d(X)$ by

$$Fix_d(X) := \{x \in X \mid d(x) = x\}$$

for all $x \in X$.

Proposition 3.11. *Let d be a derivation of X . Then $Fix_d(X)$ is a subalgebra of X .*

Proof. Clearly, $1 \in Fix_d(X)$ and so $Fix_d(X)$ is nonempty. Let $x, y \in Fix_d(X)$. Then we have $d(x) = x$ and $d(y) = y$, and so

$$d(x * y) = (x * d(y)) \vee (d(x) * y) = (x * y) \vee (x * y) = x * y.$$

This implies $x * y \in Fix_d(X)$. Hence $Fix_d(X)$ is a subalgebra of X . $\square$

Proposition 3.12. *If $x, y \in Fix_d(X)$, then $x \vee y \in Fix_d(X)$.*

Proof. Let $x, y \in Fix_d(X)$. Then we get

$$\begin{aligned} d(x \vee y) &= d((y * x) * x) \\ &= ((y * x) * d(x)) \vee (d(y * x) * y) \\ &= ((y * x) * x) \vee ((y * d(x) \vee d(y) * x) * x) \\ &= ((y * x) * x) \vee ((y * x) * x) \\ &= (y * x) * x = x \vee y. \end{aligned} \quad \square$$

Proposition 3.13. *Let d be a derivation of X . If $x \in Fix_d(X)$, then we have $(d \circ d)(x) = x$.*

Proof. Let $x \in Fix_d(X)$. Then we have

$$(d \circ d)(x) = d(d(x)) = d(x) = x.$$

This completes the proof. $\square$

Proposition 3.14. *Let d be a derivation of a self-distributive BE-algebra X . If $y \in Fix_d(X)$, then we have $x * y \in Fix_d(X)$ for all $x \in X$.*

Proof. Let $y \in Fix_d(X)$. Then we have $d(y) = y$. Hence we have

$$\begin{aligned} d(x * y) &= (x * d(y)) \vee (d(x) * y) = ((d(x) * y) * (x * y)) * (x * y) \\ &= (x * ((d(x) * y) * y)) * (x * y) \\ &= ((x * (d(x) * y)) * (x * y)) * (x * y) \\ &= ((x * d(x)) * (x * y)) * (x * y) * (x * y) \\ &= ((1 * (x * y)) * (x * y)) * (x * y) \\ &= ((x * y) * (x * y)) * (x * y) = 1 * (x * y) = x * y. \end{aligned}$$

This completes the proof. $\square$

Theorem 3.15. *Let X be a BE-algebra and let d_1 and d_2 be two isotone derivations on X . If $d(x) \in \text{Fix}_d(X)$, then $d_1 = d_2$ if and only if $\text{Fix}_{d_1}(X) = \text{Fix}_{d_2}(X)$.*

Proof. Let $d_1 = d_2$. Then $\text{Fix}_{d_1}(X) = \text{Fix}_{d_2}(X)$. Conversely, let $\text{Fix}_{d_1}(X) = \text{Fix}_{d_2}(X)$ and $d(x) \in \text{Fix}_d(X)$ for $x \in X$. Then $d^2 = d$. Thus $d_1(x) \in \text{Fix}_{d_1}(X) = \text{Fix}_{d_2}(X)$ since $d_1(d_1(x)) = d_1^2(x) = d_1(x)$. Hence $d_2(d_1(x)) = d_1(x)$. Also, $d_2(x) \in \text{Fix}_{d_2}(X) = \text{Fix}_{d_1}(X)$, and so $d_1(d_2(x)) = d_2(x)$. Since $x \leq d_1(x)$, we have $d_1(d_2(x)) = d_2(x) \leq d_2(d_1(x))$. Symmetrically, we have $d_1(d_2(x)) \leq d_2(d_1(x))$. Hence $d_1 d_2 = d_2 d_1$. It follows that $d_2(x) = d_1(d_2(x)) = d_2(d_1(x)) = d_1(x)$. $\square$

Definition 3.16. Let X be a BE-algebra and let d be a derivation of X .

- (1) d is called a *monomorphic derivation* if d is one-to-one.
- (2) d is called an *epic derivation* if d is onto.

Theorem 3.17. *Let d be a derivation of X and $d(x) \in \text{Fix}_d(X)$. Then the following conditions are equivalent:*

- (1) $d(x) = x$ for all $x \in X$,
- (2) d is a monomorphic derivation,
- (3) d is an epic derivation.

Proof. (1) $\Rightarrow$ (2) is clear.

(2) $\Rightarrow$ (1) Let d be a monomorphic derivation and $x \in X$. Since $d(x) \in \text{Fix}_d(X)$, we have $d(d(x)) = d(x)$. Since d is monomorphic, $d(x) = x$ for all $x \in X$.

(1) $\Rightarrow$ (3) is trivial.

(3) $\Rightarrow$ (1) Let d be an epic derivation and $x \in X$. Then there exists $y \in X$ such that $d(y) = x$. Since $d(x) \in \text{Fix}_d(X)$, we have $d(x) = d(d(y)) = d^2(y) = d(y) = x$. $\square$

Let d be a derivation of X . Define a $\text{Ker}d$ by

$$\text{Ker}d = \{x \mid d(x) = 1\}$$

for all $x \in X$.

Proposition 3.18. *Let d be a derivation of X . Then $\text{Ker}d$ is a subalgebra of X .*

Proof. Clearly, $1 \in \text{Ker}d$, and so $\text{Ker}d$ is non-empty. Let $x, y \in \text{Ker}d$. Then $d(x) = 1$ and $d(y) = 1$. Hence we have

$$d(x * y) = x * d(y) \vee d(x) * y = (x * 1) \vee (1 * y) = 1 \vee y = 1,$$

and so $x * y \in \text{Kerd}$. Thus Kerd is a subalgebra of X . $\square$

A BE-algebra X is said to be *commutative* if for all $x, y \in X$,

$$(y * x) * x = (x * y) * y.$$

Proposition 3.19. *Let X be a commutative BE-algebra. If $x \in \text{Kerd}$ and $x \leq y$, then we have $y \in \text{Kerd}$.*

Proof. Let $x \in \text{Kerd}$ and $x \leq y$. Then $d(x) = 1$ and $x * y = 1$.

$$\begin{aligned} d(y) &= d(1 * y) = d((x * y) * y) = d((y * x) * x) \\ &= ((y * x) * d(x)) \vee (d(y * x) * x) \\ &= ((y * x) * 1) \vee (d(y * x) * x) \\ &= 1 \vee (d(y * x) * x) = 1, \end{aligned}$$

and so $y \in \text{Kerd}$. This completes the proof. $\square$

Theorem 3.20. *Let d be an isotone derivation of a self-distributive BE-algebra X . If d is idempotent, then Kerd is a filter of X .*

Proof. Clearly, $1 \in \text{Kerd}$. Let $x \in \text{Kerd}$ and $x * y \in \text{Kerd}$. Then we have $1 = d(x * y) = x * d(y)$ from Theorem 3.8, which means $x \leq d(y)$. Hence we get $1 = d(x) \leq d^2(y) = d(y)$, i.e., $d(y) = 1$. This implies $y \in \text{Kerd}$. This completes the proof. $\square$

Proposition 3.21. *Let d be a derivation of X and $x \in \text{Kerd}$. Then we have $x \vee y \in \text{Kerd}$ for all $y \in X$.*

Proof. Let d be a derivation of X and $x \in \text{Kerd}$. Then $d(x) = 1$. Hence we have for all $y \in X$,

$$\begin{aligned} d(x \vee y) &= d((y * x) * x) = ((y * x) * d(x)) \vee (d(y * x) * x) \\ &= ((y * x) * 1) \vee (d(y * x) * x) = 1 \vee (d(y * x) * x) = 1. \end{aligned}$$

Hence we have $x \vee y \in \text{Kerd}$. $\square$

Proposition 3.22. *Let d be a derivation of X and $y \in \text{Kerd}$. Then we have $x * y \in \text{Kerd}$ for all $x \in X$.*

Proof. Let d be a derivation of X and $y \in \text{Kerd}$. Then $d(y) = 1$. Hence we have for all $x \in X$,

$$\begin{aligned} d(x * y) &= (x * d(y)) \vee (d(x) * y) \\ &= (x * 1) \vee (d(x) * y) = 1 \vee d(x) * y = 1. \end{aligned} \quad \square$$

Proposition 3.23. *Let d be a derivation of X . If d is one-to-one, then $\text{Kerd} = 1$.*

Proof. Suppose that d is one-to-one and $x \in \text{Ker}d$. Then $d(x) = 1 = d(1)$, and thus $x = 1$, i.e., $\text{Ker}d = \{1\}$. $\square$

Definition 3.24. Let X be a BE-algebra. A self-map d is *isotone* if $x \leq y$ implies $d(x) \leq d(y)$.

Proposition 3.25. Let d be a derivation of a BE-algebra X . If $d(x) \vee d(y) \leq d(x \vee y)$ for all $x, y \in X$, then d is an isotone derivation of X .

Proof. Suppose that $d(x) \vee d(y) \leq d(x \vee y)$ and $x \leq y$. Then we have $d(x) \leq d(x) \vee d(y) \leq d(x \vee y) = d(y)$. $\square$

Proposition 3.26. Let d be an isotone derivation of X . If $x \leq y$ and $x \in \text{Ker}d$, then $y \in \text{Ker}d$.

Proof. Let $x \leq y$ and $x \in \text{Ker}d$. Then we have $d(x) = 1$, and so

$$1 = d(x) \leq d(y),$$

which implies $d(y) = 1$. $\square$

Definition 3.27. Let X be a BE-algebra. A non-empty subset F of X is said to be a *d-invariant* if $d(F) \subseteq F$ where $d(F) = \{d(x) \mid x \in F\}$.

Proposition 3.28. Let X be a BE-algebra and let d be a derivation. Then every filter F is a *d-invariant*.

Proof. Let F be a filter of X . Let $y \in d(F)$. Then $y = d(x)$ for some $x \in F$. It follows that $x * y = x * d(x) = 1 \in F$, which implies $y \in F$. Thus $d(F) \subseteq F$. Hence F is a *d-invariant*. $\square$

Let X be a BE-algebra. We define the binary operation " + " as the following

$$x + y = (x * y) * y$$

for all $x, y \in X$. Clearly, X is a commutative BE-algebra if and only if $x + y = y + x$ for all $x, y \in X$.

Proposition 3.29. Let X be a commutative BE-algebra. Then the followings hold for all $x, y, z \in X$,

- (1) $x + 1 = 1 + x$.
- (2) $x + y = y + x$.
- (3) $x + (y + z) = (x + y) + z$.
- (4) $x + x = x$.

Proof. The proof is straightforward. $\square$

Proposition 3.30. Let X be a BE-algebra and let d be a derivation of X . Then the followings hold for all $x, y \in X$,

- (1) $d(x + 1) = 1$.
- (2) $d(x + x) = d(x) + x$.
- (3) $d(x) = x + d(x)$.

Proof. (1) Let $x \in X$. Then we get

$$\begin{aligned} d(x + 1) &= d((x * 1) * 1) = ((x * 1) * d(1)) \vee (d(x * 1) * 1) \\ &= (1 * 1) \vee (d(x * 1) * 1) = 1 \vee 1 \\ &= 1. \end{aligned}$$

(2) Let $x \in X$. Then we have

$$\begin{aligned} d(x + x) &= d((x * x) * x) = ((x * x) * d(x)) \vee (d(x * x) * x) \\ &= (1 * d(x)) \vee (d(1) * x) = d(x) \vee x \\ &= (x * d(x)) * d(x) = x + d(x). \end{aligned}$$

(3) Let $x \in X$. Then we have

$$\begin{aligned} d(x) &= d(1 * x) = (1 * d(x)) \vee (d(1) * x) \\ &= d(x) \vee x = (x * d(x)) * d(x) \\ &= x + d(x). \end{aligned} \quad \square$$

Definition 3.31. Let X be a BE-algebra. A non-empty set F of X is called a *normal filter* of X if it satisfies the following conditions:

- (NF1) $1 \in F$,
- (NF2) $x \in X$ and $y \in F$ imply $x * y \in F$.

Example 3.32. Let $X = \{1, a, b, c\}$ be a set in which “ $*$ ” is defined by

$*$	1	a	b	c
1	1	a	b	c
a	1	1	b	c
b	1	1	1	c
c	1	1	1	1

Then X is a BE-algebra. Let $F = \{1, a\}$. Then F is a normal filter of X .

Proposition 3.33. Let X be a self-distributive BE-algebra and let d be a derivation of X . Then $Fix_d(X)$ is a normal filter of X .

Proof. Clearly, $1 \in Fix_d(X)$. Let $x \in X$ and $y \in Fix_d(X)$. Then we have $d(y) = y$, and so

$$\begin{aligned} d(x * y) &= x * d(y) \\ &= x * y, \end{aligned}$$

which implies $x * y \in \text{Fix}_d(X)$ from Theorem 3.8. This completes the proof. $\square$

Proposition 3.34. *Let X be a self-distributive BE-algebra and let d be a derivation of X . Then $\text{Ker}d$ is a normal filter of X .*

Proof. Clearly, $1 \in \text{Ker}d$. Let $x \in X$ and $y \in \text{Ker}d$. Then we have $d(y) = 1$, and so from Theorem 3.8,

$$d(x * y) = x * d(y) = x * 1 = 1,$$

which implies $x * y \in \text{Ker}d$. Hence $\text{Ker}d$ is a normal filter of X . $\square$

Proposition 3.35. *Let X be a self-distributive BE-algebra and let d be a derivation of X . Then $F_a = \{x \in X \mid a \leq d(x)\}$ is a normal filter of X .*

Proof. Clearly, $a \leq d(1) = 1$ for any $a \in X$, and so $1 \in F_a$. Let $x \in X$ and $y \in F_a$. Then we have $a * d(y) = 1$, and so from Theorem 3.8,

$$\begin{aligned} a * d(x * y) &= a * (x * d(y)) = (a * x) * (a * d(y)) \\ &= (a * x) * 1 \\ &= 1, \end{aligned}$$

which implies $x * y \in F_a$. Hence F_a is a normal filter of X . $\square$

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