AN EQUIVALENT PROPERTY OF A NORMAL ADJACENCY OF A DIGITAL PRODUCT

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Abstract. Owing to the development of the notion of normal adjacency of a digital product [9], product properties of digital topological properties were studied efficiently. To equivalently represent a normal adjacency of a digital product, the present paper proposes an S -compatible adjacency of a digital product. This approach can be helpful to understand a normal adjacency of a digital product. Finally, using an S -compatible adjacency of a digital product, we can study product properties of digital topological properties, which improves the presentations of the normal adjacency of a digital product in [9] and [5, 6].

1. Introduction

It is well known that the study of product properties of a certain topological property is very important role in topology. Similarly, in digital topology we also need to follow this approach. For instance, the paper [9] introduced the notion of normal adjacency for a digital product and studied product property of a digital fundamental group [24, 25].

Indeed, a digital image (X, k) can be considered to be a set $X \subset \mathbf{Z}^n$ with a k -adjacency relation on $\mathbf{Z}^n$ (or an adjacency graph) [26, 31], where $\mathbf{Z}^n$ is the set of points in the Euclidean n D space with integer coordinates, $n \in \mathbf{N}$ and $\mathbf{N}$ is the set of natural numbers. To study digital topological properties of a digital image (X, k) , we have used various tools such as digital fundamental groups [4, 14, 19], digital covering spaces [8, 9, 15, 17, 20], digital homotopy equivalences [18, 23] and digital k -surface structures [2, 3, 7, 11, 25].

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Motivated by the notion of a *covering space* in algebraic topology [32], its digital version was developed in [9] (see also [15, 20]). It provides powerful tool to study multi-dimensional digital images (X, k) in $\mathbf{Z}^n$ and it has been used for calculating (digital) k -fundamental groups of some digital images (X, k) and classifying digital images from the viewpoints of digital topology and digital geometry [11, 12, 13, 14, 15, 16, 17, 18, 19, 20].

In graph theory it is well known that both the *normal* adjacency and the *Cartesian* adjacency for a Cartesian product of graphs [1, 24] have substantially contributed to the study of a Cartesian product of graphs. However, their digital versions have partially been developed. Indeed, a Cartesian product of two coverings in algebraic topology is also a covering [28, 32] and further, the classical fundamental group has the multiplicative property [28, 32]. In relation to the study of this problem, since we have often used a normal adjacency of a digital product, the present paper proposes an equivalent property of a normal adjacency which can be helpful to deal with the product problem in digital topology.

The paper has main results in Section 3 and is organized as follows: Section 2 provides basic notions. Section 3 develops an S -compatible k -adjacency for a digital product and studies its various properties which play important roles in investigating digital topological properties of a digital product (see Theorem 3.5). Finally, Section 4 concludes the paper with a summary.

2. Preliminaries

A (binary) digital image (X, k) can be regarded as a subset $X \subset \mathbf{Z}^n$ with a k -adjacency relation of $\mathbf{Z}^n$ (or an adjacency graph). For $a, b \in \mathbf{Z}$ with $a \leq b$, the set $[a, b]_{\mathbf{Z}} = \{n \in \mathbf{Z} \mid a \leq n \leq b\}$ is called a digital interval. Further, let us recall the following: let $p := (p_i)_{i \in [1, n]_{\mathbf{Z}}}$ be a point of $\mathbf{Z}^n$ and m an integer in $[1, n]_{\mathbf{Z}}$. Consider all points $q := (q_i)_{i \in [1, n]_{\mathbf{Z}}} \in \mathbf{Z}^n$ satisfying the property of (2.1) [9] such that $p \neq q$

$$\left\{ \begin{array}{l} \bullet \text{ there are at most } m \text{ indices } i \text{ such that } |p_i - q_i| = 1 \text{ and} \\ \bullet \text{ for all other indices } i, p_i = q_i. \end{array} \right\} \quad (2.1)$$

The number of such points is [16] (for more details, see [18])

$$k := k(m, n) = \sum_{i=n-m}^{n-1} 2^{n-i} C_i^n, \text{ where } C_i^n = \frac{n!}{(n-i)! i!}. \quad (2.2)$$

For consistency with the nomenclature “4-adjacent” and “8-adjacent” well established in the context of 2-dimensional integer grids, we will say that two points $p, q \in \mathbf{Z}^n$ are k -adjacent if they satisfy the condition (2.1), where $k := k(m, n)$ of (2.2) [9] (see also [16, 18]).

For instance,

$$(m, n, k) \in \left\{ \begin{array}{l} (1, 2, 4), (2, 2, 8); \\ (1, 3, 6), (2, 3, 18), (3, 3, 26); \\ (1, 4, 8), (2, 4, 32), (3, 4, 64), (4, 4, 80); \\ (1, 5, 10), (2, 5, 50), (3, 5, 130), (4, 5, 210), (5, 5, 242); \text{ and} \\ (1, 6, 12), (2, 6, 72), (3, 6, 232), (4, 6, 472), (5, 6, 664), (6, 6, 728). \end{array} \right\} \quad (2.3)$$

Owing to the phrase “at most m ” from the first bullet of (2.1), we can obviously see that the points $p = (p_1, p_2, \dots, p_n)$ and $q = (q_1, q_2, \dots, q_n) \in \mathbf{Z}^n$ may differ in as many as m coordinates. Thus, in general, we obtain the following:

Remark 2.1. [19] *If two points $x, y \in \mathbf{Z}^n$ are $k(m, n)$ -adjacent then they are obviously $k(m', n)$ -adjacent, where $m \leq m'$.*

This remark will be often used in Section 3, *e.g.* Theorem 3.5.

Using the k -adjacency relations of $\mathbf{Z}^n$ in (2.2), we can study digital topological properties of a set $X \subset \mathbf{Z}^n$ with a k -adjacency, $n \in \mathbf{N}$. This has been often used for representing digital continuity, a digital isomorphism, a digital homotopy, a digital k -surface structure, etc. Owing to the digital k -connectivity paradox in [27], we remind the reader that $k \neq \bar{k}$ except the case $(\mathbf{Z}, 2, 2, X)$. However, in this paper we are not concerned with $\bar{k}$ -adjacency between two points in $\mathbf{Z}^n \setminus X$.

Let us now recall basic notions. Hereafter, the digital interval $[a, b]_{\mathbf{Z}}$ is assumed with 2-adjacency. We say that two subsets (A, k) and (B, k) of (X, k) are k -adjacent to each other if $A \cap B = \emptyset$ and there are points $a \in A$ and $b \in B$ such that a and b are k -adjacent to each other [27]. We say that a set $X \subset \mathbf{Z}^n$ is k -connected if it is not a union of two disjoint non-empty subsets of X that are not k -adjacent to each other [27]. For (X, k) , a point $x \in X$ is called *isolated* if it is not k -connected with any point in X [27]. For an adjacency relation k of $\mathbf{Z}^n$, a simple k -path with $l + 1$ elements in $\mathbf{Z}^n$ is assumed to be an injective sequence $(x_i)_{i \in [0, l]_{\mathbf{Z}}} \subset \mathbf{Z}^n$ such that x_i and x_j are k -adjacent if and only if either $j = i + 1$ or $i = j + 1$ [27]. If $x_0 = x$ and $x_l = y$, then we say

that the length of the simple k -path is l . A simple closed k -curve with l elements in $\mathbf{Z}^n, n \geq 2$, denoted by $SC_k^{n,l}$ [9] (see also [11]), is the simple k -path $(x_i)_{i \in [0, l-1]_{\mathbf{Z}}}$, where x_i and x_j are k -adjacent if and only if $j = i + 1(\text{mod } l)$ or $i = j + 1(\text{mod } l)$ [27]. Besides, for $\mathbf{Z}^n$ we remind the following [27]:

$$\left\{ \begin{array}{l} N_k(x) := \{x' \mid x \text{ is } k\text{-adjacent to } x' \text{ in } \mathbf{Z}^n\} \text{ and} \\ N_k^*(x) := N_k(x) \cup \{x\}. \end{array} \right\} \quad (2.4)$$

As a generalization of $N_k^*(x)$ in $\mathbf{Z}^n$, for a multi-dimensional adjacency graph (or a digital image) (X, k) and a point $x \in X \subset \mathbf{Z}^n$, the notion of a (digital) k -neighborhood of a point x with radius $\varepsilon \in \mathbf{N}$ was established [9], as follows.

$$N_k(x_0, \varepsilon) := \{x \in X \mid l_k(x_0, x) \leq \varepsilon\} \cup \{x_0\}, \quad (2.5)$$

where $l_k(x_0, x)$ is the length of a shortest simple k -path from x_0 to x in X . For instance, for $x \in (X, k)$ we can observe that $N_k^*(x) \cap X$ is equal to $N_k(x, 1)$ [15]. Besides, we need to remind that while $N_k(x)$ in (2.4) does not contain the point x , the set $N_k(x_0, \varepsilon)$ has the point x_0 . This difference should be reminded in Sections 3 and 4. If a point x in a digital image (X, k) is isolated, then for any $\varepsilon \in \mathbf{N}$ we can observe that $N_k(x, \varepsilon)$ is a singleton $\{x\}$. The k -neighborhood of (2.5) will be often used for establishing several compatible adjacency relations for a Cartesian product of two digital images (see Section 3) and digital continuity. The original version of digital continuity was developed in [30]: let (X, k_0) and (Y, k_1) be digital images in $\mathbf{Z}^{n_0}$ and $\mathbf{Z}^{n_1}$, respectively. Let $f : (X, k_0) \rightarrow (Y, k_1)$ be a function. We say that f is (k_0, k_1) -continuous if the image under f of every k_0 -connected subset of X is k_1 -connected [30].

In view of the k -neighborhood of (2.5), since for each point x of a digital image (X, k) in $\mathbf{Z}^n$ there is always $N_k(x, 1) \subset (X, k)$, the notion of *digital continuity* in [30] can be represented by the following form [9] (see also [15]), which can be efficiently used in studying product properties of digital topological invariants.

Definition 1. [9] (see also [15]) *Let (X, k_0) and (Y, k_1) be digital images in $\mathbf{Z}^{n_0}$ and $\mathbf{Z}^{n_1}$, respectively. A function $f : X \rightarrow Y$ is (k_0, k_1) -continuous if for every $x \in X$ $f(N_{k_0}(x, 1)) \subset N_{k_1}(f(x), 1)$.*

We have often used the following notion of a (k_0, k_1) -isomorphism instead of a (k_0, k_1) -homeomorphism in [4]: for two adjacency graphs (or digital images) (X, k_0) in $\mathbf{Z}^{n_0}$ and (Y, k_1) in $\mathbf{Z}^{n_1}$, a map $h : X \rightarrow Y$

is called a (k_0, k_1) -isomorphism [29] (see also [5, 10, 14]) if h is a (k_0, k_1) -continuous bijection and further, $h^{-1} : Y \rightarrow X$ is (k_1, k_0) -continuous [10] (see also [15]), and we use the notation $X \approx_{(k_0, k_1)} Y$. If $n_0 = n_1$ and $k_0 = k_1$, then we call it a k_0 -isomorphism.

3. Compatible adjacency relations for a digital product

In classical graph theory two compatible adjacencies for a Cartesian product such as *normal adjacency* in [1] and *Cartesian product adjacency* in [24] play important roles in studying classical graphs. For given two adjacency graphs $(X_i, k(m_i, n_i)), i \in \{1, 2\}$ we call $(X_1 \times X_2, k(m, n_1 + n_2))$ a *digital product* with a $k(m, n_1 + n_2)$ -adjacency of $\mathbf{Z}^{n_1+n_2}$ (see (2.2)). Thus the paper [9] formulated a digital product with a normal adjacency (or a normal adjacency graph product) to study a digital fundamental group of a digital product as follows:

Definition 2. [9] For two digital images (X, k_1) in $\mathbf{Z}^{n_1}$, (Y, k_2) in $\mathbf{Z}^{n_2}$, consider the digital product $X \times Y \subset \mathbf{Z}^{n_1+n_2}$. Then we say that two points $(x, y) \in X \times Y$, $(x', y') \in X \times Y$ are normally k -adjacent to each other if and only if

- (1) x is k_1 -adjacent to x' and $y = y'$;
- (2) y is k_2 -adjacent to y' and $x = x'$; or
- (3) x is k_1 -adjacent to x' and y is k_2 -adjacent to y' .

Unfortunately, the paper [9] misused an 8-adjacency on $MSC_4 \times MSC_4 := SC_4^{2,8} \times SC_4^{2,8}$ as a normal adjacency (see [5]), where $MSC_4 := SC_4^{2,8}$ means a simple closed 4-curve with eight elements in $\mathbf{Z}^2$ (see Figure 1). Indeed, the product $SC_4^{2,8} \times SC_4^{2,8}$ does not have any normal k -adjacency. In digital geometry, to study digital product properties, we strongly need to develop a compatible k -adjacency on a digital product which makes a product of adjacency graph isomorphisms an adjacency graph isomorphism.

The following simple closed 4- and 8-curves in $\mathbf{Z}^2$ in [8, 9, 10] and simple closed 18- and 26-curves in $\mathbf{Z}^3$ in [9, 13] will be often used later in the paper (see Figure 1).

$$\left\{ \begin{array}{l} SC_4^{2,8} \approx_4 ((0,0), (1,0), (2,0), (2,1), (2,2), (1,2), (0,2), (0,1)), \\ SC_8^{2,6} := MSC_8 \approx_8 ((0,0), (1,1), (1,2), (0,3), (-1,2), (-1,1)), \\ SC_8^{2,4} \approx_8 ((0,0), (1,1), (2,0), (1,-1)), \\ SC_8^{2,8} := ((0,0), (1,1), (2,2), (1,3), (0,4), (-1,3), (-2,2), (-1,1)), \\ MSC_{18} := ((0,0,0), (1,-1,0), (1,-1,1), (2,0,1), (1,1,1), (1,1,0)), \\ SC_{18}^{3,6} := ((0,0,0), (1,0,1), (1,1,2), (0,2,2), (-1,1,2), (-1,0,1)), \\ SC_{26}^{3,4} := ((0,0,0), (1,1,1), (0,2,2), (-1,1,1)). \end{array} \right\} \quad (3.1)$$

We need to recall that the set $MSC_{18} := (c_i)_{i \in [0,5]_{\mathbf{Z}}}$ in Figure 1 is a simple closed 18-curve with six element in $\mathbf{Z}^3$ which is different from the other $SC_{18}^{3,6}$ suggested in (3.1). Comparing MSC_{18} with $SC_{18}^{3,6}$, we can see some different features at the elements c_1 and c_2 of MSC_{18} in Figure 1. MSC_{18} will be often used in Sections 3 and 4.

FIGURE 1. Various kinds of simple closed k -curves [9, 14, 16].

To represent a Cartesian product of two adjacency graphs (or a digital product) as a matrix, we use the following notation:

$SC_4^{2,8} := (a_i)_{i \in [1,8]_{\mathbf{Z}}}$, $SC_8^{2,6} := (b_j)_{j \in [1,6]_{\mathbf{Z}}}$ and $SC_8^{2,4} := (d_i)_{i \in [1,4]_{\mathbf{Z}}}$ (see (3.1)).

Then we can obtain their products in $\mathbf{Z}^4$ as sets without adjacencies at the moment as follows:

$$\left\{ \begin{array}{l} SC_4^{2,8} \times SC_8^{2,6} := (c_{ij})_{(i,j) \in [1,8]_{\mathbf{Z}} \times [1,6]_{\mathbf{Z}}}, \text{ : the matrix in (3.3), and} \\ SC_8^{2,4} \times SC_8^{2,6} := (e_{ij})_{(i,j) \in [1,4]_{\mathbf{Z}} \times [1,6]_{\mathbf{Z}}}, \text{ : the matrix in (3.4)} \end{array} \right\} \quad (3.2)$$

where $c_{ij} := (a_i, b_j)$ (see (3.3)), $e_{ij} := (d_i, b_j)$ (see (3.4)).

$$\left(\begin{array}{cccccc} (0,0,0,0) & (0,0,1,1) & (0,0,1,2) & (0,0,0,3) & (0,0,-1,2) & (0,0,-1,1) \\ (1,0,0,0) & (1,0,1,1) & (1,0,1,2) & (1,0,0,3) & (1,0,-1,2) & (1,0,-1,1) \\ (2,0,0,0) & (2,0,1,1) & (2,0,1,2) & (2,0,0,3) & (2,0,-1,2) & (2,0,-1,1) \\ (2,1,0,0) & (2,1,1,1) & (2,1,1,2) & (2,1,0,3) & (2,1,-1,2) & (2,1,-1,1) \\ (2,2,0,0) & (2,2,1,1) & (2,2,1,2) & (2,2,0,3) & (2,2,-1,2) & (2,2,-1,1) \\ (1,2,0,0) & (1,2,1,1) & (1,2,1,2) & (1,2,0,3) & (1,2,-1,2) & (1,2,-1,1) \\ (0,2,0,0) & (0,2,1,1) & (0,2,1,2) & (0,2,0,3) & (0,2,-1,2) & (0,2,-1,1) \\ (0,1,0,0) & (0,1,1,1) & (0,1,1,2) & (0,1,0,3) & (0,1,-1,2) & (0,1,-1,1) \end{array} \right) \quad (3.3)$$

$$\left(\begin{array}{cccccc} (0,0,0,0) & (0,0,1,1) & (0,0,1,2) & (0,0,0,3) & (0,0,-1,2) & (0,0,-1,1) \\ (1,1,0,0) & (1,1,1,1) & (1,1,1,2) & (1,1,0,3) & (1,1,-1,2) & (1,1,-1,1) \\ (2,0,0,0) & (2,0,1,1) & (2,0,1,2) & (2,0,0,3) & (2,0,-1,2) & (2,0,-1,1) \\ (1,-1,0,0) & (1,-1,1,1) & (1,-1,1,2) & (1,-1,0,3) & (1,-1,-1,2) & (1,-1,-1,1) \end{array} \right) \quad (3.4)$$

Let us now consider the maps p_1 and p_2 defined by

$$\left\{ \begin{array}{l} p_1 : SC_4^{2,8} \rightarrow SC_8^{2,4} \text{ for which} \\ p_1(a_i) = d_i, i \in [1,4]_{\mathbf{Z}}, p_1(a_i) = d_{i(mod 5)+1}, i \in [5,8]_{\mathbf{Z}}, \text{ and} \\ p_2 := 1_{SC_8^{2,6}} : SC_8^{2,6} \rightarrow SC_8^{2,6} \text{ as an identity map.} \end{array} \right\} \quad (3.5)$$

Obviously, we can observe that the map p_1 is $(4,8)$ -continuous and p_2 is $(8,8)$ -continuous.

Using the maps p_1 and p_2 in (3.5), let us now investigate product properties of digital continuity. Consider the natural product map

$$p := p_1 \times p_2 : SC_4^{2,8} \times SC_8^{2,6} := (c_{ij}) \rightarrow SC_8^{2,4} \times SC_8^{2,6} := (e_{ij})$$

(see (3.3) and (3.4)), defined by

$$\left\{ \begin{array}{l} p_1 \times p_2(c_{ij}) = (p_1(a_i), p_2(b_j)) = (d_i, b_j), i \in [1,4]_{\mathbf{Z}} \text{ and} \\ p_1 \times p_2(c_{ij}) = (p_1(a_i), p_2(b_j)) = (d_{i(mod 5)+1}, b_j), i \in [5,8]_{\mathbf{Z}}. \end{array} \right\}$$

Let us now investigate continuities of the given product maps with some k -adjacencies of the products $SC_4^{2,8} \times SC_8^{2,6}$ and $SC_8^{2,4} \times SC_8^{2,6}$ relative to the given factor adjacency graphs. To be specific, we observe that the Cartesian product map $p_1 \times p_2$ is not a (k, k') -continuous map for any k - and k' -adjacency relations of $\mathbf{Z}^4$, $k, k' \in \{80, 64, 32\}$.

More precisely, consider $k \in \{80, 64, 32\}$. Let us now examine if p is

(k, k') -continuous at the point $c_{44} := (2, 1, 0, 3) \in SC_4^{2,8} \times SC_8^{2,6} := (c_{ij})_{(i,j) \in [1,8]_{\mathbf{Z}} \times [1,6]_{\mathbf{Z}}} \subset \mathbf{Z}^4$ in (3.3).

First, we prove that for any $k \in \{80, 64, 32\}$ the map p is not $(k, 80)$ -continuous. Comparing two points $c_{44} := (2, 1, 0, 3)$ and $c_{24} := (1, 0, 0, 3)$ in (3.3) in terms of Definition 1, we observe that

$$p(N_k(c_{44}, 1)) \not\subseteq N_{80}(e_{44}, 1), \quad (3.6)$$

where

$$\begin{cases} p(c_{44}) = (p_1(a_4), p_2(b_4)) = (d_4, b_4) := e_{44}, \\ N_{80}(c_{44}, 1) = \{c_{ij} \mid (i, j) \in [3, 5]_{\mathbf{Z}} \times [2, 6]_{\mathbf{Z}}\}, \\ N_{64}(c_{44}, 1) = N_{80}(c_{44}, 1) \setminus \{c_{23}, c_{25}, c_{63}, c_{65}\}, \\ N_{32}(c_{44}, 1) = N_{64}(c_{44}, 1) \setminus \{c_{33}, c_{36}, c_{63}, c_{66}\}, \text{ and} \\ N_8(e_{44}, 1) = \{e_{ij} \mid (i, j) \in \{4\} \times [3, 5]_{\mathbf{Z}}\}. \end{cases}$$

Namely, the property (3.6) implies that the map p is not $(k, 80)$ -continuous for any k -adjacency on $SC_4^{2,8} \times SC_8^{2,6}$, $k \in \{80, 64, 32\}$. More precisely, take the points $c_{24} := (1, 0, 0, 3)$ and $c_{44} := (2, 1, 0, 3)$ which are k -adjacent to each other and further, $N_k(c_{44}, 1)$ in (3.3). Let us now consider their images by the map p , i.e. $e_{24} := p(c_{24})$, $e_{44} := p(c_{44})$ and $p(N_k(c_{44}, 1))$. Owing to the difference ‘2’ of the first components of the points $e_{24} := (1, 1, 0, 3)$ and $e_{44} := (1, -1, 0, 3)$, these points are not 80-adjacent to each other. Thus it turns out that the property (3.6) holds.

Second, using a method similar to that of the proof above for non- $(k, 80)$ -continuity of p , we can prove that for $k \in \{80, 64, 32\}$ the map p is not (k, k') -continuous either, where $k' \in \{64, 32\}$ (see the point $c_{44} \in SC_4^{2,8} \times SC_8^{2,6}$ in (3.3)).

Finally, let us now assume 8-adjacency on $SC_4^{2,8} \times SC_8^{2,6}$. Then we observe that

$$p(N_8(c_{44}, 1)) \subset N_{32}(p(c_{44}), 1) = N_{32}(e_{44}, 1), \quad (3.7)$$

where $N_8(c_{44}, 1) = \{c_{34}, c_{44}, c_{54}\}$ in (3.3) and $N_{32}(e_{44}, 1) = \{e_{44}, e_{43}, e_{45}, e_{14}, e_{34}\}$ in (3.4). The property (3.7) implies $(8, 32)$ -continuity of p at the point c_{44} . However, since $N_8(c_{44}, 1)$ is only related to $SC_4^{2,8} \times \{b_4\}$ instead of $SC_4^{2,8} \times SC_8^{2,6}$, $b_4 \in SC_8^{2,6}$, the choice of $N_8(c_{44}, 1)$ does not reflect the 8-connectivity of $SC_8^{2,6}$, which is a drawback of the choice of 8-adjacency on $SC_4^{2,8} \times SC_8^{2,6}$.

In view of the above discussion about the Cartesian product property of given digitally continuous maps, we obtain the following:

Remark 3.1. Let (X, k_0) , (Y, k_1) , (Z, k_2) , and (W, k_3) be adjacency graphs in $\mathbf{Z}^{n_0}$, $\mathbf{Z}^{n_1}$, $\mathbf{Z}^{n_2}$, and $\mathbf{Z}^{n_3}$, respectively. Let $f : X \rightarrow Y$ and $g : Z \rightarrow W$ be (k_0, k_1) - and (k_2, k_3) -continuous maps, respectively. Then, not every k -adjacency for $X \times Z \subset \mathbf{Z}^{n_0+n_2}$ and k' -adjacency for $Y \times W \subset \mathbf{Z}^{n_1+n_3}$ makes the Cartesian product map $f \times g : X \times Z \rightarrow Y \times W$ given by $f \times g(x, z) = (f(x), g(z))$ be (k, k') -continuous.

In view of Remark 3.1, to study a product problem of an adjacency graph isomorphism, we need a compatible adjacency such as an S -compatible k -adjacency (see Definition 3) which can be used for studying both the product problem of two adjacency graph isomorphisms and the multiplicative property of a digital fundamental group. Thus, motivated from the strong product in graph theory [1, 24], let us now develop an S -compatible adjacency for a digital product as follows.

Definition 3. For two adjacency graphs (X, k_1) in $\mathbf{Z}^{n_1}$ and (Y, k_2) in $\mathbf{Z}^{n_2}$, consider a Cartesian product $X \times Y \subset \mathbf{Z}^{n_1+n_2}$. We say that a k -adjacency on $X \times Y$ is S -compatible with the k_i -adjacency on X and Y , $i \in \{1, 2\}$ if every point (x, y) in $X \times Y$ satisfies the following property: for two distinct points (x, y) and (x', y') in $X \times Y$

$$(x', y') \in N_k((x, y), 1) \Leftrightarrow x' \in N_{k_1}(x, 1), y' \in N_{k_2}(y, 1).$$

For instance, we obtain the following S -compatible adjacency: $([a, b]_{\mathbf{Z}} \times [c, d]_{\mathbf{Z}}, 8)$ and $(SC_8^{2,6} \times [a, b]_{\mathbf{Z}}, 26)$.

In view of Definition 3, we observe the following: let $k_1 = k(m_1, n_1)$, where $m_1 = n_1$ (or $m_1 \notin [1, n_1 - 1]_{\mathbf{Z}}$ and $k_2 = k(m_2, n_2)$, where $m_2 = n_2$ (or $m_2 \notin [1, n_2 - 1]_{\mathbf{Z}}$. Then we can take the number $m = n_1 + n_2$ for establishing the $k(m, n_1 + n_2)$ -adjacency on the digital product, where $k := k(m, n_1 + n_2)$ is not smaller than $3^{n_1+n_2} - 1$.

Let us now consider several examples which do not have S -compatible adjacency for digital products.

Example 3.2. No S -compatible adjacency on $MSC_{18} \times MSC_{18}$, $SC_4^{2,8} \times SC_8^{2,6}$, and $SC_4^{2,8} \times SC_4^{2,8}$ exist.

Proof. Let us prove that $MSC_{18} \times MSC_{18} \subset \mathbf{Z}^6$ has no S -compatible adjacency. Let us consider the following matrix (a_{ij}) in (3.8) representing $MSC_{18} \times MSC_{18} := (a_{ij})_{(i,j) \in [1,6]_{\mathbf{Z}} \times [1,6]_{\mathbf{Z}}}$, where for each $(i, j) \in [1, 6]_{\mathbf{Z}} \times [1, 6]_{\mathbf{Z}}$ $a_{ij} := (c_{i-1}, c_{j-1}) \in MSC_{18} \times MSC_{18}$.

$$\begin{pmatrix}
(0, 0, 0, 0, 0, 0) & (0, 0, 0, 1, -1, 0) & (0, 0, 0, 1, -1, 1) \\
(1, -1, 0, 0, 0, 0) & (1, -1, 0, 1, -1, 0) & (1, -1, 0, 1, -1, 1) \\
(1, -1, 1, 0, 0, 0) & (1, -1, 1, 1, -1, 0) & (1, -1, 1, 1, -1, 1) \\
(2, 0, 1, 0, 0, 0) & (2, 0, 1, 1, -1, 0) & (2, 0, 1, 1, -1, 1) \\
(1, 1, 1, 0, 0, 0) & (1, 1, 1, 1, -1, 0) & (1, 1, 1, 1, -1, 1) \\
(1, 1, 0, 0, 0, 0) & (1, 1, 0, 1, -1, 0) & (1, 1, 0, 1, -1, 1) \\
(0, 0, 0, 2, 0, 1) & (0, 0, 0, 1, 1, 1) & (0, 0, 0, 1, 1, 0) \\
(1, -1, 0, 2, 0, 1) & (1, -1, 0, 1, 1, 1) & (1, -1, 0, 1, 1, 0) \\
(1, -1, 1, 2, 0, 1) & (1, -1, 1, 1, 1, 1) & (1, -1, 1, 1, 1, 0) \\
(2, 0, 1, 2, 0, 1) & (2, 0, 1, 1, 1, 1) & (2, 0, 1, 1, 1, 0) \\
(1, 1, 1, 2, 0, 1) & (1, 1, 1, 1, 1, 1) & (1, 1, 1, 1, 1, 0) \\
(1, 1, 0, 2, 0, 1) & (1, 1, 0, 1, 1, 1) & (1, 1, 0, 1, 1, 0)
\end{pmatrix} \quad (3.8)$$

Suppose an S -compatible adjacency on $MSC_{18} \times MSC_{18}$, $k \in \{728 := k(6, 6), 664 := k(5, 6), 472 := k(4, 6)\}$ (see (2.3)). Then we have a contradiction to Definition 3. More precisely, let us now examine if $MSC_{18} \times MSC_{18}$ has an S -compatible adjacency, $k \in \{728, 664, 472\}$. To be specific, consider two points $a_{33} := (1, -1, 1, 1, -1, 1)$ and $a_{13} := (0, 0, 0, 1, -1, 1)$ in (3.8) which are k -adjacent, $k \in \{728, 664, 472\}$. Then we observe that they does not satisfy Definition 3. More precisely, while $a_{13} \in N_k(a_{33}, 1)$, $c_0 := (0, 0, 0) \notin N_{18}(c_2, 1)$, where $c_2 := (1, -1, 1)$, $a_{13} := (c_0, c_2)$, $a_{33} := (c_2, c_2)$.

Therefore, it turns out that $MSC_{18} \times MSC_{18}$ has no S -compatible adjacency, $k \in \{728, 664, 472\}$.

By a method similar to that of the proof above for $MSC_{18} \times MSC_{18}$, both $SC_4^{2,8} \times SC_8^{2,6}$ and $SC_4^{2,8} \times SC_4^{2,8}$ can be proved not to have their S -compatible adjacencies. $\square$

In view of Definition 3, Remark 3.1 and Example 3.2, we observe the following:

Remark 3.3. In view of Definition 3 and the above report, for the $k_i := k(m_i, n_i)$ -adjacency of (X, k_1) and (Y, k_2) , $i \in \{1, 2\}$ we can observe that an S -compatible $k := k(m, n_1 + n_2)$ -adjacency of the Cartesian product $X \times Y$ should at least have the number $m \in [m_1 + m_2, n_1 + n_2]_{\mathbf{Z}}$ [6]. However, not every number $m \in [m_1 + m_2, n_1 + n_2]_{\mathbf{Z}}$ formulates an S -compatible $k := k(m, n_1 + n_2)$ -adjacency on the digital product. While this study was partially done in [6], we should study the property more precisely (see Example 3.2). Namely, for $SC_{k_i}^{n_i, l_i}$, $i \in \{1, 2\}$ the existence

of an S -compatible $k := k(m, n_1 + n_2)$ -adjacency on $SC_{k_1}^{n_1, l_1} \times SC_{k_2}^{n_2, l_2}$ depends on the situation (see Theorem 3.5).

In view of Definitions 2 and 3, we obtain the following:

Proposition 3.4. *For a digital product a normal adjacency is equivalent to an S -compatible adjacency.*

Proof. In view of the notions of $N_{k_1}(x, 1)$, $N_{k_2}(y, 1)$ and $N_k((x, y), 1)$, we can easily see that the property of Definition 2 implies that of Definition 3. Conversely, the property of Definition 3 clearly implies that of Definition 2. $\square$

[Merits of an S -compatible adjacency] Comparing the normal adjacency of a digital product in Definition 2 with an S -compatible adjacency in Definition 3, we can observe some utility at the latter because it uses a digital k -neighborhood. Thus, this kind of representation of the original normal adjacency of a digital product can be very helpful to study product properties of digital topological property.

Motivated by this review above and Remark 2.1, we obtain the following two cases:

Theorem 3.5. *Consider $SC_{k_i}^{n_i, l_i}$, $n_i \geq 2$, where $k_i := k(m_i, n_i)$ in (2.2) and $i \in \{1, 2\}$.*

(1) *Let $k_1 := k_1(m_1, n_1)$ be equal to $3^{n_1} - 1$ (or $m_1 \notin [1, n_1 - 1]_{\mathbf{Z}}$) and let the number m_2 taken from the $k_2(m_2, n_2)$ -adjacency be equal to $n_2 - 1$ instead of n_2 . Further, assume that for each element $x_i \in SC_{k_2}^{n_2, l_2} := (x_i)_{i \in [0, l_2 - 1]_{\mathbf{Z}}}$, $n_2 \geq 3$ the set $N_{3^{n_2} - 1}^*(x_i) \cap SC_{k_2}^{n_2, l_2}$ has the only three points $x_{i-1(\text{mod } l_2)}$, x_i and $x_{i+1(\text{mod } l_2)}$. Then the product $SC_{k_1}^{n_1, l_1} \times SC_{k_2}^{n_2, l_2}$ has an S -compatible $k(m_0, n_1 + n_2)$ -adjacency, where $m_0 \in [n_1 + n_2 - 1, n_1 + n_2]_{\mathbf{Z}}$.*

(2) *Let $k_i := k_i(m_i, n_i)$ be not equal to $3^{n_i} - 1$, $i \in \{1, 2\}$. Assume that for each element $x_i \in SC_{k_i}^{n_i, l_i} := (x_i)_{i \in [0, l_i - 1]_{\mathbf{Z}}}$, $n_i \geq 3$ the set $N_{3^{n_i} - 1}^*(x_i) \cap SC_{k_i}^{n_i, l_i}$ has the only three points $x_{i-1(\text{mod } l_i)}$, x_i and $x_{i+1(\text{mod } l_i)}$. If the number m_i taken from the $k_i := k(m_i, n_i)$ -adjacency is equal to $n_i - 1$ instead of n_i , $i \in \{1, 2\}$, then the product $SC_{k_1}^{n_1, l_1} \times SC_{k_2}^{n_2, l_2}$ has an S -compatible $k(m_0, n_1 + n_2)$ -adjacency, where $m_0 \in [n_1 + n_2 - 2, n_1 + n_2]_{\mathbf{Z}}$.*

Before proving this theorem, we need to recall that in general, for $x_i \in SC_k^{n, l} := (x_i)_{i \in [0, l - 1]_{\mathbf{Z}}}$ the set $N_{3^n - 1}^*(x_i) \cap SC_k^{n, l}$ need not have the only three points $x_{i-1(\text{mod } l)}$, x_i and $x_{i+1(\text{mod } l)}$. For instance, we can

consider the sets $SC_4^{2,8}$, MSC_{18} and so forth. This observation plays an important role in proving this theorem (see Theorem 3.5 (1) and (2)). Besides, if $k_i = 3^{n_i} - 1$ (or $m_i \notin [1, n_i - 1]_{\mathbf{Z}}$), then the product $SC_{k_1}^{n_1, l_1} \times SC_{k_2}^{n_2, l_2}$ has an S -compatible k -adjacency, where $k := k(m_0, n_1 + n_2) = 3^{n_1 + n_2} - 1$ (or $m_0 \notin [1, n_1 + n_2 - 1]_{\mathbf{Z}}$). The converse holds. This case was already discussed in [6]. More precisely, assume the hypothesis that $k_i := k_i(m_i, n_i)$ is equal to $3^{n_i} - 1$ (or $m_i \notin [1, n_i - 1]_{\mathbf{Z}}$), $i \in \{1, 2\}$. If $k_i = 3^{n_i} - 1$ (or $m_i \notin [1, n_i - 1]_{\mathbf{Z}}$), then in terms of Definition 3 we observe that the product $SC_{k_1}^{n_1, l_1} \times SC_{k_2}^{n_2, l_2}$ has an S -compatible $k := k(m_0, n_1 + n_2) = (3^{n_1 + n_2} - 1)$ -adjacency, where $m_0 = n_1 + n_2$ (or $m_0 \notin [1, n_1 + n_2 - 1]_{\mathbf{Z}}$).

Conversely, it is clearly that the given product $SC_{k_1}^{n_1, l_1} \times SC_{k_2}^{n_2, l_2}$ has an S -compatible $k := k(m_0, n_1 + n_2) = (3^{n_1 + n_2} - 1)$ -adjacency, where $m_0 = n_1 + n_2$ (or $m_0 \notin [1, n_1 + n_2 - 1]_{\mathbf{Z}}$), then in terms of Definition 3 we easily obtain that $k_i = 3^{n_i} - 1$ (or $m_i \notin [1, n_i - 1]_{\mathbf{Z}}$).

Proof. (1) Before proving this theorem, we need to explain the phrase in (1) such as the hypothesis of $SC_{k_2}^{n_2, l_2} := (x_i)_{i \in [0, l_2 - 1]_{\mathbf{Z}}}$: for each element $x_i \in SC_{k_2}^{n_2, l_2} := (x_i)_{i \in [0, l_2 - 1]_{\mathbf{Z}}}$ the set $N_{3^{n_2} - 1}^*(x_i) \cap SC_{k_2}^{n_2, l_2}$ has the only three points $x_{i-1(\text{mod } l_2)}$, x_i and $x_{i+1(\text{mod } l_2)}$. For instance, consider $SC_{18}^{3,6}$ in Figure 1 as an example. Namely, we can obtain that for each element $x_i \in SC_{18}^{3,6}$ the set $N_{26}^*(x_i) \cap SC_{18}^{3,6}$ has the only three points $x_{i-1(\text{mod } 6)}$, x_i and $x_{i+1(\text{mod } 6)}$. However, MSC_{18} does not satisfy this property. Thus, without this hypothesis, the theorem is not valid. To be specific, consider the following product $SC_8^{2,4} \times MSC_{18} := (e_{ij})_{(i,j) \in [1,4]_{\mathbf{Z}} \times [1,6]_{\mathbf{Z}}} \subset \mathbf{Z}^5$ from the adjacency graphs in Figure 1, where $SC_8^{2,4} := (y_j)_{j \in [0,3]_{\mathbf{Z}}}$ and $MSC_{18} := (c_i)_{i \in [0,5]_{\mathbf{Z}}}$ and for each $(i, j) \in [1, 4]_{\mathbf{Z}} \times [1, 6]_{\mathbf{Z}}$ we use the notation $e_{ji} := (y_{j-1}, c_{i-1})$. Then we can observe that $SC_8^{2,4} \times MSC_{18}$ has no S -compatible k -adjacency because for each point c_i of $MSC_{18} := (c_i)_{i \in [0,5]_{\mathbf{Z}}}$ the set $N_{26}^*(c_i) \cap MSC_{18}$ has another points except the three points $c_{i-1(\text{mod } 6)}$, c_i and $c_{i+1(\text{mod } 6)}$, e.g. see the points c_0, c_1, c_2 of MSC_{18} in Figure 1.

By Remark 3.3, let us now examine if the number $m \in [4, 5]_{\mathbf{Z}}$ formulates an S -compatible $k(m, 5)$ -adjacency of $SC_8^{2,4} \times MSC_{18} \subset \mathbf{Z}^5$. First, take the number 4. Then we easily see that the number 4 is not suitable for establishing an S -compatible $k(4, 5)$ -adjacency of $SC_8^{2,4} \times MSC_{18} := (e_{ij})_{(i,j) \in [1,4]_{\mathbf{Z}} \times [1,6]_{\mathbf{Z}}}$, where for $(i, j) \in [1, 4]_{\mathbf{Z}} \times [1, 6]_{\mathbf{Z}}$ $e_{ij} = (y_{i-1}, c_{j-1}) \in SC_8^{2,4} \times MSC_{18}$. More precisely, consider the two points

$e_{23} := (1, 1, 1, -1, 1)$ and $e_{32} := (2, 0, 1, -1, 0)$ taken from the given k -curves in (3.1).

Even though the first two coordinates $(2, 0)$ and $(1, 1)$ from the given two points e_{32} and e_{24} are 8-adjacent in $SC_8^{2,4}$, the points $e_{32} := (2, 0, 1, -1, 0)$ and $e_{24} := (1, 1, 2, 0, 1)$ are $k(5, 5) := 242$ -adjacent instead of $k(4, 5) := 210$ -adjacent. Thus the $k(4, 5) := 210$ -adjacency is not suitable for an S -compatible adjacency of $SC_8^{2,4} \times MSC_{18}$.

Similarly, using the similar method shown above, we can observe that the $k(5, 5)$ -adjacency does not satisfy Definition 3 either.

Under the hypothesis, let us now prove the assertion. Since $k_1 := k_1(m_1, n_1)$ is equal to $3^{n_1} - 1$ (or $m_1 \notin [1, n_1 - 1]_{\mathbf{Z}}$), if $m_2 = n_2 - 1$, in terms of Definition 3, then we obtain that the $k(m_0, n_1 + n_2)$ -adjacency of $SC_{k_1}^{m_1, l_1} \times SC_{k_2}^{m_2, l_2}$ is an S -compatible adjacency and $m_0 \in [n_1 + n_2 - 1, n_1 + n_2]_{\mathbf{Z}}$.

For instance, both $SC_8^{2,4} \times SC_{18}^{3,6}$ and $SC_8^{2,6} \times SC_{18}^{3,6}$ have an S -compatible k -adjacency, where $k \in \{k(5, 6) := 664, k(6, 6) := 728\}$.

(2) Before proving this theorem, we need to explain the phrase in (2) such as the hypothesis that for each $x_i \in SC_{k_i}^{m_i, l_i}$ the set $N_{3^{n_i-1}}^*(x_i) \cap SC_{k_i}^{m_i, l_i}$ has the only three points $x_{i-1(mod l_i)}$, x_i and $x_{i+1(mod l_i)}$ is also required. To be specific, consider the following digital product $MSC_{18} \times MSC_{18} := (a_{ij})_{(i,j) \in [1,6]_{\mathbf{Z}} \times [1,6]_{\mathbf{Z}}} \subset \mathbf{Z}^6$ (see the matrix of (3.8)), where $MSC_{18} := (c_i)_{i \in [0,5]_{\mathbf{Z}}}$ in Figure 1 and for $(i, j) \in [1, 6]_{\mathbf{Z}} \times [1, 6]_{\mathbf{Z}}$ $a_{ij} := (c_{i-1}, c_{j-1})$. Then we can observe that any k -adjacency of $\mathbf{Z}^6$ is not an S -compatible adjacency of $MSC_{18} \times MSC_{18}$ because $MSC_{18} := (c_i)_{i \in [0,5]_{\mathbf{Z}}}$ does not satisfy the hypothesis. Namely, for some element $c_i \in MSC_{18}$ the set $N_{26}^*(c_i) \cap MSC_{18}$ has another points except the three points $c_{i-1(mod 6)}$, c_i and $c_{i+1(mod 6)}$ (see the points c_1 and c_3 of MSC_{18} in Figure 1). By Remark 3.3, let us now examine if the number $m \in [4, 6]_{\mathbf{Z}}$ formulates an S -compatible $k(m, 6)$ -adjacency of $MSC_{18} \times MSC_{18}$. First, take the number 4. Then we easily see that $k(4, 6)$ -adjacency is not an S -compatible adjacency of $MSC_{18} \times MSC_{18}$ (see (3.8)). More precisely, consider the two points $a_{33} := (1, -1, 1, 1, -1, 1)$ and $a_{41} := (2, 0, 1, 0, 0, 0)$. Even though the first three coordinates, *i.e.* $(1, -1, 1)$ and $(2, 0, 1)$ from the two points a_{33} and a_{41} are 18 := $k(2, 3)$ -adjacent in MSC_{18} , the points a_{33} and a_{41} are $k(5, 6) := 664$ -adjacent instead of $k(4, 6) := 472$ -adjacent in $MSC_{18} \times MSC_{18}$.

Similarly, for the number $m \in [5, 6]_{\mathbf{Z}}$. Then, using the same method used in the proof of the case of $k(4, 6)$ -adjacency above, we easily see that $k(m, 6)$ -adjacency does not satisfy Definition 3 either.

Let us now prove the assertion. Owing to the hypothesis of $m_i = n_i - 1, n_i \geq 3, i \in \{1, 2\}$, we obtain that $k_i := k_i(n_i - 1, n_i)$ is equal to $3^{n_i} - 2^{n_i} - 1$ [18]. With the hypothesis, if $m_i = n_i - 1$, then the $k = (m_0, n_1 + n_2)$ -adjacency of $SC_{k_1}^{n_1, l_1} \times SC_{k_2}^{n_2, l_2}$ is an S -compatible adjacency, where $m_0 \in [n_1 + n_2 - 2, n_1 + n_2]_{\mathbf{Z}}$. Namely, if $m_i = n_i - 1$, then the adjacency $k = 3^{n_1+n_2} - 2^{n_1+n_2} - (n_1 + n_2)2^{n_1+n_2-1} - 1$ [18] is an S -compatible adjacency of $SC_{k_1}^{n_1, l_1} \times SC_{k_2}^{n_2, l_2} \subset \mathbf{Z}^{n_1+n_2}$.

More precisely, for each element $c_{ij} := (x_i, x_j) \in SC_{k_1}^{n_1, l_1} \times SC_{k_2}^{n_2, l_2}$ take the point $d_{ij} \in N_{3^{n_1+n_2}-1}(c_{ij}, 1) \setminus \{c_{ij}\} \subset SC_{k_1}^{n_1, l_1} \times SC_{k_2}^{n_2, l_2}$. Due to the hypothesis that for each element $x_i \in SC_{k_i}^{n_i, l_i}$ the set $N_{3^{n_i}-1}^*(x_i) \cap SC_{k_i}^{n_i, l_i}$ has the only three points $x_{i-1(\text{mod } l_i)}, x_i$ and $x_{i+1(\text{mod } l_i)}$, the two points d_{ij} and c_{ij} are also $k(m_0, n_1 + n_2)$ -adjacent, where $m_0 \in [n_1 + n_2 - 2, n_1 + n_2]_{\mathbf{Z}}$.

For instance, we observe that $SC_{18}^{3,6} \times SC_{18}^{3,6} \subset \mathbf{Z}^6$ has an S -compatible $k := k(m, 6)$ -adjacency, where $m \in \{4, 5, 6\}$. To be specific, let us consider the following digital product of $SC_{18}^{3,6}$ in Figure 1, *i.e.* $SC_{18}^{3,6} \times SC_{18}^{3,6} := (d_{ij})_{(i,j) \in [1,6]_{\mathbf{Z}} \times [1,6]_{\mathbf{Z}}}$ can be represented via the following matrix (see (3.9)), where for each $(i, j) \in [1, 6]_{\mathbf{Z}} \times [1, 6]_{\mathbf{Z}}$ $d_{ij} := (c_{i+1}, c_{j+1})$ and $SC_{18}^{3,6} := (c_i)_{i \in [0,5]_{\mathbf{Z}}}$ in Figure 1.

$$\begin{pmatrix} (0, 0, 0, 0, 0, 0) & (0, 0, 0, 1, 0, 1) & (0, 0, 0, 1, 1, 2) \\ (1, 0, 1, 0, 0, 0) & (1, 0, 1, 1, 0, 1) & (1, 0, 1, 1, 1, 2) \\ (1, 1, 2, 0, 0, 0) & (1, 1, 2, 1, 0, 1) & (1, 1, 2, 1, 1, 2) \\ (0, 2, 2, 0, 0, 0) & (0, 2, 2, 1, 0, 1) & (0, 2, 2, 1, 1, 2) \\ (-1, 1, 2, 0, 0, 0) & (-1, 1, 2, 1, 0, 1) & (-1, 1, 2, 1, 1, 2) \\ (-1, 0, 1, 0, 0, 0) & (-1, 0, 1, 1, 0, 1) & (-1, 0, 1, 1, 1, 2) \end{pmatrix} \quad (3.9)$$

$$\begin{pmatrix} (0, 0, 0, 0, 2, 2) & (0, 0, 0, -1, 1, 2) & (0, 0, 0, -1, 0, 1) \\ (1, 0, 1, 0, 2, 2) & (1, 0, 1, -1, 1, 2) & (1, 0, 1, -1, 0, 1) \\ (1, 1, 2, 0, 2, 2) & (1, 1, 2, -1, 1, 2) & (1, 1, 2, -1, 0, 1) \\ (0, 2, 2, 0, 2, 2) & (0, 2, 2, -1, 1, 2) & (0, 2, 2, -1, 0, 1) \\ (-1, 1, 2, 0, 2, 2) & (-1, 1, 2, -1, 1, 2) & (-1, 1, 2, -1, 0, 1) \\ (-1, 0, 1, 0, 2, 2) & (-1, 0, 1, -1, 1, 2) & (-1, 0, 1, -1, 0, 1) \end{pmatrix}$$

Then we can observe that $SC_{18}^{3,6} \times SC_{18}^{3,6}$ has an S -compatible $k(4, 6) := 472$ -adjacency. Besides, using the matrix (3.9), we can observe that both $k(5, 6) := 664$ and $k(6, 6) := 728$ -adjacency are also S -compatible adjacencies on $SC_{18}^{3,6} \times SC_{18}^{3,6}$, which supports the assertion. $\square$

Since both $SC_8^{2,6}$ and $SC_{18}^{3,6}$ satisfy the hypothesis of Theorem 3.5, we obtain the following:

Example 3.6. *We obtain the following S -compatible adjacencies for the given digital products.*

- (1) $(SC_{18}^{3,6} \times SC_8^{2,4}, k), k \in \{210 := k(4, 5), 242 := k(5, 5)\}$ and
- (2) $(SC_8^{2,6} \times SC_{18}^{3,6}, k), k \in \{210 := k(4, 5), 242 := k(5, 5)\}$.

By Definitions 1 and 3, we obtain the following:

Corollary 3.7. *Assume an S -compatible adjacency on $X_1 \times X_2$. Then we obtain that the natural projection map $P_i : X_1 \times X_2 \rightarrow X_i$ is a (k, k_i) -continuous map, $i \in \{1, 2\}$.*

4. Summary

To study a digital version of the classical graph product isomorphism problem, the paper have studied an S -compatible adjacency on a digital product. In particular, it turns out that an S -compatible adjacency can be a suitable condition addressing a product problem of two adjacency graph isomorphisms. Further, an S -compatible adjacency on a digital product leads to the multiplicative property of the digital k -fundamental group. Besides, these compatible k -adjacency relations on a digital product can be used for finding product properties in adjacency graph theory and digital geometry.

As a further work, using the an S -compatible adjacency, we can study multiplicative properties of a digital topological k -number [3] and a digital k -linking number [2, 3], etc.

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