GRAPHICAL ARRANGEMENTS OF COMPRESSED GRAPHS

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Abstract. We show that if a graph G is compressed, then the proper part of the intersection poset of the corresponding graphical arrangement A_G has the homotopy type of a wedge of spheres. Furthermore, we also indicate the number of spheres in the wedge, based on the number of adjacent edges of vertices in G .

1. Introduction

A *hyperplane arrangement* $\mathcal{A}$ is a finite set of affine hyperplanes in some vector space $V \cong K^n$, where K is a field. There are a lot of examples of hyperplane arrangements in K^n such as braid arrangements, Shi arrangements, and Catalan arrangements.

Graphical arrangements are special hyperplane arrangements obtained from finite simple undirected graphs. They are studied in the connection with graph theory [5, 9]. Many of the invariants associated with a graphical arrangement can be computed directly from the corresponding graph. For example, the characteristic polynomial of a graphical arrangement is the chromatic polynomial of the corresponding graph (see [8, section 2.4], [10]). The braid arrangement B_n is the graphical arrangement of the complete graph K_n . We also know that the graphical arrangement of a general graph G on vertex set $[n] := \{1, 2, \dots, n\}$ is a subarrangement of the braid arrangement B_n .

The *intersection poset* $L(\mathcal{A})$ of a hyperplane arrangement $\mathcal{A}$ is the set of all nonempty intersections of hyperplanes in $\mathcal{A}$, including V itself

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as the intersection over the empty set, ordered by reverse inclusion. The intersection poset is an important invariant of the arrangement. The number of regions and bounded regions of a real hyperplane arrangement is determined by its intersection poset [13] and the Betti numbers of the complement of a complex hyperplane arrangement can be expressed using the intersection lattice [7]. The topological property of the proper part of the intersection poset is related to its order complex, where the *order complex* $\Delta(P)$ of a poset P is the abstract simplicial complex whose vertices are all elements of P and whose simplices are all finite chains of P , including the empty chain. A property of a poset studied a lot is shellability [3, 4]. A shellable poset has the homotopy type of a wedge of spheres ([3, theorem 4.1]). One nice way to show the shellability of a poset is the lexicographic shellability [1, 2, 3, 4, 6]. This method also provides the number of spheres in the wedge by counting decreasing maximal chains in the intersection poset.

It is well-known that the intersection poset of the braid arrangement B_n is isomorphic to the partition lattice Π_n of the n -element set ordered by refinement [8, proposition 2.9]. Since Π_n is known to be EL-shellable [1, 11], one can compute the homotopy type of the proper part of the intersection poset of the braid arrangement B_n (see [12]). We will show this result again by using the number of decreasing maximal chains.

In this paper, we study the topological property of the graphical arrangement of a *compressed* graph ([4, p.3952]). It is known that compressed graphs are supersolvable, and the intersection poset of supersolvable graphs are *EL*-shellable ([9]). In this paper, we provide a new *EL*-shelling which can be easily obtained from graphs. Furthermore, we also describe the number of spheres in the wedge, based on adjacent edges of vertices in G .

The rest of the paper is organized as follows. Section 2 provides the basic notions and definitions which will be used throughout the paper. In Section 3, an edge-labeling is defined and shown to be an EL-labeling. The last section describes the number of spheres in the wedge using the number of decreasing chains. The enumerative results for special cases will be given as well.

2. Basic notions and definitions

Let G be a simple undirected graph on a vertex set $[n]$ and $E(G)$ denote the set of edges of G . We will use the notation ij for an edge of

G with two vertices i and j . Let $V = K^n$ be a vector space over a field K with coordinates $x_1, x_2, \dots, x_n$.

Definition 2.1. The *graphical arrangement* $\mathcal{A}_G$ in a vector space V is the hyperplane arrangement

$$\{x_i - x_j = 0 \mid ij \in E(G)\}$$

In the intersection poset of the graphical arrangement $\mathcal{A}_G$, we also label each hyperplane $x_i - x_j = 0$ by ij . There are two cases to label an intersection of two hyperplanes. If two hyperplanes are ij and kl , where $\{i, j\} \cap \{k, l\} = \emptyset$, then the common subspace is labeled by $ij - kl$. Otherwise, we may assume that $l = j$. Then the common subspace will be labeled by ijk . A general subspace in the intersection poset is denoted by $A = A_1 - A_2 - \dots - A_r$, where A_i , $i = 1, 2, \dots, r$, are the sets of indices of coordinates. We call A_i , $i = 1, 2, 3, \dots, r$, *blocks* of A , and those indices of coordinates *indices* of A . When we get the intersection of two subspaces $H = H_1 - H_2 - \dots - H_p$ and $K = K_1 - K_2 - \dots - K_q$, where H_i and K_i are blocks of H and K , we also have two cases to label their common subspace. If there is a common index in two blocks H_i of H and K_j of K , then in the common subspace we get the union of H_i and K_j as union of two sets. Otherwise, the common space is denoted by $H_1 - H_2 - \dots - H_p - K_1 - K_2 - \dots - K_q$. Furthermore, we say $K < H$ if each block K_j of K is contained in some block H_i of H .

Example 2.2. Consider a graph G_1 shown in Figure 1. The graphical arrangement $\mathcal{A}_G$ of G consists of hyperplanes $x_1 - x_3 = 0$, $x_1 - x_4 = 0$, and $x_2 - x_5 = 0$. In the intersection poset of $\mathcal{A}_G$, we label hyperplane $x_1 - x_3 = 0$ by 13, $x_1 - x_4 = 0$ by 14, and $x_2 - x_5 = 0$ by 25. Figure 1 shows the intersection poset P_{G_1} of $\mathcal{A}_G$.

Definition 2.3. A simple undirected graph G on a vertex set $[n]$ is *compressed* if $i < j < k$ and $ik \in E(G)$ imply that $ij \in E(G)$.

Example 2.4. Complete graphs K_n are compressed graphs. Paths P_n with the edges $\{i, i + 1\}$ for all $i = 1, 2, \dots, n - 1$ are compressed graphs. Furthermore, Figure 2 shows other examples of compressed and noncompressed graphs.

Definition 2.5. A *partially order set* P (or *poset*, for short) is a set together with a relation $\leq$ (or $\leq_P$) satisfying the following three axioms:

- For all $t \in P$, $t \leq t$.
- If $s \leq t$ and $t \leq s$, then $s = t$.
- If $s \leq t$ and $t \leq u$, then $s \leq u$.

FIGURE 1. A graph G_1 and the intersection poset P_{G_1} FIGURE 2. A compressed graph G_2 and a noncompressed graph G_3

For $s, t \in P$, the (closed) *interval* $[s, t] = \{u \in P : s \leq u \leq t\}$. We say that t *covers* s or s is *covered by* t (denote $s \lessdot t$), if $s < t$ and no element $u \in P$ satisfies $s < u < t$. P *has a* $\hat{0}$ if there exists an element $\hat{0} \in P$ such that $t \geq \hat{0}$ for all $t \in P$. Similarly, P *has a* $\hat{1}$ if there exists an element $\hat{1} \in P$ such that $t \leq \hat{1}$ for all $t \in P$. A poset is *bounded* if it contains both $\hat{0}$ and $\hat{1}$. The *Hasse diagram* of a finite poset P is the graph whose vertices are the elements of P , whose edges are the cover relations, and

such that if $s < t$ then t is drawn *above* s (i.e., with a higher vertical coordinate). See P_{G_1} as Figure 1 for the Hasse diagram of a poset. A *chain* is a poset in which any two elements are comparable. A subset C of a poset P is called a *chain* if C is a chain when regarded as a subposet of P . The chain C of P is called *maximal* if it is not contained in a larger chain of P . The *length* $l(C)$ of a finite chain is defined by $\|C\| - 1$. The *length* of a finite poset P is $l(P) := \max\{l(C) : C \text{ is a chain of } P\}$.

Definition 2.6. An *edge-labeling* of a poset P is a map $\lambda : \mathcal{E}(P) \rightarrow \Lambda$, where $\mathcal{E}(P)$ is the set of edges of the Hasse diagram of poset P and Λ is an ordered set. For an edge $\{s, t\} \in \mathcal{E}(P)$, we denote $\lambda(s, t)$ for $\lambda(\{s, t\})$.

Definition 2.7. An edge-labeling is called an *EL-labeling* (edge lexicographical labeling) if for every interval $[x, y]$ in poset P , there is a unique increasing maximal chain, which lexicographically precedes all other maximal chains of $[x, y]$.

Björner and Wachs [3] showed the following fundamental result about the EL-labeling.

Theorem 2.8. *If a bounded poset P admits an EL-labeling, then the order complex $\Delta(\overline{P})$ has the homotopy type of a wedge of spheres. Furthermore, the number of i -spheres is the number of decreasing maximal $(i + 2)$ -chains of P .*

3. The edge-labeling

In this section, we will define an edge-labeling of the intersection poset of the graphical arrangement for a compressed graph, and show that it is an EL-labeling.

Let P_G be the intersection poset of the graphical arrangement $\mathcal{A}_G$ of a compressed graph G . We now describe an edge-labeling $\lambda : \mathcal{E}(P_G) \rightarrow (\mathbb{Z}, \mathbb{Z})$, where $(\mathbb{Z}, \mathbb{Z})$ is a partially ordered set with the order defined by $(a, b) < (c, d)$ if $a < c$ or $a = c$ and $b < d$. We can identify $(a, 0) \in (\mathbb{Z}, \mathbb{Z})$ with $a \in \mathbb{Z}$.

Let H and K be two elements in P_G where H covers K . There are three kinds of coverings in P_G which are described in the following labeling rule:

Case 1: when $H = H_1 - \dots - H_i x - \dots - H_m$ and $K = H_1 - \dots - H_m$ for some $i (1 \leq i \leq m)$

$$\lambda(K, H) := \min\{(x, y) | xy \in E(G) \text{ and } y \in H_i\}$$

FIGURE 3. A compressed graph G_4 and the intersection poset P_{G_4}

Case 2: when $H = H_1 - \dots - H_m - xy$ and $K = H_1 - \dots - H_m$

$$\lambda(K, H) := \max\{x, y\}$$

Case 3: when $H = H_1 - \dots - H_i \cup H_j - \dots - H_m$ and $K = H_1 - \dots - H_i - \dots - H_j - \dots - H_m$ for some $i, j (1 \leq i < j \leq m)$

$$\lambda(K, H) := \min\{(u, v) | uv \in E(G) \text{ and } u \in H_i, v \in H_j \text{ or } v \in H_i, u \in H_j\}$$

Example 3.1. Figure 3 shows an example of compressed graph G_4 and the intersection poset P_{G_4} with some labeling values. In the poset P_{G_4} , 12 is covered by 123, and the labeling value $\lambda(12, 123)$ is given by Case 3. When we find the labeling value for the edge $\{K, H\}$ where $K = \hat{0}$, we use Case 3 to label that edge. And the last case is used to find the labeling value of the edge $\{123 - 45, 12345\}$ of the example.

Let M and N be two elements in the poset P_G such that $N < M$.

Lemma 3.2. *There exists an increasing maximal chain from N to M .*

Proof. Assume that $N = N_1 - N_2 - \dots - N_p$ and $M = M_1 - M_2 - \dots - M_q$. A covering element of N in $[N, M]$ must have one of the following forms:

- (Form 1) $N_1 - N_2 - \dots - N_i x - N_p$, or
- (Form 2) $N_1 - N_2 - \dots - N_p - xy$, or
- (Form 3) $N_1 - N_2 - \dots - N_i \cup N_j - \dots - N_p$.

Let $A_1 := N_1 - N_2 - \dots - N_i x - N_p$ where $\lambda(N, A_1)$ is minimal among all covering elements of N of the form 1. There is only one element A_1 of form 1 satisfying this condition.

Let $A_2 := N_1 - N_2 - \dots - N_p - xy (x < y)$, where $\lambda(N, A_2) = y$ is minimal among all covering elements of N of the form 2. Assume that there exists $A'_2 = N_1 - N_2 - \dots - N_p - x'y (x' < y)$, where $N \triangleleft A'_2 < M$ and $\lambda(N, A'_2) = \lambda(N, A_2) = y$. Since $x < y$ and $x' < y$, we have $x < x' < y$ or $x' < x < y$. Since G is compressed, there exists edge xx' . Therefore, we have another element $A''_2 = N_1 - N_2 - \dots - N_p - xx'$ satisfying $N \triangleleft A''_2 < M$. Since $\lambda(N, A''_2) = x$ or x' and both of these are less than y , this contradicts the condition of A_2 . So there exists only one element of form 2 that satisfies the condition.

Let $A_3 := N_1 - N_2 - \dots - N_i \cup N_j - \dots - N_p$ where $\lambda(N, A_3) = (u, v)$ is minimal among all covering elements of N of the form 3. We also have only one element A_3 of form 3 satisfying the above condition.

Choose the covering element N_1 of N so that

$$\lambda(N, N_1) = \min\{\lambda(N, A_1), \lambda(N, A_2), \lambda(N, A_3)\}.$$

Since $\lambda(N, A_1)$, $\lambda(N, A_2)$, and $\lambda(N, A_3)$ are pairwise distinct, we have only one choice of N_1 . We find the covering element of N_1 in the similar way with finding the covering element of N . Continue this process until we get

$$\mathcal{C} := N \triangleleft N_1 \triangleleft \dots \triangleleft N_\ell = M.$$

See Example 3.5 for the choosing.

Now we claim that $\mathcal{C}$ is an increasing chain.

Let A , B , and C be three elements in $\mathcal{C}$ satisfying $A \triangleleft B \triangleleft C$. It is enough to show that $\lambda(A, B) < \lambda(B, C)$. Assume that A has form $A = A_1 - A_2 - \dots - A_k$. The proof is divided into small cases based on the form of B and C . Since B is a covering element of A , there are three possible cases for B :

- If B is $A_1 - A_2 - \dots - A_i x - A_k$, then C can get forms
 - $A_1 - A_2 - \dots - A_i xy - A_k$, or
 - $A_1 - A_2 - \dots - A_i x - A_j y - A_k$, or
 - $A_1 - A_2 - \dots - A_i x - A_k - yz$, or

- $A_1 - A_2 - \dots - A_i x - A_j \cup A_l - A_k$, or
- $A_1 - A_2 - \dots - A_i x \cup A_j - A_k$.
- If B is $A_1 - A_2 - \dots - A_i - A_k - xy$, then C can get the following forms
 - $A_1 - A_2 - \dots - A_i - A_k - xy - zt$,
 - $A_1 - A_2 - \dots - A_i - A_k - xyz$,
 - $A_1 - A_2 - \dots - A_i z - A_k - xy$,
 - $A_1 - A_2 - \dots - A_i \cup A_j - A_k - xy$,
 - $A_1 - A_2 - \dots - A_i \cup xy - A_k$.
- For the final cases, when B is $A_1 - A_2 - \dots - A_i \cup A_j - A_k$, then C will have 5 forms. Those are
 - $A_1 - A_2 - \dots - A_i \cup A_j - A_l x - A_k$,
 - $A_1 - A_2 - \dots - A_i \cup A_j x - A_k$,
 - $A_1 - A_2 - \dots - A_i \cup A_j - \dots - A_h \cup A_l - A_k$,
 - $A_1 - A_2 - \dots - A_i \cup A_j \cup A_l - \dots - A_k$, and
 - $A_1 - A_2 - \dots - A_i \cup A_j - \dots - A_k - xy$.

In all cases, we find another covering element B_1 of A , and $B_1 < C$. It is obvious that $B_1 < M$. Then find $\lambda(A, B_1)$. We choose B_1 such that $\lambda(A, B_1) \leq \lambda(B, C)$, or the first component of $\lambda(A, B_1)$ equals the first component of $\lambda(B, C)$. Since B covers A in $\mathcal{C}$ and by the rule of choosing, we have $\lambda(A, B) < \lambda(A, B_1)$. Therefore, $\lambda(A, B) < \lambda(B, C)$. $\square$

We proved that $\mathcal{C}$ is an increasing chain. To prove λ is an EL-labeling, we also need to show that $\mathcal{C}$ is the unique increasing maximal chain from N to M .

Lemma 3.3. *The increasing maximal chain from N to M is unique.*

Proof. The main idea of the proof is getting a unique covering element N_1 of N in $\mathcal{C}$. If N'_1 is another covering element of N in $[N, M]$, we try to show that the maximal chain $N < N'_1 < \dots < M$ is not increasing by indicating one labeling value in the interval $[N'_1, M]$ which is less than $\lambda(N, N'_1)$. That indication depends on the form of N_1 and N'_1 . There are nine cases to consider and we omit straightforward proof. $\square$

Theorem 3.4. *The labeling λ is an EL-labeling.*

Proof. By Lemma 3.2 and Lemma 3.3, we can see there is only one increasing maximal chain $\mathcal{C}$ from N to M . By the rule we choose an increasing chain, it is easy to see that $\mathcal{C}$ lexicographically precedes all other maximal chains of $[N, M]$. Therefore, the labeling λ is an EL-labeling. $\square$

Example 3.5. Consider the intersection poset P_{G_4} of G_4 in Figure 3. We use the above way to find an increasing chain $\mathcal{C}$ from 12 to 12345. Firstly, we find the covering element of 12. There are three elements in P_G covering 12, and the labeling value $\lambda(12, 123) = (3, 1)$, $\lambda(12, 124) = (4, 1)$ and $\lambda(12, 12 - 45) = 5$. Since $\lambda(12, 123)$ is the smallest value, we choose 123 for the covering element of 12 in $\mathcal{C}$. Next, we find the covering element of 123 in $\mathcal{C}$. We see that 123 is covered by 1234 and 123 - 45, and the labeling value $\lambda(123, 1234) = (4, 1)$, $\lambda(123, 123 - 45) = 5$. We must choose 1234 for the covering of 123 in $\mathcal{C}$. Finally, we choose 12345 for the covering of 1234 in $\mathcal{C}$, $\lambda(1234, 12345) = (5, 4)$. Obviously, $\mathcal{C}$ is an increasing chain.

We proved that the edge-labeling above is an *EL*-labeling. Theorem 2.8 and Theorem 3.4 imply the following corollary.

Corollary 3.6. *The order complex $\Delta(\overline{P_G})$ has the homotopy type of a wedge of spheres.*

Since P_G is the intersection poset of a hyperplane arrangement, all maximal chains in P_G have the same cardinality [8, Lemma 2.3]. Therefore, the decreasing maximal chains of P_G have the same length, and hence $\Delta(\overline{P_G})$ has the homotopy type of a wedge of spheres of the same dimension. That dimension is also the length of $\overline{P_G}$ or $l(P_G) - 2$ where $l(P_G)$ is the length of P_G . Let $c(G)$ be the number of connected components in G . It is not difficult to see that the length of P_G is equal to $n - c(G)$. Therefore, the dimension of spheres in the wedge is equal to $n - c(G) - 2$.

We already showed the existence of an increasing maximal chain in a closed interval $[N, M]$. Now we consider the existence of a decreasing maximal chain in that interval.

Lemma 3.7. *There is a decreasing maximal chain from N to M .*

Proof. Let $\mathcal{D}$ be the decreasing maximal chain we need to find. We choose the covering element X_1 of N in $\mathcal{D}$ if

$$\lambda(\{N, X_1\}) = \max\{\lambda(\{N, X\}) | N \lessdot X < M\}.$$

We continue to choose X_2 for covering element of X_1 in $\mathcal{D}$ if

$$\lambda(\{X_1, X_2\}) = \max\{\lambda(\{X_1, X\}) | X_1 \lessdot X < M\}.$$

Use this way to find other elements in the maximal chain $\mathcal{D}$ until we get M .

Now we claim that $\mathcal{D}$ is a decreasing chain.

Let A , B , and C be three elements in the maximal chain $\mathcal{D}$ such that $A \triangleleft B \triangleleft C$. There are fifteen cases to consider which are shown in the proof of Lemma 3.2. In all cases, the common way to prove $\lambda(\{A, B\}) > \lambda(\{B, C\})$ is also similar to the way used to prove the chain $\mathcal{C}$ is increasing. But the difference is choosing B_1 such that $\lambda(\{A, B_1\}) \geq \lambda(\{B, C\})$ or the first component of $\lambda(\{A, B_1\})$ equals the first component of $\lambda(\{B, C\})$. Since B covers A in $\mathcal{D}$ and by the rule of choosing, we have $\lambda(\{A, B\}) > \lambda(\{A, B_1\})$. Therefore, $\lambda(\{A, B\}) > \lambda(\{B, C\})$. $\square$

Remark 3.8. It is easy to see that a compressed graph is a chordal graph and Stanley mentions in [9] that a chordal graph is a supersolvable graph. Therefore, the intersection poset of the graphical arrangement of a chordal graph is a supersolvable lattice. Stanley also gives an edge-labeling on supersolvable lattice, and proves that it is an *EL*-labeling (see [9]). The labeling introduced in this paper can be easily obtained from the graph and is different from the labeling by Stanley. For example, we consider two labelings for the intersection poset of the graphical arrangement of the complete graph K_4 so that their unique increasing chains are the same. Then $\hat{0} < 34 < 134 < 1234$ is a decreasing chain in Stanley's labelling while it is not decreasing in this paper's labelling.

4. The number of decreasing maximal chains

In this section, we will find the number of decreasing maximal chains of P_G , where G is a compressed graph.

Let $\mathcal{M} := \hat{0} \triangleleft X_1 \triangleleft X_2 \triangleleft \cdots \triangleleft X_l \triangleleft \hat{1}$ be a maximal chain in $[\hat{0}, \hat{1}]$. We associate $\mathcal{M}$ with the ordered string

$$\lambda(\mathcal{M}) = [\lambda(\hat{0}, X_1), \lambda(X_1, X_2), \dots, \lambda(X_l, \hat{1})].$$

Since P_G is the intersection poset of a hyperplane arrangement, the labeling values $\lambda(\hat{0}, X_1), \lambda(X_1, X_2), \dots, \lambda(X_l, \hat{1})$ are corresponding to hyperplanes or edges of G . The labeling values are based on the corresponding hyperplanes or edges of G . If the intersections of a subspace S and different hyperplanes give us the same common subspace T , then the labeling value $\lambda(S, T)$ is found by using the hyperplane with smaller indices. For example, consider the maximal chain $\mathcal{M} = \hat{0} \triangleleft 12 \triangleleft 123 \triangleleft 1234 \triangleleft 12345 = \hat{1}$ in Figure 3. We associate $\mathcal{M}$ with the ordered string of labeling values $\lambda(\mathcal{M}) = [2, (3, 1), (4, 1), (5, 4)]$. Whenever we get an intersection of a subspace and a new hyperplane, we also

get a labeling value. So we can see that the labeling value 2 is corresponding to the hyperplane $x_1 - x_2 = 0$ or the edge 12. Similarly, the labeling values (3, 1), (4, 1), and (5, 4) are corresponding to the edges 13, 14, and 45 respectively. We can associate the string of labeling values $\lambda(\mathcal{M}) = [2, (3, 1), (4, 1), (5, 4)]$ with the string of edges [12, 13, 14, 45]. In other words, we can associate the maximal chain $\mathcal{M}$ with the string of edges [12, 13, 14, 45].

The following lemma, whose straightforward proof is omitted, gives the condition when an edge is corresponding to a labeling value.

Lemma 4.1. *Let X be a string of edges of a compressed graph G , and x be an edge in X . Then x is corresponding to a labeling value (or used to find a labeling value) if and only if both of the following situations do not happen:*

- I. *Both vertices of x already appear in X before the edge x and there is another edge y in G satisfying*
 - *there exists a cycle containing x and y such that all the other edges appear in X before x ,*
 - *$(e, f) < (a, b)$ if $x = ab(a < b)$ and $y = ef(e < f)$.*
- II. *One of the vertices of x , say a , already appear in X before x and there exists another edge $y = be$ in G containing the other vertex b of x satisfying*
 - *there exists a cycle containing x and y such that all the other edges appear in X before x ,*
 - *$e < a$.*

A *spanning forest* of a graph G is a union of spanning trees in all connected components of G .

Definition 4.2. Let X be a string of edges of a graph G . We say X is *associated with a maximal chain* if

- The edges in X form a spanning forest of G .
- Each edge in X gives us a labeling value.
- The number of edges in the string is the length of poset P_G .

Furthermore, X is *associated with a decreasing maximal chain* if the labeling values corresponding to the edges in X are decreasing.

Given an arbitrary string of edges of length $l(P_G)$, we cannot say that it is associated with a maximal chain of P_G . Also, if we change the order of edges in the string associated with a maximal chain, it can be not related to any maximal one. On the other hand, we also can rearrange edges in a string of edges to associate it with a maximal chain

FIGURE 4. A compressed graph G_5

or a decreasing maximal chain. The following examples will show us this issue.

Example 4.3. See Figure 4 for another compressed graph. Let $X = [16, 23, 24, 25, 12]$ be a string of edges. The edges in X form a spanning forest of G_5 . Furthermore, X is associated with the string of labeling values $[6, 3, (4, 2), (5, 2), (1, 2)]$ or the maximal chain $\hat{0} < 16 < 16 - 23 < 16 - 234 < 16 - 2345 < 123456$. Now we change the order of edges in X to get the string $X' = [16, 12, 23, 24, 25]$. In order to see whether X' is associated with a maximal chain or not, we have to check whether all the edges are used to find labeling values. Firstly, edge 16 is used to find the labeling value when we get the intersection of hyperplane 16 and $\mathbb{R}^6$, and the corresponding labeling value $\lambda(\hat{0}, 16) = 6$. Since the intersection of subspace 16 and hyperplane 12 gives us the subspace 126 and the labeling value $\lambda(16, 126) = (2, 1)$, the edge 12 is also used to find a labeling value. Now we consider the intersection of 126 and hyperplane 23. This intersection gives us subspace 1236, and the labeling value $\lambda(126, 1236) = (3, 1)$. Therefore, the edge 23 is not used to find a labeling value in this case. Hence X' is not associated with a maximal chain.

The following lemma provides some conditions for a string of edges to be associated with a decreasing maximal chain.

Lemma 4.4. *Let $X = [x_1, x_2, \dots, x_l]$ be a string of edges of a compressed graph G where l is the length of the poset P_G . Then, the edges in X form a spanning forest of G and there do not exist any two edges ab and cb in X where $a, c < b$ if and only if the edges in X can be rearranged to get a new string X' associated with a decreasing maximal chain in $[\hat{0}, \hat{1}]$. Furthermore, X' is unique.*

Proof. Let $X = [x_1, x_2, \dots, x_l]$ be a string of edges satisfying the given conditions. Since the edges in X form a spanning forest of G , we can find a path using edges in X to connect any two vertices in a connected component of G .

Suppose there exist edges ab and cb in X where $a, c < b$. Since G is compressed, G must have the edge ac . Since any labeling value obtained from ac is less than any labeling value got by edges ab or cb , the latter of ab and cb in X does not correspond to a labeling value. Thus there is no rearrangement of edges in X associated with a decreasing maximal chain.

Now assume that X does not have edges ab and cb such that $a, c < b$. Then we can rearrange edges in X to get a new string X' that is associated with a decreasing maximal chain by the following rule.

Let m be the largest non-isolated vertex of G . By the given condition of X , m appears in only one edge, say $xm(x < m)$. We choose xm for the first position in the string X' .

Assume that we already have edges $x_1, x_2, \dots, x_k$ in X' . Then there are three cases:

- (1) If $\{\text{vertices of } x_1, x_2, \dots, x_k\} \neq [n]$ and there is an edge in X both of whose vertices appear in $x_1, \dots, x_k$ and greater than $\max([n] \setminus \{\text{vertices of } x_1, x_2, \dots, x_k\})$, then we add an edge $x = ab(a < b)$ where (a, b) is maximal among those edges.
- (2) If $\{\text{vertices of } x_1, x_2, \dots, x_k\} \neq [n]$ and there does not exist any edge in X such that both of whose vertices appear in $x_1, \dots, x_k$ and greater than $\max([n] \setminus \{\text{vertices of } x_1, x_2, \dots, x_k\})$, then we add an edge $x = ab$ where b is the maximal new vertex and a is maximal among the vertices adjacent to b .
- (3) If $\{\text{vertices of } x_1, x_2, \dots, x_k\} = [n]$, then we add an edge $x = ab(a < b)$ where (a, b) is maximal among the remaining edges.

We see that an edge ab appears before an edge cd in X' when there is at least one vertex in ab that is greater than one vertex in cd .

Now we claim that every edge in X' is corresponding to a labeling value. Suppose that there is an edge $x = ab(a < b)$ in X' not corresponding to a labeling value. There are two cases from Lemma 4.1.

- I. In this case, there exist an edge $y = ef (e < f)$ of $G \setminus X$ and a cycle containing x and y . Also, note that $e < f < a < b$ and all the edges of the cycle except x and y appear in X' before x . There are two sub cases:
- (a) When e is closer to b than to a in the cycle:
In this case, the vertices in the path from b to e in the cycle must be increasing because of the condition of X . But this implies $b < e$ which is impossible.
 - (b) When e is closer to a than to b in the cycle:
In this case, the vertex adjacent to a in the cycle must be bigger than a since the edge connecting this vertex and a appears before x in X' by the construction of X' . Then the path from a to e is increasing which implies $a < e$. This is also impossible.
- II. In this case, there is an edge y in $G \setminus X$ adjacent to x and a cycle containing x and y . Also note that the vertex of $y \setminus x$ must be smaller than the vertex in $x \setminus y$ and all the edges of the cycle except x and y appear in X' before x . We omit the proof for this case which is similar to the previous case.

It is obvious that the choosing gives us a decreasing chain. In each step of choosing, we have only one choice. So we can get only one string X' from X that is associated with a decreasing maximal chain. $\square$

Example 4.5. We use the compressed graph G_5 shown in Figure 4. Let $X = [12, 13, 34, 36, 45]$ be a string of edges of G_5 . The edges in X form a spanning forest of G_5 . Now we use the rule above to find a new string X' that can be associated with a decreasing maximal chain. Firstly, we choose 36 for the first position in X' . Since $5 = \max\{[6] \setminus \{3, 6\}\}$ and there is only one edge containing 5 in X , we choose 45 for the next position. Although $2 = \max\{[6] \setminus \{3, 6, 4, 5\}\}$, there is an edge 34 that has $2 < 3 < 4$. We add 34 into the next position of X' . After that, 12 is added into X' . Finally, we add 13 to combine blocks. The new string $X' = [36, 45, 34, 12, 13]$, and the string of labeling values is $[6, 5, (3, 4), 2, (1, 3)]$. It is associated with the decreasing maximal chain $\hat{0} \triangleleft 36 \triangleleft 36 - 45 \triangleleft 3456 \triangleleft 12 - 3456 \triangleleft 123456$.

Now consider adjacent edges of vertices of G . Let

$$\alpha(i) = \{xi | xi \in E(G), x < i\}$$

for all $i = 2, \dots, n$.

Theorem 4.6. *The number of decreasing maximal chains of the intersection poset P_G is*

$$\prod_{i=2, |\alpha(i)| \neq 0}^{i=n} |\alpha(i)|$$

where $|\alpha(i)|$ is the number of edges in $\alpha(i)$ for $i = 1, 2, \dots, n$. Furthermore, the order complex $\Delta(\overline{P_G})$ has the homotopy type of a wedge of $\prod_{i=1, |\alpha(i)| \neq 0}^{i=n} |\alpha(i)|$ spheres of dimension $n - c(G) - 2$, where $c(G)$ is the number of connected components in G .

Proof. Lemma 3.7 shows that there is a string X of edges that is associated with a decreasing maximal chain.

Let $X = [x_1, x_2, \dots, x_{l-1}, x_l]$ be a string of edges that can be associated with a decreasing maximal chain. By Lemma 4.4, X does not contain two edges ab and cb such that $a, c < b$, and the number of edges in X is equal to the length, $n - c(G)$, of P_G . It is easy to see that the vertices v with $|\alpha(v)| = 0$ are precisely the vertices with the smallest label in each connected component of G . So the length of P_G (or the number of edges in X) is equal to the number of vertices v that have $|\alpha(v)| \neq 0$.

In other words, if $|\alpha(v)| \neq 0$, then there is only one edge in X whose larger vertex is v .

Let $gm (g < m)$ be an edge of X where m is the largest vertex appear in X . Since the set $X' = X \setminus \{gm\} \cup \{g'm\}$, where $g' < m$, satisfies the condition for Lemma 4.4, we can rearrange the edges in X' to get another maximal chain. There are $|\alpha(m)|$ ways to do this. Since the smaller vertices in other edges in X can be replaced in similar way, the number of decreasing chains in P_G is

$$\prod_{i=1, |\alpha(i)| \neq 0}^{i=n} |\alpha(i)|$$

The second part of the theorem follows from Theorem 2.8. $\square$

Example 4.7. Let G be a compressed graph as Figure 4. Use Theorem 4.6, we can indicate that the order complex $\Delta(\overline{P_G})$ has the homotopy type of a wedge of 48 spheres of dimension 3.

Corollary 4.8. *The order complex $\Delta(\overline{P_{K_n}})$ has the homotopy type of a wedge of $(n-1)!$ spheres of dimension $n-3$, where K_n is a complete graph.*

Proof. It is easy to see that $|\alpha(i)| = i - 1$ for all $i = 1, 2, \dots, n$. Theorem 4.6 indicates that the number of decreasing maximal chains in P_{K_n} is $(n - 1)!$. Therefore, the order complex $\Delta(\overline{P_{K_n}})$ has the homotopy type of a wedge of $(n - 1)!$ spheres of dimension $n - 3$. $\square$

Corollary 4.9. *If G is a compressed graph with no cycle, then there is only one decreasing maximal chain from $\hat{0}$ to $\hat{1}$. Furthermore, the order complex $\Delta(\overline{P})$ has the homotopy type of a sphere of dimension $n - c(G) - 2$, where $c(G)$ is the number of connected components of G .*

Proof. If G is a compressed graph with no cycle, then it is obvious that $|\alpha(i)| = 0$ or 1 for all vertex i . Therefore, the number of decreasing maximal chains in P_G is 1. $\square$

The next lemma will give some conditions to help us find the number of decreasing maximal chains of P_G , where G is a compressed graph with one cycle.

Lemma 4.10. *If G is a compressed graph with only one cycle, then the cycle must be a 3-cycle with vertices a, b , and $b + 1$ for some $a < b$.*

Proof. Assume that the cycle has more than 3 edges. Let $i_1 i_2$ be an edge of the cycle such that $i_1 < i_2$ and $|i_1 - i_2| = \max\{|i_x - i_y| : i_x i_y \in E(G)\}$.

Firstly, we claim that there exists i_3 in the cycle such that $i_1 < i_3 < i_2$. Suppose there is no such vertex in the cycle. Then one can label the vertices of the cycle so that $i_k i_{k+1}$ are edges for $k = 1, \dots, m - 1$ and $i_m i_1$ is an edge. One can see that $i_k > i_2$ for all $k = 3, \dots, m$ because of the choice of $i_1 i_2$. But then $|i_m - i_1| > |i_1 - i_2|$ which is impossible.

Since G is compressed, it has the edge $i_1 i_3$. Since G has only one cycle, $i_1 i_3$ must be contained in a cycle. Now suppose i_2 and i_3 are not adjacent. Let j be the vertex adjacent to i_3 in the cycle. We can choose edge $i_3 j$ since the cycle has more than 3 edges. We get four cases, $j < i_1 < i_3 < i_2$ or $i_1 < j < i_3 < i_2$, or $i_1 < i_3 < j < i_2$, or $i_1 < i_3 < i_2 < j$. In all four cases, G also has more than one cycle. Therefore, the unique cycle of G must be a 3-cycle.

Furthermore, assume that there is d such that $a < b < d < c$, then we get edges ad and bd and so G will have more than one cycle. So $c = b + 1$. $\square$

Corollary 4.11. *If G is a compressed graph with one cycle, then the order complex $\Delta(\overline{P_G})$ has the homotopy type of a wedge of two spheres.*

Proof. If the vertices of the unique cycle of G are a, b , and $b + 1$, then $|\alpha(b + 1)| = 2$ and $|\alpha(i)| = 0$ or 1 for all $i \neq b + 1$. Therefore,

$$\prod_{i=1, |\alpha(i)| \neq 0}^{i=n} |\alpha(i)| = 2$$

and the order complex $\Delta(\overline{P_G})$ has the homotopy type of a wedge of two spheres. $\square$

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