WALLMAN SUBLATTICES AND QUASI- F COVERS

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Abstract. In this paper, we first will show that for any space X and any Wallman sublattice $\mathcal{A}$ of $\mathcal{R}(X)$ with $Z(X)^\# \subseteq \mathcal{A}$, $(\Phi_{\mathcal{A}}^{-1}(X), \Phi_{\mathcal{A}})$ is the minimal quasi- F cover of X if and only if $(\Phi_{\mathcal{A}}^{-1}(X), \Phi_{\mathcal{A}})$ is a quasi- F cover of X and $\mathcal{A} \subseteq \mathcal{Q}_X$. Using this, if X is a locally weakly Lindelöf space, the set $\{\mathcal{A} \mid \mathcal{A} \text{ is a Wallman sublattice of } \mathcal{R}(X) \text{ with } Z(X)^\# \subseteq \mathcal{A} \text{ and } \Phi_{\mathcal{A}}^{-1}(X) \text{ is the minimal quasi-}F \text{ cover of } X\}$, when partially ordered by inclusion, has the minimal element $Z(X)^\#$ and the maximal element $\mathcal{Q}_X$. Finally, we will show that any Wallman sublattice $\mathcal{A}$ of $\mathcal{R}(X)$ with $Z(X)^\# \subseteq \mathcal{A} \subseteq \mathcal{Q}_X$, $\Phi_{\mathcal{A}_X} : \Phi_{\mathcal{A}}^{-1}(X) \rightarrow X$ is $z^\#$ -irreducible if and only if $\mathcal{A} = \mathcal{Q}_X$.

1. Introduction

All spaces in this paper are assumed to be Tychonoff spaces and $(\beta X, \beta_X)$ denotes the Stone-Čech compactification of a space X .

Gleason [5] showed that the projective objects in the category of compact spaces and continuous maps are precisely the extremally disconnected space and that every compact space has essentially unique projective cover, namely its extremally disconnected cover. Iliadis [9] (Banaschewski [2], resp.) proved similar results for the category of Hausdorff spaces and perfect continuous maps (the category of regular spaces and perfect continuous maps, resp.). To generalize extremally disconnected spaces, basically disconnected spaces, quasi- F spaces and cloz-spaces have been introduced and their minimal covers have been studied by various authors ([3], [6], [7], [8], [10], [13]).

In particular, Henriksen and Gillman introduced the concept of quasi- F spaces which is a generalization of F -spaces, in which every cozero-set

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is C^* -embedded and proved that each space X has the minimal quasi- F cover (QFX, Φ_X) ([6]).

Henriksen, Vermeer and Woods introduced the concept of Wallman covers for comapct spaces and proved that if X is a compact space, then QFX is given by the Wallman cover $\mathcal{L}(Z(X)^\#)$ of X with respect to $Z(X)^\#$ and $\Phi_X(\alpha) = \cap\{A \mid A \in \alpha\}$ ($\alpha \in \mathcal{L}(Z(X)^\#)$) ([7]). In [10], the Wallman sublattice $\mathcal{Q}_X$ of $\mathcal{R}(X)$ was introduced and it was shown that QFX is the subspace $\{\alpha \mid \alpha \text{ is a fixed } \mathcal{Q}_X - \text{ultrafilter}\}$ of the Wallman cover $\mathcal{L}(\mathcal{Q}_X)$ of βX with respect to $\mathcal{Q}_X$.

In this paper, we first will show that for any space X and any Wallman sublattice $\mathcal{A}$ of $\mathcal{R}(X)$ with $Z(X)^\# \subseteq \mathcal{A}$, $(\Phi_{\mathcal{A}}^{-1}(X), \Phi_{\mathcal{A}})$ is a minimal quasi- F cover of X if and only if $(\Phi_{\mathcal{A}}^{-1}(X), \Phi_{\mathcal{A}})$ is a quasi- F cover of X and $\mathcal{A} \subseteq \mathcal{Q}_X$. Using this, if X is a locally weakly Lindelöf space, the set $\{\mathcal{A} \mid \mathcal{A} \text{ is a Wallman sublattice of } \mathcal{R}(X) \text{ with } Z(X)^\# \subseteq \mathcal{A} \text{ and } \Phi_{\mathcal{A}}^{-1}(X) \text{ is the minimal quasi-}F \text{ cover of } X\}$, when partially ordered by inclusion, has the minimal element $Z(X)^\#$ and the maximal element $\mathcal{Q}_X$. Finally, we show that any Wallman sublattice $\mathcal{A}$ of $\mathcal{R}(X)$ with $Z(X)^\# \subseteq \mathcal{A} \subseteq \mathcal{Q}_X$, $\Phi_{\mathcal{A}_X} : \Phi_{\mathcal{A}}^{-1}(X) \rightarrow X$ is $z^\#$ -irreducible if and only if $\mathcal{A} = \mathcal{Q}_X$.

For the terminology, we refer to [1], [4] and [12].

2. Minimal quasi- F covers

The set $\mathcal{R}(X)$ of all regular closed sets in a space X , when partially ordered by inclusion, becomes a complete Boolean algebra, in which the join, meet, and complementation operations are defined as follows : for any $A \in \mathcal{R}(X)$ and any $\{A_i \mid i \in I\} \subseteq \mathcal{R}(X)$,

$$\begin{aligned} \vee\{A_i \mid i \in I\} &= cl_X(\cup\{A_i \mid i \in I\}), \\ \wedge\{A_i \mid i \in I\} &= cl_X(int_X(\cap\{A_i \mid i \in I\})), \text{ and} \\ A' &= cl_X(X - A) \end{aligned}$$

and a sublattice of $\mathcal{R}(X)$ is a subset of $\mathcal{R}(X)$ that contains $\emptyset$, X and is closed under finite joins and meets.

We recall that a map $f : Y \rightarrow X$ is called a *covering map* if it is a continuous, onto, perfect, and irreducible map.

Lemma 2.1. ([6], [10]) (1) Let $f : Y \rightarrow X$ be a covering map. Then the map $\psi : \mathcal{R}(Y) \rightarrow \mathcal{R}(X)$, defined by $\psi(A) = A \cap X$, is a Boolean isomorphism and the inverse map ψ^{-1} of ψ is given by $\psi^{-1}(B) = cl_Y(f^{-1}(int_X(B))) = cl_Y(int_Y(f^{-1}(B)))$.

(2) Let X be a dense subspace of a space K . Then the map $\phi : \mathcal{R}(K) \rightarrow$

$\mathcal{R}(X)$, defined by $\phi(A) = A \cap X$, is a Boolean isomorphism and the inverse map ϕ^{-1} of ϕ is given by $\phi^{-1}(B) = cl_K(B)$.

A subspace Y of a space X is C^* -embedded in X if for any continuous map $f : Y \rightarrow \mathbb{R}$, there is a continuous map $g : X \rightarrow \mathbb{R}$ such that $g|_Y = f$, where $\mathbb{R}$ is the space of all real numbers endowed with the usual topology.

Definition 2.2. A space X is called a *quasi- F space* if for any zero-sets A, B in X , $cl_X(int_X(A \cap B)) = cl_X(int_X(A)) \cap cl_X(int_X(B))$, equivalently, every dense cozero-set in X is C^* -embedded in X .

It is well-known that a space X is a quasi- F space if and only if βX is a quasi- F space.

Definition 2.3. Let X be a space. Then a pair (Y, f) is called

- (1) a *cover of X* if $f : X \rightarrow Y$ is a covering map,
- (2) a *quasi- F cover of X* if (Y, f) is a cover of X and Y is a quasi- F space, and
- (3) a *minimal quasi- F cover of X* if (Y, f) is a quasi- F cover of X and for any quasi- F cover (Z, g) of X , there is a covering map $h : Z \rightarrow Y$ such that $f \circ h = g$.

M. Henriksen, J. Vermeer, and R. G. Woods showed that every space X has the minimal quasi- F cover (QFX, Φ_X) ([6]).

Definition 2.4. ([7]) Let X be a space and $\mathcal{A}$ a sublattice of $\mathcal{R}(X)$. Then $\mathcal{A}$ is said to be a *Wallman sublattice* if

- (1) for any $A \in \mathcal{A}$ and $x \in X - A$, there is a $B \in \mathcal{A}$ such that $x \in int_X(B)$ and $A \wedge B = \emptyset$, and
- (2) for any $A, B \in \mathcal{A}$ with $A \wedge B = \emptyset$, there are $C, D \in \mathcal{A}$ such that $A \wedge C = B \wedge D = \emptyset$ and $C \vee D = X$.

For any space X , $Z(X)^\# = \{cl_X(int_X(Z)) \mid Z \text{ is a zero-set in } X\}$ is a Wallman sublattice of $\mathcal{R}(X)$.

First, suppose that X is a compact space and $\mathcal{A}$ is a Wallman sublattice of $\mathcal{R}(X)$ such that $\mathcal{A}$ is a base for closed sets in X . Let $\mathcal{L}(\mathcal{A}) = \{\alpha \mid \alpha \text{ is an } \mathcal{A}\text{-ultrafilter}\}$ and for any $A \in \mathcal{A}$, $\sum_{\mathcal{A}}^A = \{\alpha \in \mathcal{L}(\mathcal{A}) \mid A \in \alpha\}$. Then $\{\sum_{\mathcal{A}}^A \mid A \in \mathcal{A}\}$ is a base for closed sets in some topology on $\mathcal{L}(\mathcal{A})$. Let $\mathcal{L}(\mathcal{A})$ be the space, equipped with the topology for which $\{\sum_{\mathcal{A}}^A \mid A \in \mathcal{A}\}$ is a base for closed sets and define a map $\Phi_{\mathcal{A}} : \mathcal{L}(\mathcal{A}) \rightarrow X$ by $\Phi_{\mathcal{A}}(\alpha) = \cap\{A \mid A \in \alpha\}$. Then the pair $(\mathcal{L}(\mathcal{A}), \Phi_{\mathcal{A}})$, called a *Wallman cover of X with respect to $\mathcal{A}$* , is a cover of X and for any $A, B \in \mathcal{A}$, $\sum_{\mathcal{A}}^A \wedge \sum_{\mathcal{A}}^B = \sum_{\mathcal{A}}^{A \wedge B}$. Note that $Z(X)^\#$ is a base for closed sets in

X and the Wallman cover- F $(\mathcal{L}(Z(X)^\#), \Phi_{Z(X)^\#})$ of X with respect to $Z(X)^\#$ is the minimal quasi- F cover of X ([7]).

Throughout in this paper, for any space X , $(QF(\beta X), \Phi_\beta)$ denotes the minimal quasi- F cover of βX . Since βX is a compact space and $Z(\beta X)^\#$ is a base for closed sets in βX , $QF(\beta X) = \mathcal{L}(Z(\beta X)^\#)$ and $\Phi_\beta(\alpha) = \cap\{A \in Z(\beta X)^\# \mid A \in \alpha\}$.

Now, suppose that X is a space, and $\mathcal{A}$ is a Wallman sublattice of $\mathcal{R}(X)$ such that $\mathcal{A}^\beta$ is a base for closed sets in βX , where $\mathcal{A}^\beta = \{cl_{\beta X}(A) \mid A \in \mathcal{A}\}$. Then the pair $(\mathcal{L}(\mathcal{A}^\beta), \Phi_{\mathcal{A}^\beta})$, denoted by $(\mathcal{L}(\mathcal{A}), \Phi_{\mathcal{A}})$, is the minimal quasi- F cover of βX . Since $Z(X)^\#$ is a Wallman sublattice of $\mathcal{R}(X)$ and X is C^* -embedded in βX , by Lemma 2.1, $Z(X)^\#{}^\beta = Z(\beta X)^\#$. Hence $(\mathcal{L}(Z(X)^\#), \Phi_{Z(X)^\#})$ is the minimal quasi- F cover of βX .

For any space X and any Wallman sublattice $\mathcal{A}$ of $\mathcal{R}(X)$, let

$$q(X, \mathcal{A}) = \{\alpha \in \mathcal{L}(\mathcal{A}) \mid \alpha \text{ is fixed}\}$$

be the subspace of $\mathcal{L}(\mathcal{A})$ and in particular, let $\mathcal{Q}_X = \{\Phi_X(A) \mid A \in Z(QFX)^\#\}$ and

$$qX = q(X, \mathcal{Q}_X).$$

For any space $f : Y \rightarrow X$ and $U \subseteq X$, let $f_U : f^{-1}(U) \rightarrow U$ be the restriction and corestriction of f with respect to U and $f^{-1}(U)$, respectively.

Lemma 2.5. [10] *Let X be a space. Then there is a homeomorphism $g_X : qX \rightarrow QFX$ and $(qX, \Phi_X \circ g_X)$ is the minimal quasi- F cover of X .*

By Lemma 2.1, we have the following proposition :

Proposition 2.6. *Let X be a space. Then we have the following :*

- (1) $\mathcal{Q}_X$ is a Wallman sublattice of $\mathcal{R}(X)$ and $Z(X)^\# \subseteq \mathcal{Q}_X$.
- (2) Suppose that $\mathcal{A}$ is a Wallman sublattice of $\mathcal{R}(X)$ containing $Z(X)^\#$. Then $\mathcal{L}(\mathcal{A})$ is a compactification of $q(X, \mathcal{A})$.

Proof. (1) Since $Z(QFX)^\#$ is a Wallman sublattice of $\mathcal{R}(X)$, by Lemma 2.1, $\mathcal{Q}_X$ is a Wallman sublattice of $\mathcal{R}(X)$ and since $\Phi_X : QFX \rightarrow X$ is a covering map, $Z(X)^\# \subseteq \mathcal{Q}_X$.

(2) Let $V = q(X, \mathcal{A})$ and $W = \mathcal{L}(\mathcal{A})$. Note that $\{\Sigma_{\mathcal{A}}^A \mid A \in \mathcal{A}\}$ is a base for closed sets in W . Let $A \in \mathcal{A}$ such that $W - \Sigma_{\mathcal{A}}^A \neq \emptyset$. Suppose that $(W - \Sigma_{\mathcal{A}}^A) \cap V = \emptyset$. Then $V \subseteq \Sigma_{\mathcal{A}}^A$ and $\Sigma_{\mathcal{A}}^A \neq W$. Pick $\alpha \in W - \Sigma_{\mathcal{A}}^A$. Since α is an $\mathcal{A}$ -ultrafilter and $A \notin \alpha$, there is a $B \in \alpha$ such that $A \wedge B = \emptyset$ and so $\Sigma_{\mathcal{A}}^A \cap \Sigma_{\mathcal{A}}^B = \emptyset$. Since $V \subseteq \Sigma_{\mathcal{A}}^A$, $V \cap \Sigma_{\mathcal{A}}^B = \emptyset$. Since B is an element of a filter, $B \neq \emptyset$ and pick $x \in B$. Since $Z(X)^\#$ is a base for closed sets

of X , $\mathcal{A}$ is a base for closed sets of X and so there is an $\mathcal{A}$ -ultrafilter δ such that $x \in \cap\{C \mid C \in \delta\}$. Since $B \in \delta$, $\Sigma_{\mathcal{A}}^B \cap V \neq \emptyset$ and this is a contradiction. Hence $(W - \Sigma_{\mathcal{A}}^A) \cap V \neq \emptyset$ and so V is dense in W . Thus W is a compactification of V . $\square$

It is well-known that for any compactification Y of X , X is C^* -embedded in Y if and only if $\beta X = Y$ ([4]).

Theorem 2.7. *Let X be a space. Then $\beta QFX = \mathcal{L}(\mathcal{Q}_X)$.*

Proof. By Lemma 2.5, $\mathcal{L}(\mathcal{Q}_X)$ is a compactification of QFX . Let A, B be disjoint zero-sets in QFX . Then there are C, D in $Z(QFX)^\#$ such that $C \cap D = \emptyset$, $A \subseteq C$ and $B \subseteq D$. By the definition of $\mathcal{Q}_X$, $\Phi_X(C) \in \mathcal{Q}_X$ and $\Phi_X(D) \in \mathcal{Q}_X$. Note that $\sum_{\mathcal{Q}_X}^{\Phi_X(C)} \cap QFX = \Phi_X(C)$ and $\sum_{\mathcal{Q}_X}^{\Phi_X(D)} \cap QFX = \Phi_X(D)$. Since $C \cap D = C \wedge D = \emptyset$, $(\sum_{\mathcal{Q}_X}^{\Phi_X(C)} \wedge \sum_{\mathcal{Q}_X}^{\Phi_X(D)}) \cap QFX = \emptyset$ and since $\Phi_X(C) \in \mathcal{Q}_X$ and $\Phi_X(D) \in \mathcal{Q}_X$, $\sum_{\mathcal{Q}_X}^{\Phi_X(C)} \cap \sum_{\mathcal{Q}_X}^{\Phi_X(D)} \cap QFX = \emptyset$. Since $\sum_{\mathcal{Q}_X}^{\Phi_X(C)}, \sum_{\mathcal{Q}_X}^{\Phi_X(D)} \in Z(\mathcal{L}(\mathcal{Q}_X))^\#$, $\sum_{\mathcal{Q}_X}^{\Phi_X(C)} \cap \sum_{\mathcal{Q}_X}^{\Phi_X(D)} = \emptyset$. Since $\Phi_{\mathcal{Q}_X}(\sum_{\mathcal{Q}_X}^{\Phi_X(C)}) = \Phi_{\mathcal{Q}_X}(C)$, by Lemma 2.1, $C \subseteq \sum_{\mathcal{Q}_X}^{\Phi_X(C)}$ and similarly, $D \subseteq \sum_{\mathcal{Q}_X}^{\Phi_X(D)}$. Since $\sum_{\mathcal{Q}_X}^{\Phi_X(C)}$ and $\sum_{\mathcal{Q}_X}^{\Phi_X(D)}$ are closed in $\mathcal{L}(\mathcal{Q}_X)$, $cl_{\mathcal{L}(\mathcal{Q}_X)}(C) \cap cl_{\mathcal{L}(\mathcal{Q}_X)}(D) = \emptyset$ and by the Urysohn's extension theorem, QFX is C^* -embedded in $\mathcal{L}(\mathcal{Q}_X)$. Hence $\beta QFX = \mathcal{L}(\mathcal{Q}_X)$. $\square$

Let X be a compact space and $\mathcal{A}, \mathcal{B}$ Wallman sublattices of $\mathcal{R}(X)$ with $Z(X)^\# \subseteq \mathcal{B} \subseteq \mathcal{A}$. Since $Z(X)^\#$ is a base for closed sets in X and $Z(X)^\#{}^\beta = Z(\beta X)^\#$ is a base for closed sets in βX , there is a covering map $l : \mathcal{L}(\mathcal{A}) \rightarrow \mathcal{L}(\mathcal{B})$ such that $\Phi_{\mathcal{B}} \circ l = \Phi_{\mathcal{A}}([7])$.

Proposition 2.8. *Let X be a space and $\mathcal{A}$ a Wallman sublattice of $\mathcal{R}(X)$ with $Z(X)^\# \subseteq \mathcal{A}$. Then a pair $(\Phi_{\mathcal{A}}^{-1}(X), \Phi_{\mathcal{A}})$ is the minimal quasi- F cover of X if and only if $(\Phi_{\mathcal{A}}^{-1}(X), \Phi_{\mathcal{A}})$ is a quasi- F cover of X and $\mathcal{A} \subseteq \mathcal{Q}_X$.*

Proof. Let $T = \Phi_{\mathcal{A}}^{-1}(X)$. Suppose that $(\Phi_{\mathcal{A}}^{-1}(X), \Phi_{\mathcal{A}})$ is the minimal quasi- F cover of X . Then there is a homeomorphism $g : T \rightarrow QFX$ such that $\Phi_X \circ g = \Phi_{\mathcal{A}_X}$. By Theorem 2.7, $\beta QFX = \mathcal{L}(\mathcal{Q}_X)$ and so there is a continuous map $f : \beta QFX \rightarrow \mathcal{L}(\mathcal{A})$ such that $f \circ \beta_{QFX} \circ g = j$, where $j : T \rightarrow \mathcal{L}(\mathcal{A})$ is the inclusion map. Since βQFX and $\mathcal{L}(\mathcal{A})$ are compact spaces, $f : \beta QFX \rightarrow \mathcal{L}(\mathcal{A})$ is a covering map. Clearly, $\Phi_{\mathcal{A}} \circ f = \Phi_{\mathcal{Q}_X}$.

Let $A \in \mathcal{A}$. Since $\Phi_{\mathcal{A}} \circ f = \Phi_{\mathcal{Q}_X} : \beta QFX \longrightarrow \beta X$ is a covering map, by Lemma 2.1, $cl_{\beta QFX}(\Phi_{\mathcal{Q}_X}^{-1}(int_{\beta X}(cl_{\beta X}(A)))) \in Z(\beta QFX)^{\#}$. Hence

$$\begin{aligned}
 & \Phi_X[cl_{\beta QFX}(\Phi_{\mathcal{Q}_X}^{-1}(int_{\beta X}(cl_{\beta X}(A)))) \cap QFX] \\
 (2.1) \quad &= \Phi_{\mathcal{Q}_X}[cl_{\beta QFX}(\Phi_{\mathcal{Q}_X}^{-1}(int_{\beta X}(cl_{\beta X}(A))))] \cap X \\
 &= cl_{\beta X}(A) \cap X = A
 \end{aligned}$$

and $cl_{\beta QFX}(\Phi_{\mathcal{Q}_X}^{-1}(int_{\beta X}(A))) \cap QFX \in Z(QF)^{\#}$. Thus $\mathcal{A} \subseteq \mathcal{Q}_X$.

Suppose that $(\Phi_{\mathcal{A}}^{-1}(X), \Phi_{\mathcal{A}})$ is a quasi- F cover of X and $\mathcal{A} \subseteq \mathcal{Q}_X$. Note that $\mathcal{A}^{\beta} = \{G \in \mathcal{R}(\beta X) \mid G \cap X \in \mathcal{A}\}$ is lattice isomorphic to $\mathcal{A}$ and $\mathcal{Q}_{X\beta} = \{G \in \mathcal{R}(\beta X) \mid G \cap X \in \mathcal{Q}_X\}$ is Boolean isomorphic to $\mathcal{Q}_X$. Since $\mathcal{A}^{\beta} \subseteq \mathcal{Q}_{X\beta}^{\beta}$ and $\mathcal{L}(\mathcal{A}^{\beta})$ and $\mathcal{L}(\mathcal{A})$ are homeomorphic, there is a covering map $l : \beta QFX \longrightarrow \mathcal{L}(\mathcal{A})$ such that $\Phi_{\mathcal{A}} \circ l = \Phi_{\mathcal{Q}_X}$. So there is a covering map $k : QFX \longrightarrow T$ such that $l \circ \beta_{QFX} = j \circ k$ and $k \circ \Phi_{\mathcal{A}_X} = \Phi_{QFX}$. Since $(\Phi_{\mathcal{A}}^{-1}(X), \Phi_{\mathcal{A}})$ is a quasi- F cover of X , there is a covering map $m : T \longrightarrow QFX$ such that $\Phi_X \circ m = \Phi_{\mathcal{A}_X}$. Hence $k \circ m = 1_T$ and so $m : T \longrightarrow QFX$ is a homeomorphism. Thus $(\Phi_{\mathcal{A}}^{-1}(X), \Phi_{\mathcal{A}})$ is the minimal quasi- F cover of X . $\square$

For any space X , let $\mathbb{K}(X) = \{\mathcal{A} \mid \mathcal{A} \text{ is a Wallman sublattice of } \mathcal{R}(X) \text{ with } Z(X)^{\#} \subseteq \mathcal{A} \text{ and } \Phi_{\mathcal{A}}^{-1}(X) \text{ is the minimal quasi-}F \text{ cover of } X\}$.

Corollary 2.9. *Let X be a space. Then the poset $(\mathbb{K}(X), \subseteq)$ has the maximal element.*

We recall that a space X is called a *weakly Lindelöf space* if for any open cover $\mathcal{U}$, there is a countable subset $\mathcal{V}$ of $\mathcal{U}$ such that $\cup \mathcal{V}$ is dense in X and that X is called a *locally weakly Lindelöf space* if every element of X has a weakly Lindelöf neighborhood.

It was shown that for any locally weakly Lindelöf space X , $(\Phi_{\beta}^{-1}(X), \Phi_{\beta X})$ is the minimal quasi- F cover of X ([10]).

Corollary 2.10. *Let X be a locally weakly Lindelöf space. Then $(\mathbb{K}(X), \subseteq)$ has the minimal element $Z(X)^{\#}$ and the maximal element $\mathcal{Q}_X$.*

Note that for any compact space X , $\Phi_X(Z(QFX)^{\#}) = Z(X)^{\#}$. Hence for any space X , $\Phi_{\beta}(Z(QF\beta X)^{\#}) = Z(\beta X)^{\#}$ and using these, we have the following proposition :

Proposition 2.11. *Let X be a space. Then $\mathbb{K}(X) = \{Z(X)^{\#}\}$ if and only if $QF\beta X = \beta QFX$.*

Proof. Suppose that $\mathcal{Q}_X = Z(X)^\#$. By Theorem 2.7, $\mathcal{L}(\mathcal{Q}_X)$ is a compactification of $QF(X)$ and $\beta QFX = \mathcal{L}(\mathcal{Q}_X)$.

We claim that QFX is C^* -embedded in $QF\beta X$. Let A, B be disjoint zero-sets in QFX . Then there are C, D in $Z(QFX)^\#$ such $C \cap D = \emptyset$, $A \subseteq C$ and $B \subseteq D$. By the definition of $\mathcal{Q}_X$, $\Phi_X(C), \Phi_X(D) \in \mathcal{Q}_X = Z(X)^\#$. Then there are $C_0, D_0 \in Z(QF\beta X)^\#$ such that $C_0 \cap X = C$ and $D_0 \cap X = D$. Since $\Phi_\beta : QF\beta X \rightarrow \beta X$, $\Phi_\beta^{-1}(C_0), \Phi_\beta^{-1}(D_0) \in Z(QF\beta X)$. Note that $\Phi_\beta(\sum_{Z(\beta X)^\#}^{\Phi_\beta(C_0)} \cap QFX) = \Phi_X(C)$ and $\Phi_\beta(\sum_{Z(\beta X)^\#}^{\Phi_\beta(D_0)} \cap QFX) = \Phi_X(D)$. Since $C \cap D = C \wedge D = \emptyset$, $(\sum_{Z(\beta X)^\#}^{\Phi_\beta(C_0)} \wedge \sum_{Z(\beta X)^\#}^{\Phi_\beta(D_0)}) \cap QFX = \emptyset$ and since $\Phi_X(C) \in Z(X)^\#$ and $\Phi_X(D) \in Z(X)^\#$, $\sum_{Z(\beta X)^\#}^{\Phi_\beta(C_0)} \cap \sum_{Z(\beta X)^\#}^{\Phi_\beta(D_0)} \cap QFX = \emptyset$. Since $\sum_{Z(\beta X)^\#}^{\Phi_\beta(C_0)}, \sum_{Z(\beta X)^\#}^{\Phi_\beta(D_0)} \in Z(QF\beta X)^\#$, $\sum_{Z(\beta X)^\#}^{\Phi_\beta(C_0)} \cap \sum_{Z(\beta X)^\#}^{\Phi_\beta(D_0)} = \emptyset$. Since $\Phi_\beta(\sum_{Z(\beta X)^\#}^{\Phi_\beta(C_0)} \cap QFX) = \Phi_X(C)$, by Lemma 2.1, $C \subseteq \sum_{Z(\beta X)^\#}^{\Phi_\beta(C_0)}$ and similarly, $D \subseteq \sum_{Z(\beta X)^\#}^{\Phi_\beta(D_0)}$. Since $\sum_{Z(\beta X)^\#}^{\Phi_\beta(C_0)}$ and $\sum_{Z(\beta X)^\#}^{\Phi_\beta(D_0)}$ are closed in $QF\beta X$, $cl_{QF\beta X}(C) \cap cl_{QF\beta X}(D) = \emptyset$ and by the Urysohn's extension theorem, QFX is C^* -embedded in $QF\beta X$. Hence $QF\beta X = \beta QFX$.

Suppose that $QF\beta X = \beta QFX$. Then $QF(X)$ is C^* -embedded in $QF\beta X$. By Lemma 2.1, $Z(\beta QFX)^\# = Z(QF\beta X)^\#$ and since $QF\beta X = \mathcal{L}(Z(X)^\#) = \mathcal{L}(Z(\beta X)^\#)$,

$$\begin{aligned} \mathcal{Q}_X &= \Phi_X(Z(QFX)^\#) = \Phi_\beta(Z(QF\beta X)^\# \cap QFX) \\ &= \Phi_\beta(Z(QF\beta X)^\#) \cap X = Z(\beta X)^\# \cap X \\ &= Z(X)^\#. \end{aligned}$$

□

We recall that a covering map $f : Y \rightarrow X$ is called a $z^\#$ -irreducible map if $\{f(A) \mid A \in Z(Y)^\#\} = Z(X)^\#$. Note that for any $B \in Z(X)^\#$, $cl_Y(f^{-1}(int_X(B))) \in Z(Y)^\#$ and $f(cl_Y(f^{-1}(int_X(B)))) = B$. Hence a covering map $f : Y \rightarrow X$ is $z^\#$ -irreducible if and only if for any $A \in Z(Y)^\#$, $f(A) \in Z(X)^\#$.

It is well-known that for any compact space X , $\Phi_X : QFX \rightarrow X$ is $z^\#$ -irreducible [6].

Theorem 2.12. *Let X be a space and $\mathcal{A}$ a Wallman sublattice of $\mathcal{R}(X)$ with $Z(X)^\# \subseteq \mathcal{A} \subseteq \mathcal{Q}_X$. Then $\Phi_{\mathcal{A}_X} : \Phi_{\mathcal{A}}^{-1}(X) \rightarrow X$ is $z^\#$ -irreducible if and only if $\mathcal{A} = \mathcal{Q}_X$.*

Proof. Suppose that $\Phi_{\mathcal{A}_X} : \Phi_{\mathcal{A}}^{-1}(X) \rightarrow X$ is $z^\#$ -irreducible. Let $F \in Z(\mathcal{L}(\mathcal{A}))^\#$. By Lemma 2.1, $F \cap \Phi_{\mathcal{A}}^{-1}(X) \in Z(\Phi_{\mathcal{A}}^{-1}(X))^\#$ and so

$\Phi_{\mathcal{A}_X}(F \cap \Phi_{\mathcal{A}}^{-1}(X)) = \Phi_{\mathcal{A}}(F) \cap X \in Z(X)^{\#}$. Hence $\Phi_{\mathcal{A}}(F) \in Z(\beta X)^{\#}$ and thus $\Phi_{\mathcal{A}} : \mathcal{L}(\mathcal{A}) \rightarrow \beta X$ is $z^{\#}$ -irreducible. Since $Z(X)^{\#} \subseteq \mathcal{A}$, there is a covering map $g : \mathcal{L}(\mathcal{A}) \rightarrow QF\beta X$ such that $\Phi_{\beta} \circ g = \Phi_{\mathcal{A}}$. $\Phi_{\mathcal{A}}$ is $z^{\#}$ -irreducible, g is $z^{\#}$ -irreducible.

Take any $A, B \in Z(\mathcal{L}(\mathcal{A}))^{\#}$ such that $A \wedge B = \emptyset$. Since g is $z^{\#}$ -irreducible, $g(A), g(B) \in Z(QF\beta X)^{\#}$ and by Lemma 2.1, $g(A) \wedge g(B) = \emptyset$. Since $QF\beta X$ is a quasi- F space, $g(A) \cap g(B) = \emptyset$ and so $A \cap B = \emptyset$. Hence $\mathcal{L}(\mathcal{A})$ is a quasi- F space and since g is $z^{\#}$ -irreducible and $QF\beta X$ is a quasi- F space, g is a homeomorphism.

Let $C \in Z(\Phi_{\mathcal{A}}^{-1}(X))^{\#}$. Since $\Phi_{\mathcal{A}_X} : \Phi_{\mathcal{A}}^{-1}(X) \rightarrow X$ is $z^{\#}$ -irreducible and X is C^* -embedded in βX , there is a $D \in Z(\mathcal{L}(\mathcal{A}))^{\#}$ such that $C = D \cap \Phi_{\mathcal{A}}^{-1}(X)$. Using this, we can show that $\Phi_{\mathcal{A}}^{-1}(X)$ is a quasi- F space and $\beta\Phi_{\mathcal{A}}^{-1}(X) = \mathcal{L}(\mathcal{A})$. By Proposition 2.8, $(\Phi_{\mathcal{A}}^{-1}(X), \Phi_{\mathcal{A}_X})$ is the minimal quasi- F cover of X , that is $\Phi_{\mathcal{A}}^{-1}(X) = QFX$. Hence $\beta QFX = QF\beta X$ and by Proposition 2.11, $\mathcal{A} = \mathcal{Q}_X$. $\square$

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