ARITHMETIC SUMS SUBJECT TO LINEAR AND CONGRUENT CONDITIONS AND SOME APPLICATIONS

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Abstract. We investigate the explicit evaluation for the sum $\sum_{\substack{(a,b,x,y) \in \mathbb{N}^4, \\ ax+by=n, \\ C(x,y)}} ab$ in terms of various divisor functions (where $C(x,y)$ is the set of residue conditions on x and y) for various fixed $C(x,y)$. We also obtain some identities and congruences as interesting applications.

1. Introduction

For $d, n, u \in \mathbb{N}$ and $s \in \mathbb{N} \cup \{0\}$, we define some divisor functions for our use in the sequel.

$$\sigma_s(n) = \sum_{d|n} d^s, \quad \sigma_s^*(n; u) := \sum_{\substack{d|n, \\ \frac{n}{d} \not\equiv 0 \pmod{u}}} d^s, \quad \sigma_s^*(n) := \sigma_s^*(n; 2)$$

and we define the arithmetic function $c_8(n)$ by the expression:

$$\sum_{n=1}^{\infty} c_8(n) q^n := q \prod_{m=1}^{\infty} (1 - q^{2m})^4 (1 - q^{4m})^4 \text{ with } q \in \mathbb{C}, |q| < 1.$$

For $k, n \in \mathbb{N}$, we define the arithmetic function $F_k(n)$ by

$$F_k(n) = \begin{cases} 1, & \text{if } k|n, \\ 0, & \text{if } k \nmid n. \end{cases}$$

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For $k \in \mathbb{N}$ and $n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$, we denote the number of representations of n as the sum of k squares by $r_k(n)$, that is,

$$r_k(n) := \text{card}\{(x_1, \dots, x_k) \in \mathbb{Z}^k \mid n = x_1^2 + \dots + x_k^2\}.$$

As there is only one way of expressing 0 as a sum of k squares, namely, $0 = 0^2 + \dots + 0^2$, we have $r_k(0) = 1$ for all $k \in \mathbb{N}$.

Let $n \in \mathbb{N}$ and let $f : \mathbb{Z} \rightarrow \mathbb{C}$ be an even function. In 1858 (see [21, p. 247], [22, p. 275]), Liouville asserted the surprising result that the sum

$$(1) \quad S_1 := \sum_{\substack{(a,b,x,y) \in \mathbb{N}^4, \\ ax+by=n}} \{f(a-b) - f(a+b)\}$$

can be expressed in terms of sums over the positive divisors d of n . More precisely, we have

$$(2) \quad S_1 = f(0) (\sigma_1(n) - \sigma_0(n)) + \sum_{\substack{d \in \mathbb{N}, \\ d|n}} \left(1 + \frac{2n}{d} - d\right) f(d) - 2 \sum_{\substack{d \in \mathbb{N}, \\ d|n}} \left(\sum_{v=1}^d f(v)\right).$$

In the literature, several proofs of (2) are available (for example see [23], [25], [27, p. 93] and [28]). The identity (2) of Liouville has a lot of important applications. Bouniakowsky's theorem on the number of specific representations of a prime number $q \equiv 7 \pmod{16}$, Besge's formula for the twisted convolution sums (for $s = 1, 2$) (see [31]),

$$A_s(n) := \sum_{\substack{k \in \mathbb{N}, \\ k < n/s}} \sigma_1(k) \sigma_1(n - sk)$$

are some of them to be quoted as applications. We note that (see [31])

$$A_1(n) = \frac{5}{12} \sigma_3(n) + \left(\frac{1}{12} - \frac{1}{2}n\right) \sigma_1(n)$$

and

$$A_2(n) = \frac{1}{12} \sigma_3(n) + \frac{1}{3} \sigma_3\left(\frac{n}{2}\right) + \left(\frac{1}{24} - \frac{1}{8}n\right) \sigma_1(n) + \left(\frac{1}{24} - \frac{1}{4}n\right) \sigma_1\left(\frac{n}{2}\right).$$

The extension of the Liouville's identity (2) to three variables is due to McAfee and Williams (see [24, Theorem 1.2, p. 35]) and the four variable extension is due to Alaca, Alaca, McAfee and Williams (see [1, Theorem 6.4, p. 20]). In 2000, Huard, Ou, Spearman and Williams obtained a far reaching generalization of the Liouville's identity (2) (which is stated as Proposition 2.1 in the sequel).

For $a, b, n \in \mathbb{N}$, we define the convolution sum

$$S_{a,b}(n) := \sum_{m=1}^{n-1} \sigma_a(m) \sigma_b(n-m).$$

It is plain that $S_{a,b}(n) = S_{b,a}(n)$ for all $a, b, n \in \mathbb{N}$. Ramanujan showed that $S_{a,b}(n)$ can be evaluated in terms of $\sigma_{a+b+1}(n), \sigma_{a+b-1}(n), \dots, \sigma_3(n), \sigma_1(n)$ for the nine pairs $(a, b) \in \mathbb{N}^2$ satisfying $a + b = 2, 4, 6, 8, 12, a \leq b, a \equiv b \equiv 1 \pmod{2}$.

As a nice application to their general identity, Huard, Ou, Spearman and Williams established that all these nine Ramanujan's evaluations can be obtained by suitable choices of the function $f(a, b, x, y)$ in their general identity. It should be noted that in a certain sense, the nine evaluations of $S_{a,b}(n)$ with a and b odd are the only ones. This fact was proved by Levitt (see [20]) using the theory of modular forms. An interesting and a some what difficult question is to investigate a sum of the sort :

$$S_1^* := \sum_{\substack{(a,b,x,y) \in \mathbb{N}^4, \\ ax+by=n, \\ x \equiv l_1 \pmod{k_1}, \\ y \equiv l_2 \pmod{k_2}}} \{f(a-b) - f(a+b)\},$$

where, as before, $f : \mathbb{Z} \rightarrow \mathbb{C}$ is an even function and $n \in \mathbb{N}$. Here k_1 and k_2 are integers ≥ 2 .

Let $C(x, y) := \{(x, y) \mid x \equiv l_1 \pmod{k_1}, y \equiv l_2 \pmod{k_2}\}$.

The aim of this paper is to obtain explicit evaluation in terms of divisor functions of the sum :

$$S(l_1, l_2; k_1, k_2) := \sum_{\substack{(a,b,x,y) \in \mathbb{N}^4, \\ ax+by=n, \\ C(x,y)}} ab$$

for various fixed residue classes set $C(x, y)$. More precisely, we prove :

Theorem 1.1. For $n \in \mathbb{N}$, we have

	$C(x, y) =$	$\sum_{\substack{(a,b,x,y) \in \mathbb{N}^4, \\ ax+by=n, \\ C(x,y)}} ab =$
(i)	$x \equiv 0 \pmod{2}, y \equiv 0 \pmod{2}$	$\frac{1}{12} (5\sigma_3(\frac{n}{2}) + (1-3n)\sigma_1(\frac{n}{2}))$
(ii)	$x \not\equiv 0 \pmod{2}, y \equiv 0 \pmod{2}$ or $x \equiv 0 \pmod{2}, y \not\equiv 0 \pmod{2}$	$\frac{1}{24} (2\sigma_3^*(n) + \sigma_1^*(n) - 3n\sigma_1(n))$
(iii)	$x \not\equiv 0 \pmod{2}, y \not\equiv 0 \pmod{2}$	$\frac{1}{4} (\sigma_3^*(n) - n\sigma_1^*(n))$

TABLE 1. Formula for modulo 2

Theorem 1.2. For $n \in \mathbb{N}$, we have

	$C(x, y) =$	$\sum_{\substack{(a,b,x,y) \in \mathbb{N}^4, \\ ax+by=n, \\ C(x,y)}} ab =$
(i)	$x \equiv 0 \pmod{4}, y \equiv 0 \pmod{4}$	$\frac{1}{12}(5\sigma_3(\frac{n}{4}) + (1 - \frac{3}{2}n)\sigma_1(\frac{n}{4}))$
(ii)	$x \equiv 2 \pmod{4}, y \equiv 2 \pmod{4}$	$\frac{1}{8}(2\sigma_3^*(\frac{n}{2}) - n\sigma_1^*(\frac{n}{2}))$
(iii)	$x \equiv 0 \pmod{4}, y \equiv 2 \pmod{4}$	$\frac{1}{24}\{2\sigma_3^*(\frac{n}{2}) + \sigma_1^*(\frac{n}{2}) - \frac{3}{2}n\sigma_1(\frac{n}{2})\}$
(iv)	$x \not\equiv 0 \pmod{4}, y \not\equiv 0 \pmod{4}$	$\frac{1}{8}[3\sigma_3^*(n) + 2\sigma_3^*(\frac{n}{2}) - 3n\{\sigma_1^*(n) + \sigma_1^*(\frac{n}{2})\}]$
(v)	$x \equiv 0 \pmod{4}, y \not\equiv 0 \pmod{4}$	$\frac{1}{48}[\sigma_3^*(n) + 4\sigma_3^*(\frac{n}{2}) - (3n - 2)\{\sigma_1^*(n) + \sigma_1^*(\frac{n}{2})\} - 9n\sigma_1(\frac{n}{4})]$
(vi)	$x \equiv 0 \pmod{4}, y \equiv 1 \pmod{2}$	$\frac{1}{48}[\sigma_3^*(n) - (3n - 2)\sigma_1^*(n) - 6n\sigma_1(\frac{n}{4})]$

TABLE 2. Formula for modulo 4 and 2

Theorem 1.3. For $n \in \mathbb{N}$, we have

(i)

$$\sum_{\substack{(a,b,x,y) \in \mathbb{N}^4, \\ ax+by=n, \\ x \equiv 0 \pmod{2}, \\ y \equiv 2 \pmod{4}}} ab = \frac{1}{48} \left(16\sigma_3^*(\frac{n}{2}) - 2(3n - 1)\sigma_1^*(\frac{n}{2}) - 3n\sigma_1(\frac{n}{2}) \right).$$

(ii)

$$\sum_{\substack{(a,b,x,y) \in \mathbb{N}^4, \\ ax+by=n, \\ x \equiv 0 \pmod{4}, \\ y \equiv 0 \pmod{2}}} ab = \frac{1}{24} \left(2\sigma_3(\frac{n}{2}) + 8\sigma_3(\frac{n}{4}) + \frac{1}{2}(2 - 3n)\sigma_1(\frac{n}{2}) + (1 - 3n)\sigma_1(\frac{n}{4}) \right).$$

Theorem 1.4. For $n \in \mathbb{N}$, we have

	$C(x, y) =$	$\sum_{\substack{(a,b,x,y) \in \mathbb{N}^4, \\ ax+by=n, \\ C(x,y)}} ab =$
(i)	$x \equiv 0 \pmod{8}, y \equiv 0 \pmod{4}$	$\frac{1}{96}[8\{\sigma_3(\frac{n}{4}) + 4\sigma_3(\frac{n}{8})\} - (3n - 4)\sigma_1(\frac{n}{4}) - 2(3n - 2)\sigma_1(\frac{n}{8})]$
(ii)	$x \equiv 4 \pmod{8}, y \equiv 0 \pmod{4}$	$\frac{1}{96}[32\sigma_3^*(\frac{n}{4}) - (9n - 4)\sigma_1^*(\frac{n}{4}) - 3n\sigma_1(\frac{n}{8})]$
(iii)	$x \equiv 0 \pmod{8}, y \equiv 0 \pmod{2}$	$\frac{1}{48}(\sigma_3(\frac{n}{2}) + (2 - \frac{3n}{2})\sigma_1(\frac{n}{2}) + 3\sigma_3(\frac{n}{4}) + 16\sigma_3(\frac{n}{8}) + (2 - 6n)\sigma_1(\frac{n}{8}))$
(iv)	$x \equiv 0 \pmod{8}, y \equiv 2 \pmod{4}$	$\frac{1}{96}\{2\sigma_3^*(\frac{n}{2}) - (3n - 4)\sigma_1^*(\frac{n}{2}) - 6n\sigma_1(\frac{n}{8})\}$
(v)	$x \equiv 4 \pmod{8}, y \equiv 0 \pmod{2}$	$\frac{1}{96}\{6\sigma_3^*(\frac{n}{2}) + 32\sigma_3^*(\frac{n}{4}) - 3n\sigma_1(\frac{n}{2}) - 4(3n - 1)\sigma_1^*(\frac{n}{4})\}$
(vi)	$x \equiv 4 \pmod{8}, y \equiv 2 \pmod{4}$	$\frac{1}{32}\{2\sigma_3^*(\frac{n}{2}) - n\sigma_1^*(\frac{n}{2}) - 2n\sigma_1(\frac{n}{4})\}$

TABLE 3. Formula for modulo 8, 4 and 2

Theorem 1.5. For $n \in \mathbb{N}$, we have

	$C(x, y) =$	$\sum_{\substack{(a,b,x,y) \in \mathbb{N}^4, \\ ax+by=n, \\ C(x,y)}} ab =$
(i)	$x \equiv 0 \pmod{8}, y \equiv 0 \pmod{8}$	$\frac{1}{48} \{20\sigma_3(\frac{n}{8}) - (3n-4)\sigma_1(\frac{n}{8})\}$
(ii)	$x \equiv 4 \pmod{8}, y \equiv 0 \pmod{8}$	$\frac{1}{96} \{8\sigma_3^*(\frac{n}{4}) + 4\sigma_1^*(\frac{n}{4}) - 3n\sigma_1(\frac{n}{4})\}$
(iii)	$x \equiv 4 \pmod{8}, y \equiv 4 \pmod{8}$	$\frac{1}{16} \{4\sigma_3^*(\frac{n}{4}) - n\sigma_1^*(\frac{n}{4})\}$
(iv)	$x \not\equiv 0 \pmod{8}, y \not\equiv 0 \pmod{8}$	$\frac{1}{32} [13\sigma_3^*(n) + 12\sigma_3^*(\frac{n}{2}) + 8\sigma_3^*(\frac{n}{4}) - 14n\{\sigma_1^*(n) + \sigma_1^*(\frac{n}{2}) + \sigma_1^*(\frac{n}{4})\} + c_8(n)]$
(v)	$x \equiv 0 \pmod{8}, y \not\equiv 0 \pmod{8}$	$\frac{1}{192} \{\sigma_3^*(n) + 4\sigma_3^*(\frac{n}{2}) + 16\sigma_3^*(\frac{n}{4}) - 2(3n-4)\sigma_1(n) - 4(9n+2)\sigma_1(\frac{n}{8}) - 3c_8(n)\}$

TABLE 4. Formula for modulo 8

Theorem 1.6. For $n \in \mathbb{N}$, we have

$$\sum_{\substack{(a,b,x,y) \in \mathbb{N}^4, \\ ax+by=n, \\ x \equiv 0 \pmod{8}, \\ y \equiv 1 \pmod{2}}} ab = \frac{1}{192} \{\sigma_3^*(n) - 2(3n-4)\sigma_1^*(n) - 24n\sigma_1(\frac{n}{8}) - 3c_8(n)\}.$$

As applications to the above theorems, we obtain explicit formulae for several twisted convolution sums (see Theorem 4.8 in section 4); we establish that the congruence relation

$$\sum_{k=1}^{2n-1} c_8(k)c_8(2n-k) \equiv \sigma_3^*(n) \pmod{8}$$

and the identity

$$\sum_{k=1}^{2n} c_8(k)c_8(2n+1-k) = 0$$

hold. We also prove several congruence relations in connection with the arithmetic function $c(n)$ where $c(n)$ is given by the expression :

$$\sum_{n=1}^{\infty} c(n)q^n := q \prod_{m=1}^{\infty} (1-q^m)^8 (1-q^{2m})^8 \quad \text{with } q \in \mathbb{C}, |q| < 1.$$

As an another interesting application, we obtain the identity :

$$r_8(4n) = 16 \{56\sigma_3^*(n) + 15\sigma_3(n)\}$$

which holds for all $n \in \mathbb{N}$.

For results closely related to the content of the paper, we refer to Glaisher [10, 11, 12]; B. C. Berndt [4]; H. Hahn [13]; J. G. Huard, Z. M. Ou, B. K. Spearman, and K. S. Williams [14]; G. Melfi [26]; B. Cho, D.

Kim, and J.-K. Koo [6, 7, 8]; D. Kim, A. Kim, and N. Y. Ikikardes [18]; D. Kim, M.-S. Kim [19]; and A. Alaca, S. Alaca, and K. S. Williams [2, 3].

2. A more general Liouville sum

In 2000 Huard, Ou, Spearman and Williams proved a far reaching generalization of Liouville's identity. We state this generalization below as :

Proposition 2.1. *Let $f : \mathbb{Z}^4 \rightarrow \mathbb{C}$ be such that*

$$(3) \quad f(a, b, x, y) - f(x, y, a, b) = f(-a, -b, x, y) - f(x, y, -a, -b)$$

for all integers a, b, x and y . For $n \in \mathbb{N}$, we have

$$\begin{aligned} & \sum_{\substack{(a,b,x,y) \in \mathbb{N}^4, \\ ax+by=n}} \{f(a, b, x, -y) - f(a, -b, x, y) + f(a, a-b, x+y, y) \\ & \quad - f(a, a+b, y-x, y) + f(b-a, b, x, x+y) - f(a+b, b, x, x-y)\} \\ &= \sum_{\substack{d \in \mathbb{N}, \\ d|n}} \sum_{\substack{x \in \mathbb{N}, \\ x < d}} \left\{ f(0, \frac{n}{d}, x, d) + f(\frac{n}{d}, 0, d, x) + f(\frac{n}{d}, \frac{n}{d}, d-x, -x) \right. \\ & \quad \left. - f(x, x-d, \frac{n}{d}, \frac{n}{d}) - f(x, d, 0, \frac{n}{d}) - f(d, x, \frac{n}{d}, 0) \right\}. \end{aligned}$$

Proof. See [31, p. 137]. □

3. Proof of theorems

series Proof of the Theorem 1.1. :

(i) We note that

$$\sum_{\substack{(a,b,x,y) \in \mathbb{N}^4, \\ ax+by=n, \\ x \equiv 0 \pmod{2}, \\ y \equiv 0 \pmod{2}}} ab = \sum_{\substack{(a,b,x',y') \in \mathbb{N}^4, \\ 2ax'+2by'=n}} ab = \sum_{\substack{(a,b,x',y') \in \mathbb{N}^4, \\ ax'+by'=\frac{n}{2}}} ab = A_1\left(\frac{n}{2}\right).$$

(ii) Let us take $f(a, b, x, y) = abF_2(x)$ in Proposition 2.1. Then (3) is satisfied, so the left hand side becomes

$$\begin{aligned}
& \sum_{\substack{(a,b,x,y) \in \mathbb{N}^4, \\ ax+by=n}} \{abF_2(x) + abF_2(x) + a(a-b)F_2(x+y) \\
& \quad - a(a+b)F_2(y-x) + (b-a)bF_2(x) - (a+b)bF_2(x)\} \\
(4) \quad &= \sum_{\substack{(a,b,x,y) \in \mathbb{N}^4, \\ ax+by=n}} \{-abF_2(x+y) - abF_2(y-x)\} \\
&= -2 \sum_{\substack{(a,b,x,y) \in \mathbb{N}^4, \\ ax+by=n, \\ x \equiv 0 \pmod{2}, \\ y \equiv 0 \pmod{2}}} ab - 2 \sum_{\substack{(a,b,x,y) \in \mathbb{N}^4, \\ ax+by=n, \\ x \not\equiv 0 \pmod{2}, \\ y \not\equiv 0 \pmod{2}}} ab.
\end{aligned}$$

We note that

$$\begin{aligned}
& \sum_{\substack{(a,b,x,y) \in \mathbb{N}^4, \\ ax+by=n, \\ x \not\equiv 0 \pmod{2}, \\ y \not\equiv 0 \pmod{2}}} ab = \sum_{\substack{(a,b,x,y) \in \mathbb{N}^4, \\ ax+by=n}} ab - \sum_{\substack{(a,b,x,y) \in \mathbb{N}^4, \\ ax+by=n, \\ x \equiv 0 \pmod{2}, \\ y \equiv 0 \pmod{2}}} ab - \sum_{\substack{(a,b,x,y) \in \mathbb{N}^4, \\ ax+by=n, \\ x \not\equiv 0 \pmod{2}, \\ y \equiv 0 \pmod{2}}} ab \\
& \quad - \sum_{\substack{(a,b,x,y) \in \mathbb{N}^4, \\ ax+by=n, \\ x \equiv 0 \pmod{2}, \\ y \equiv 0 \pmod{2}}} ab \\
&= \sum_{\substack{(a,b,x,y) \in \mathbb{N}^4, \\ ax+by=n}} ab - \sum_{\substack{(a,b,x,y) \in \mathbb{N}^4, \\ ax+by=n, \\ x \equiv 0 \pmod{2}}} ab - \sum_{\substack{(a,b,x,y) \in \mathbb{N}^4, \\ ax+by=n, \\ x \not\equiv 0 \pmod{2}, \\ y \equiv 0 \pmod{2}}} ab \\
&= A_1(n) - A_2(n) - \sum_{\substack{(a,b,x,y) \in \mathbb{N}^4, \\ ax+by=n, \\ x \not\equiv 0 \pmod{2}, \\ y \equiv 0 \pmod{2}}} ab.
\end{aligned}$$

Therefore the right hand side of (4) is equal to

$$(5) \quad 2 \sum_{\substack{(a,b,x,y) \in \mathbb{N}^4, \\ ax+by=n, \\ x \not\equiv 0 \pmod{2}, \\ y \equiv 0 \pmod{2}}} ab - 2A_1\left(\frac{n}{2}\right) - 2A_1(n) + 2A_2(n).$$

However, the right hand side of Proposition 2.1 is

$$\begin{aligned}
(6) \quad & \sum_{\substack{d \in \mathbb{N}, \\ d|n}} \sum_{\substack{x \in \mathbb{N}, \\ x < d}} \left(\left(\frac{n}{d} \right)^2 F_2(d-x) - x(x-d) F_2 \left(\frac{n}{d} \right) - x d F_2(0) - d x F_2 \left(\frac{n}{d} \right) \right) \\
&= \sum_{\substack{d \in \mathbb{N}, \\ d|n}} \sum_{\substack{x \in \mathbb{N}, \\ x < d}} \left(\left(\frac{n}{d} \right)^2 F_2(d-x) - x^2 F_2 \left(\frac{n}{d} \right) - x d \right)
\end{aligned}$$

since 0 is divisible by 2 and it applies that $F_2(0) = 1$. Then the first sum in (6) is

$$\begin{aligned}
(7) \quad & \sum_{\substack{d \in \mathbb{N}, \\ d|n}} \sum_{\substack{x \in \mathbb{N}, \\ x < d}} \left(\frac{n}{d} \right)^2 F_2(d-x) \\
&= \sum_{\substack{d|n, \\ 2|d}} \left(\frac{n}{d} \right)^2 \sum_{x=1}^{d-1} F_2(d-x) + \sum_{\substack{d|n, \\ 2 \nmid d}} \left(\frac{n}{d} \right)^2 \sum_{x=1}^{d-1} F_2(d-x) \\
&= \sum_{\substack{d|n, \\ 2|d}} \left(\frac{n}{d} \right)^2 \sum_{\substack{x=1, \\ 2|x}}^{d-1} 1 + \sum_{\substack{d|n, \\ 2 \nmid d}} \left(\frac{n}{d} \right)^2 \sum_{\substack{x=1, \\ 2 \nmid x}}^{d-1} 1 \\
&= \sum_{\substack{d|n, \\ 2|d}} \left(\frac{n}{d} \right)^2 \left(\frac{d}{2} - 1 \right) + \sum_{\substack{d|n, \\ 2 \nmid d}} \left(\frac{n}{d} \right)^2 \left(\frac{d-1}{2} \right).
\end{aligned}$$

Writing $e = \frac{n}{d}$, then the right hand side of (7) is

$$\begin{aligned}
&= \sum_{\substack{\frac{n}{e}|n, \\ 2|\frac{n}{e}}} e^2 \left(\frac{1}{2} \cdot \frac{n}{e} - 1 \right) + \frac{1}{2} \sum_{\substack{\frac{n}{e}|n, \\ 2 \nmid \frac{n}{e}}} e^2 \left(\frac{n}{e} - 1 \right) \\
&= \frac{1}{2} \sum_{\substack{\frac{n}{e}|n, \\ 2|\frac{n}{e}}} n e - \sum_{\substack{\frac{n}{e}|n, \\ 2|\frac{n}{e}}} e^2 + \frac{1}{2} \left\{ \sum_{\frac{n}{e}|n} (n e - e^2) - \sum_{\substack{\frac{n}{e}|n, \\ 2|\frac{n}{e}}} (n e - e^2) \right\} \\
&= \frac{1}{2} \left\{ n \sum_{e|n} e - \sum_{e|n} e^2 - \sum_{e|\frac{n}{2}} e^2 \right\} \\
&= \frac{1}{2} \left\{ n \sigma_1(n) - \sigma_2(n) - \sigma_2 \left(\frac{n}{2} \right) \right\}.
\end{aligned}$$

The second sum in (6) is

$$\begin{aligned}
 \sum_{\substack{d \in \mathbb{N}, \\ d|n}} \sum_{\substack{x \in \mathbb{N}, \\ x < d}} x^2 F_2\left(\frac{n}{d}\right) &= \sum_{d|n} F_2\left(\frac{n}{d}\right) \sum_{x=1}^{d-1} x^2 \\
 &= \sum_{d|n} F_2\left(\frac{n}{d}\right) \cdot \frac{(d-1)d(2d-1)}{6} \\
 &= \frac{1}{6} \sum_{d|\frac{n}{2}} (2d^3 - 3d^2 + d) \\
 &= \frac{1}{3} \sigma_3\left(\frac{n}{2}\right) - \frac{1}{2} \sigma_2\left(\frac{n}{2}\right) + \frac{1}{6} \sigma_1\left(\frac{n}{2}\right).
 \end{aligned}$$

Finally, the third sum in (6) is

$$\begin{aligned}
 \sum_{\substack{d \in \mathbb{N}, \\ d|n}} \sum_{\substack{x \in \mathbb{N}, \\ x < d}} xd &= \sum_{d|n} d \sum_{x=1}^{d-1} x = \sum_{d|n} d \cdot \frac{(d-1)d}{2} = \frac{1}{2} \sum_{d|n} (d^3 - d^2) \\
 &= \frac{1}{2} \{\sigma_3(n) - \sigma_2(n)\}.
 \end{aligned}$$

Therefore the right hand side of (6) is

$$(8) \quad \frac{n}{2} \sigma_1(n) - \frac{1}{3} \sigma_3\left(\frac{n}{2}\right) - \frac{1}{6} \sigma_1\left(\frac{n}{2}\right) - \frac{1}{2} \sigma_3(n).$$

Equating (5) with (8) we obtain the formula
$$\sum_{\substack{(a,b,x,y) \in \mathbb{N}^4, \\ ax+by=n, \\ x \not\equiv 0 \pmod{2}, \\ y \equiv 0 \pmod{2}}} ab.$$

For the proof of
$$\sum_{\substack{(a,b,x,y) \in \mathbb{N}^4, \\ ax+by=n, \\ x \equiv 0 \pmod{2}, \\ y \not\equiv 0 \pmod{2}}} ab,$$
 we observe that

$$\begin{aligned}
 \sum_{\substack{(a,b,x,y) \in \mathbb{N}^4, \\ ax+by=n, \\ x \equiv 0 \pmod{2}, \\ y \not\equiv 0 \pmod{2}}} ab &= \sum_{\substack{(a,b,x,y) \in \mathbb{N}^4, \\ bx+ay=n, \\ x \not\equiv 0 \pmod{2}, \\ y \equiv 0 \pmod{2}}} ba &= \sum_{\substack{(a,b,x,y) \in \mathbb{N}^4, \\ ax+by=n, \\ x \not\equiv 0 \pmod{2}, \\ y \equiv 0 \pmod{2}}} ab.
 \end{aligned}$$

(iii) The sum $\sum_{\substack{(a,b,x,y) \in \mathbb{N}^4, \\ ax+by=n, \\ x \not\equiv 0 \pmod{2}, \\ y \not\equiv 0 \pmod{2}}} ab$ can be separated as

$$\sum_{\substack{(a,b,x,y) \in \mathbb{N}^4, \\ ax+by=n, \\ x \not\equiv 0 \pmod{2}, \\ y \not\equiv 0 \pmod{2}}} ab = \sum_{\substack{(a,b,x,y) \in \mathbb{N}^4, \\ ax+by=n}} ab - \left\{ \sum_{\substack{(a,b,x,y) \in \mathbb{N}^4, \\ ax+by=n, \\ x \equiv 0 \pmod{2}, \\ y \equiv 0 \pmod{2}}} ab + \sum_{\substack{(a,b,x,y) \in \mathbb{N}^4, \\ ax+by=n, \\ x \not\equiv 0 \pmod{2}, \\ y \equiv 0 \pmod{2}}} ab + \sum_{\substack{(a,b,x,y) \in \mathbb{N}^4, \\ ax+by=n, \\ x \equiv 0 \pmod{2}, \\ y \not\equiv 0 \pmod{2}}} ab \right\}.$$

Thus we use (i), (ii) of Theorem 1.1. $\square$

Proof of the Theorem 1.2. :

(i) We can deduce that

$$\sum_{\substack{(a,b,x,y) \in \mathbb{N}^4, \\ ax+by=n, \\ x \equiv 0 \pmod{4}, \\ y \equiv 0 \pmod{4}}} ab = \sum_{\substack{(a,b,x',y') \in \mathbb{N}^4, \\ 4ax'+4by'=n}} ab = \sum_{\substack{(a,b,x',y') \in \mathbb{N}^4, \\ ax'+by'=\frac{n}{4}}} ab = A_1\left(\frac{n}{4}\right).$$

(ii) We choose $f(a, b, x, y) = abF_2(x)F_4(y)$ in Proposition 2.1, and note that the condition (3) is satisfied. So the left hand side of Proposition 2.1 becomes (since $F_4(y) = F_4(-y)$),

$$\begin{aligned} (9) \quad & \sum_{\substack{(a,b,x,y) \in \mathbb{N}^4, \\ ax+by=n}} (2abF_2(x)F_4(y) + a(a-b)F_2(x+y)F_4(y) - a(a+b)F_2(y-x)F_4(y) \\ & + (b-a)bF_2(x)F_4(x+y) - (a+b)bF_2(x)F_4(x-y)) \\ = & \sum_{\substack{(a,b,x,y) \in \mathbb{N}^4, \\ ax+by=n, \\ x \equiv 0 \pmod{2}, \\ y \equiv 0 \pmod{2}}} \{2abF_4(y) + a(a-b)F_4(y) - a(a+b)F_4(y) \\ & + (b-a)bF_4(x+y) - (a+b)bF_4(x-y)\} \\ = & -2 \sum_{\substack{(a,b,x,y) \in \mathbb{N}^4, \\ ax+by=n, \\ x \equiv 0 \pmod{4}, \\ y \equiv 0 \pmod{4}}} ab - 2 \sum_{\substack{(a,b,x,y) \in \mathbb{N}^4, \\ ax+by=n, \\ x \equiv 2 \pmod{4}, \\ y \equiv 2 \pmod{4}}} ab. \end{aligned}$$

When $x \equiv 1 \pmod{2}$ and $y \equiv 1 \pmod{2}$, note that we have used $F_4(y) = 0$ and $F_2(x) = 0$.

Then the left part of this proof is similar to the previous (ii) of Theorem 1.1.

(iii) We observe that

$$\sum_{\substack{(a,b,x,y) \in \mathbb{N}^4, \\ ax+by=n, \\ x \equiv 0 \pmod{2}, \\ y \equiv 0 \pmod{2}}} ab = \left\{ \sum_{\substack{(a,b,x,y) \in \mathbb{N}^4, \\ ax+by=n, \\ x \equiv 0 \pmod{4}, \\ y \equiv 0 \pmod{4}}} + \sum_{\substack{(a,b,x,y) \in \mathbb{N}^4, \\ ax+by=n, \\ x \equiv 0 \pmod{4}, \\ y \equiv 2 \pmod{4}}} + \sum_{\substack{(a,b,x,y) \in \mathbb{N}^4, \\ ax+by=n, \\ x \equiv 2 \pmod{4}, \\ y \equiv 0 \pmod{4}}} + \sum_{\substack{(a,b,x,y) \in \mathbb{N}^4, \\ ax+by=n, \\ x \equiv 2 \pmod{4}, \\ y \equiv 2 \pmod{4}}} \right\} ab.$$

Thus

$$\sum_{\substack{(a,b,x,y) \in \mathbb{N}^4, \\ ax+by=n, \\ x \equiv 0 \pmod{2}, \\ y \equiv 2 \pmod{4}}} ab = \frac{1}{2} \left\{ \sum_{\substack{(a,b,x,y) \in \mathbb{N}^4, \\ ax+by=n, \\ x \equiv 0 \pmod{2}, \\ y \equiv 0 \pmod{2}}} ab - \sum_{\substack{(a,b,x,y) \in \mathbb{N}^4, \\ ax+by=n, \\ x \equiv 0 \pmod{4}, \\ y \equiv 0 \pmod{4}}} ab - \sum_{\substack{(a,b,x,y) \in \mathbb{N}^4, \\ ax+by=n, \\ x \equiv 2 \pmod{4}, \\ y \equiv 2 \pmod{4}}} ab \right\}.$$

So we refer to (i) of Theorem 1.1, (i) and (ii) of Theorem 1.2.

(iv) We have

$$\begin{aligned} \sum_{\substack{(a,b,x,y) \in \mathbb{N}^4, \\ ax+by=n, \\ x \not\equiv 0 \pmod{4}, \\ y \not\equiv 0 \pmod{4}}} ab &= \sum_{k=1}^{n-1} \left(\sum_{\substack{a|k, \\ 4 \nmid \frac{k}{a}}} a \right) \left(\sum_{\substack{b|n-k, \\ 4 \nmid (\frac{n-k}{b})}} b \right) \\ &= \sum_{k=1}^{n-1} \left(\sum_{a|k} a - \sum_{\substack{a|k, \\ 4 \mid \frac{k}{a}}} a \right) \left(\sum_{b|n-k} b - \sum_{\substack{b|n-k, \\ 4 \mid (\frac{n-k}{b})}} b \right) \\ &= \sum_{k=1}^{n-1} \left(\sigma_1(k) - \sigma_1\left(\frac{k}{4}\right) \right) \left(\sigma_1(n-k) - \sigma_1\left(\frac{n-k}{4}\right) \right) \\ &= \sum_{k=1}^{n-1} \sigma_1(k) \sigma_1(n-k) - 2 \sum_{k < \frac{n}{4}} \sigma_1(k) \sigma_1(n-4k) + \sum_{k < \frac{n}{4}} \sigma_1(k) \sigma_1\left(\frac{n}{4} - k\right) \\ &= A_1(n) - 2A_4(n) + A_1\left(\frac{n}{4}\right) \end{aligned}$$

and we use

$$\begin{aligned} A_4(n) &= \sum_{k < n/4} \sigma(k) \sigma(n-4k) \\ (10) \quad &= \frac{1}{48} \left(\sigma_3(n) + (2-3n) \sigma_1(n) + 3 \sigma_3\left(\frac{n}{2}\right) + 16 \sigma_3\left(\frac{n}{4}\right) \right. \\ &\quad \left. + (2-12n) \sigma_1\left(\frac{n}{4}\right) \right) \end{aligned}$$

from [14, Theorem 4].

(v) Since

$$\sum_{ax+by=n} ab = \sum_{\substack{(a,b,x,y) \in \mathbb{N}^4, \\ ax+by=n, \\ x \equiv 0 \pmod{4}, \\ y \equiv 0 \pmod{4}}} ab + 2 \sum_{\substack{(a,b,x,y) \in \mathbb{N}^4, \\ ax+by=n, \\ x \equiv 0 \pmod{4}, \\ y \not\equiv 0 \pmod{4}}} ab + \sum_{\substack{(a,b,x,y) \in \mathbb{N}^4, \\ ax+by=n, \\ x \not\equiv 0 \pmod{4}, \\ y \not\equiv 0 \pmod{4}}} ab,$$

we obtain

$$\sum_{\substack{(a,b,x,y) \in \mathbb{N}^4, \\ ax+by=n, \\ x \equiv 0 \pmod{4}, \\ y \not\equiv 0 \pmod{4}}} ab = \frac{1}{2} \{A_1(n) - A_1\left(\frac{n}{4}\right) - \sum_{\substack{(a,b,x,y) \in \mathbb{N}^4, \\ ax+by=n, \\ x \not\equiv 0 \pmod{4}, \\ y \not\equiv 0 \pmod{4}}} ab\}.$$

Now, we use (i) and (iv) of Theorem 1.2 to obtain the result.

(vi) Let us consider

$$\begin{aligned} \sum_{\substack{(a,b,x,y) \in \mathbb{N}^4, \\ ax+by=n, \\ x \equiv 0 \pmod{4}, \\ y \not\equiv 0 \pmod{4}}} ab &= \sum_{\substack{(a,b,x,y) \in \mathbb{N}^4, \\ ax+by=n, \\ x \equiv 0 \pmod{4}, \\ y \equiv 1 \pmod{4}}} ab + \sum_{\substack{(a,b,x,y) \in \mathbb{N}^4, \\ ax+by=n, \\ x \equiv 0 \pmod{4}, \\ y \equiv 2 \pmod{4}}} ab + \sum_{\substack{(a,b,x,y) \in \mathbb{N}^4, \\ ax+by=n, \\ x \equiv 0 \pmod{4}, \\ y \equiv 3 \pmod{4}}} ab \\ &= \sum_{\substack{(a,b,x,y) \in \mathbb{N}^4, \\ ax+by=n, \\ x \equiv 0 \pmod{4}, \\ y \equiv 1 \pmod{2}}} ab + \sum_{\substack{(a,b,x,y) \in \mathbb{N}^4, \\ ax+by=n, \\ x \equiv 0 \pmod{4}, \\ y \equiv 2 \pmod{4}}} ab. \end{aligned}$$

Note that both the conditions $y \equiv 1 \pmod{4}$ and $y \equiv 3 \pmod{4}$ are merged in $y \equiv 1 \pmod{2}$. Then we use (iii) and (v) of Theorem 1.2 to obtain the result. $\square$

Proof of the Theorem 1.3. :

(i) We note that

$$\sum_{\substack{(a,b,x,y) \in \mathbb{N}^4, \\ ax+by=n, \\ x \equiv 0 \pmod{2}, \\ y \equiv 2 \pmod{4}}} ab = \sum_{\substack{(a,b,x,y) \in \mathbb{N}^4, \\ ax+by=n, \\ x \equiv 2 \pmod{4}, \\ y \equiv 2 \pmod{4}}} ab + \sum_{\substack{(a,b,x,y) \in \mathbb{N}^4, \\ ax+by=n, \\ x \equiv 0 \pmod{4}, \\ y \equiv 2 \pmod{4}}} ab.$$

Therefore (i) follows from (ii) and (iii) of Theorem 1.2.

(ii) Follows in a similar fashion using (i) and (iii) of Theorem 1.2. $\square$

Proof of the Theorem 1.4. :

(i) It is easily checked that

$$\begin{aligned} \sum_{\substack{(a,b,x,y) \in \mathbb{N}^4, \\ ax+by=n, \\ x \equiv 0 \pmod{8}, \\ y \equiv 0 \pmod{4}}} ab &= \sum_{\substack{(a,b,x,y) \in \mathbb{N}^4, \\ 8ax+4by=n}} ab = \sum_{\substack{(a,b,x,y) \in \mathbb{N}^4, \\ 2ax+by=\frac{n}{4}}} ab \\ &= \sum_{k < \frac{n}{8}} \left(\sum_{a|k} a \right) \left(\sum_{b | (\frac{n}{4}-2k)} b \right) = \sum_{k < \frac{n}{8}} \sigma_1(k) \sigma_1 \left(\frac{n}{4} - 2k \right) = A_2 \left(\frac{n}{4} \right). \end{aligned}$$

(ii) Since

$$\sum_{\substack{(a,b,x,y) \in \mathbb{N}^4, \\ ax+by=n, \\ x \equiv 0 \pmod{4}, \\ y \equiv 0 \pmod{4}}} ab = \sum_{\substack{(a,b,x,y) \in \mathbb{N}^4, \\ ax+by=n, \\ x \equiv 4 \pmod{8}, \\ y \equiv 0 \pmod{4}}} ab + \sum_{\substack{(a,b,x,y) \in \mathbb{N}^4, \\ ax+by=n, \\ x \equiv 0 \pmod{8}, \\ y \equiv 0 \pmod{4}}} ab,$$

we use (i) of Theorem 1.4 and (i) of Theorem 1.2 to obtain the result.

(iii) We write

$$\begin{aligned} \sum_{\substack{(a,b,x,y) \in \mathbb{N}^4, \\ ax+by=n, \\ x \equiv 0 \pmod{8}, \\ y \equiv 0 \pmod{2}}} ab &= \sum_{\substack{(a,b,x,y) \in \mathbb{N}^4, \\ 8ax+2by=n}} ab = \sum_{\substack{(a,b,x,y) \in \mathbb{N}^4, \\ 4ax+by=\frac{n}{2}}} ab \\ &= \sum_{k < \frac{n}{8}} \left(\sum_{a|k} a \right) \left(\sum_{b | (\frac{n}{2}-4k)} b \right) = \sum_{k < \frac{n}{8}} \sigma_1(k) \sigma_1 \left(\frac{n}{2} - 4k \right) = A_4 \left(\frac{n}{2} \right), \end{aligned}$$

and so we use (10) to obtain the result.

(iv) Note that

$$\sum_{\substack{(a,b,x,y) \in \mathbb{N}^4, \\ ax+by=n, \\ x \equiv 0 \pmod{8}, \\ y \equiv 0 \pmod{2}}} ab = \sum_{\substack{(a,b,x,y) \in \mathbb{N}^4, \\ ax+by=n, \\ x \equiv 0 \pmod{8}, \\ y \equiv 0 \pmod{4}}} ab + \sum_{\substack{(a,b,x,y) \in \mathbb{N}^4, \\ ax+by=n, \\ x \equiv 0 \pmod{8}, \\ y \equiv 2 \pmod{4}}} ab,$$

and hence, we use (i) and (iii) of Theorem 1.4 to obtain the result.

(v) We observe that

$$\sum_{\substack{(a,b,x,y) \in \mathbb{N}^4, \\ ax+by=n, \\ x \equiv 0 \pmod{4}, \\ y \equiv 0 \pmod{2}}} ab = \sum_{\substack{(a,b,x,y) \in \mathbb{N}^4, \\ ax+by=n, \\ x \equiv 0 \pmod{8}, \\ y \equiv 0 \pmod{2}}} ab + \sum_{\substack{(a,b,x,y) \in \mathbb{N}^4, \\ ax+by=n, \\ x \equiv 4 \pmod{8}, \\ y \equiv 0 \pmod{2}}} ab,$$

and so we use (ii) of Theorem 1.3 and (iii) of Theorem 1.4 to obtain the result.

(vi) Since

$$\sum_{\substack{(a,b,x,y) \in \mathbb{N}^4, \\ ax+by=n, \\ x \equiv 4 \pmod{8}, \\ y \equiv 0 \pmod{2}}} ab = \sum_{\substack{(a,b,x,y) \in \mathbb{N}^4, \\ ax+by=n, \\ x \equiv 4 \pmod{8}, \\ y \equiv 0 \pmod{4}}} ab + \sum_{\substack{(a,b,x,y) \in \mathbb{N}^4, \\ ax+by=n, \\ x \equiv 4 \pmod{8}, \\ y \equiv 2 \pmod{4}}} ab,$$

the result follows from (ii) and (v) of Theorem 1.4. $\square$

Proof of the Theorem 1.5 :

(i) Follows from

$$\sum_{\substack{(a,b,x,y) \in \mathbb{N}^4, \\ ax+by=n, \\ x \equiv 0 \pmod{8}, \\ y \equiv 0 \pmod{8}}} ab = \sum_{\substack{(a,b,x,y) \in \mathbb{N}^4, \\ 8ax+8by=n}} ab = \sum_{\substack{(a,b,x,y) \in \mathbb{N}^4, \\ ax+by=\frac{n}{8}}} ab = A_1\left(\frac{n}{8}\right).$$

(ii) We subtract (i) of Theorem 1.5 from (i) of Theorem 1.4. That is,

$$\sum_{\substack{(a,b,x,y) \in \mathbb{N}^4, \\ ax+by=n, \\ x \equiv 4 \pmod{8}, \\ y \equiv 0 \pmod{8}}} ab = \sum_{\substack{(a,b,x,y) \in \mathbb{N}^4, \\ ax+by=n, \\ x \equiv 0 \pmod{8}, \\ y \equiv 4 \pmod{8}}} ab = \sum_{\substack{(a,b,x,y) \in \mathbb{N}^4, \\ ax+by=n, \\ x \equiv 0 \pmod{8}, \\ y \equiv 0 \pmod{4}}} ab - \sum_{\substack{(a,b,x,y) \in \mathbb{N}^4, \\ ax+by=n, \\ x \equiv 0 \pmod{8}, \\ y \equiv 0 \pmod{8}}} ab.$$

(iii) We subtract (ii) of Theorem 1.5 from (ii) of Theorem 1.4. That is,

$$\sum_{\substack{(a,b,x,y) \in \mathbb{N}^4, \\ ax+by=n, \\ x \equiv 4 \pmod{8}, \\ y \equiv 4 \pmod{8}}} ab = \sum_{\substack{(a,b,x,y) \in \mathbb{N}^4, \\ ax+by=n, \\ x \equiv 4 \pmod{8}, \\ y \equiv 0 \pmod{4}}} ab - \sum_{\substack{(a,b,x,y) \in \mathbb{N}^4, \\ ax+by=n, \\ x \equiv 4 \pmod{8}, \\ y \equiv 0 \pmod{8}}} ab.$$

(iv) We have

$$\begin{aligned}
\sum_{\substack{(a,b,x,y) \in \mathbb{N}^4, \\ ax+by=n, \\ x \not\equiv 0 \pmod{8}, \\ y \not\equiv 0 \pmod{8}}} ab &= \sum_{k=1}^{n-1} \left(\sum_{\substack{a|k, \\ 8 \nmid \frac{k}{a}}} a \right) \left(\sum_{\substack{b|(n-k), \\ 8 \nmid (\frac{n-k}{b})}} b \right) \\
&= \sum_{k=1}^{n-1} \left(\sum_{a|k} a - \sum_{\substack{a|k, \\ 8 \mid \frac{k}{a}}} a \right) \left(\sum_{b|(n-k)} b - \sum_{\substack{b|(n-k), \\ 8 \mid (\frac{n-k}{b})}} b \right) \\
&= \sum_{k=1}^{n-1} \left(\sigma_1(k) - \sigma_1\left(\frac{k}{8}\right) \right) \left(\sigma_1(n-k) - \sigma_1\left(\frac{n-k}{8}\right) \right) \\
&= \sum_{k=1}^{n-1} \sigma_1(k) \sigma_1(n-k) - 2 \sum_{k < \frac{n}{8}} \sigma_1(k) \sigma_1(n-8k) + \sum_{k < \frac{n}{8}} \sigma_1(k) \sigma_1\left(\frac{n}{8} - k\right) \\
&= A_1(n) - 2A_8(n) + A_1\left(\frac{n}{8}\right).
\end{aligned}$$

Thus we use the formula for $A_1(n)$ and

$$\begin{aligned}
A_8(n) &:= \sum_{k < n/8} \sigma_1(k) \sigma_1(n-8k) = \frac{1}{192} \sigma_3(n) + \frac{1}{64} \sigma_3\left(\frac{n}{2}\right) + \frac{1}{16} \sigma_3\left(\frac{n}{4}\right) \\
&\quad + \frac{1}{3} \sigma_3\left(\frac{n}{8}\right) + \left(\frac{1}{24} - \frac{n}{32}\right) \sigma_1(n) + \left(\frac{1}{24} - \frac{n}{4}\right) \sigma_1\left(\frac{n}{8}\right) - \frac{1}{64} c_8(n)
\end{aligned}$$

from [3, (1.1)].

(v) Note that

$$\sum_{\substack{(a,b,x,y) \in \mathbb{N}^4, \\ ax+by=n, \\ x \equiv 0 \pmod{8}, \\ y \not\equiv 0 \pmod{8}}} ab = \frac{1}{2} \{ A_1(n) - A_1\left(\frac{n}{8}\right) - \sum_{\substack{(a,b,x,y) \in \mathbb{N}^4, \\ ax+by=n, \\ x \not\equiv 0 \pmod{8}, \\ y \equiv 0 \pmod{8}}} ab \},$$

and hence the result follows from (iv) of Theorem 1.5 and the formula for $A_1(n)$. $\square$

Proof of the Theorem 1.6. :

We subtract (iv) of Theorem 1.4 and (ii) of Theorem 1.5 from (v) of Theorem 1.5, since note that

$$\begin{aligned}
 \sum_{\substack{(a,b,x,y) \in \mathbb{N}^4, \\ ax+by=n, \\ x \equiv 0 \pmod{8}, \\ y \equiv 1 \pmod{2}}} ab &= \sum_{\substack{(a,b,x,y) \in \mathbb{N}^4, \\ ax+by=n, \\ x \equiv 0 \pmod{8}, \\ y \not\equiv 0 \pmod{8}}} ab - \sum_{\substack{(a,b,x,y) \in \mathbb{N}^4, \\ ax+by=n, \\ x \equiv 0 \pmod{8}, \\ y \equiv 2 \pmod{8}}} ab - \sum_{\substack{(a,b,x,y) \in \mathbb{N}^4, \\ ax+by=n, \\ x \equiv 0 \pmod{8}, \\ y \equiv 4 \pmod{8}}} ab - \sum_{\substack{(a,b,x,y) \in \mathbb{N}^4, \\ ax+by=n, \\ x \equiv 0 \pmod{8}, \\ y \equiv 6 \pmod{8}}} ab \\
 &= \sum_{\substack{(a,b,x,y) \in \mathbb{N}^4, \\ ax+by=n, \\ x \equiv 0 \pmod{8}, \\ y \not\equiv 0 \pmod{8}}} ab - \sum_{\substack{(a,b,x,y) \in \mathbb{N}^4, \\ ax+by=n, \\ x \equiv 0 \pmod{8}, \\ y \equiv 2 \pmod{8}}} ab - \sum_{\substack{(a,b,x,y) \in \mathbb{N}^4, \\ ax+by=n, \\ x \equiv 0 \pmod{8}, \\ y \equiv 4 \pmod{8}}} ab.
 \end{aligned}$$

□

4. Applications

4.1. Application to the Coefficient $c_8(n)$:

Lemma 4.1. *If $n = 2^m p_1^{e_1} \cdots p_r^{e_r}$ with $m = 1$ or 2 and odd prime p_i , then we have*

$$\sigma_3^*(n) \equiv 2(3n - 4)\sigma_1^*(n) \pmod{192}.$$

Proof. Since $m = 1$ or 2 , and $n = 2^m p_1^{e_1} \cdots p_r^{e_r}$, we note that $c_8(n) = 0$ and $\sigma_1\left(\frac{n}{8}\right) = 0$. Therefore from Corollary 1.6, we obtain

$$\begin{aligned}
 192 \sum_{\substack{(a,b,x,y) \in \mathbb{N}^4, \\ ax+by=n, \\ x \equiv 0 \pmod{8}, \\ y \equiv 1 \pmod{2}}} ab &= \sigma_3^*(n) - 2(3n - 4)\sigma_1^*(n).
 \end{aligned}$$

Thus the proof is complete. □

Lemma 4.2. *Let n be an odd positive integer. Then we have*

(i)

$$c_8(n) \equiv -13\sigma_3(n) + 14n\sigma_1(n) \pmod{64}.$$

(ii)

$$3c_8(n) \equiv \sigma_3(n) - 2(3n - 4)\sigma_1(n) \pmod{192}.$$

(iii)

$$c_8(n) \equiv \sigma_1(n) \pmod{8}.$$

In particular, if $p_i \equiv -1 \pmod{8}$ is a prime with $p_i^{2e_i-1} | n$ and $p_i^{2e_i} \nmid n$, then $c_8(n) \equiv 0 \pmod{8}$.

Proof. (i) For odd positive integer n , (iv) of Theorem 1.5 can be written as

$$32 \sum_{k=1}^{n-1} \sigma_1^*(k; 8) \sigma_1^*(n-k; 8) = 13\sigma_3(n) - 14n\sigma_1(n) + c_8(n).$$

Then, since $\sigma_1^*(k) \sigma_1^*(n-k) = \sigma_1^*(n-k) \sigma_1^*(n-(n-k))$, we have

$$64 \sum_{k=1}^{\frac{n-1}{2}} \sigma_1^*(k; 8) \sigma_1^*(n-k; 8) = 13\sigma_3(n) - 14n\sigma_1(n) + c_8(n).$$

Therefore $c_8(n) \equiv -13\sigma_3(n) + 14n\sigma_1(n) \pmod{64}$.

(ii) In a similar manner, (v) of Theorem 1.5 becomes

$$192 \sum_{k=1}^{n-1} \sigma_1(k; 8; 0) \sigma_1^*(n-k; 8) = \sigma_3(n) - 2(3n-4)\sigma_1(n) - 3c_8(n).$$

Thus $3c_8(n) \equiv \sigma_3(n) - 2(3n-4)\sigma_1(n) \pmod{192}$.

(iii) Multiplying (ii) by 13 and subtracting (i), we obtain

$$(11) \quad 64 \left\{ 39 \sum_{k=1}^{n-1} \sigma_1(k; 8; 0) \sigma_1^*(n-k; 8) - \sum_{k=1}^{\frac{n-1}{2}} \sigma_1^*(k; 8) \sigma_1^*(n-k; 8) \right\} \\ = (104 - 64n)\sigma_1(n) - 40c_8(n).$$

(11) implies that

$$5c_8(n) \equiv 5\sigma_1(n) \pmod{8}.$$

Since $(5, 8) = 1$, we have $c_8(n) \equiv \sigma_1(n) \pmod{8}$. $\square$

Remark 4.3. Glaisher [9, p. 300] and Ramanujan [29] proved that

$$(12) \quad \sigma(1)\sigma(2n-1) + \sigma(3)\sigma(2n-3) + \cdots + \sigma(2n-1)\sigma(1) = \frac{1}{8}[\sigma_3(2n) - \sigma_3(n)].$$

We note that

$$\sigma_s^*(n) = \sigma_s(n) - \sigma_s\left(\frac{n}{2}\right)$$

(see [31, Theorem 3.3]). Thus (12) becomes

$$\sigma(1)\sigma(2n-1) + \sigma(3)\sigma(2n-3) + \cdots + \sigma(2n-1)\sigma(1) \\ = \frac{1}{8}\sigma_3^*(2n) = \frac{1}{8} \sum_{\substack{d \in \mathbb{N}, \\ d|2n, \\ \frac{2n}{d} \text{ odd}}} d^3 = \frac{1}{8} \sum_{\substack{e \in \mathbb{N}, \\ e|n, \\ \frac{n}{e} \text{ odd}}} (2e)^3 = \frac{1}{8} \cdot 8 \sum_{\substack{e \in \mathbb{N}, \\ e|n, \\ \frac{n}{e} \text{ odd}}} e^3 = \sigma_3^*(n)$$

by letting $e = \frac{d}{2}$. From (iii) of Lemma 4.2 we have

$$(13) \quad c_8(1)c_8(2n-1) + c_8(3)c_8(2n-3) + \cdots + c_8(2n-1)c_8(1) \equiv \sigma_3^*(n) \pmod{8}.$$

4.2. Application Involving the Coefficient $D(n)$:

Theorem 4.4. *Let $\sum_{n=1}^{\infty} D(n)q^n = q^2 \prod_{m=1}^{\infty} (1 - q^{2m})^8 (1 - q^{4m})^8$. Then we have*

(i)

$$D(2n) \equiv \sigma_3^*(n) \pmod{8}.$$

(ii)

$$\sum_{k=1}^{2n} c_8(k)c_8(2n+1-k) = 0.$$

Proof. (i) Since $c_8(2k) = 0$, we have

$$\begin{aligned} D(2n) &= \sum_{k=1}^{2n-1} c_8(k)c_8(2n-k) \\ &= c_8(1)c_8(2n-1) + c_8(3)c_8(2n-3) + \cdots + c_8(2n-1)c_8(1) \\ &\equiv \sigma_3^*(n) \pmod{8} \end{aligned}$$

by (13).

(ii) We note that if k is even, then $c_8(k) = 0$ and if k is odd then $c_8(2n+1-k) = 0$. Therefore $c_8(k)c_8(2n+1-k) = 0$ for all k satisfying $1 \leq k \leq 2n$. Hence

$$\sum_{k=1}^{2n} c_8(k)c_8(2n+1-k) = 0. \quad \square$$

Let $\Lambda_\tau = \mathbb{Z} + \tau\mathbb{Z}$ ($\tau \in \mathcal{H}$ the complex upper half plane) be a lattice and $z \in \mathbb{C}$. The Weierstrass $\wp$ function related to Λ_τ is defined by the series

$$\wp(z; \Lambda_\tau) = \frac{1}{z^2} + \sum_{\substack{\omega \in \Lambda_\tau, \\ \omega \neq 0}} \left\{ \frac{1}{(z-\omega)^2} - \frac{1}{\omega^2} \right\},$$

and the Eisenstein series of weight $2k$ for Λ_τ for any integer $k \geq 2$ is given by the series

$$G_{2k}(\Lambda_\tau) = \sum_{\substack{\omega \in \Lambda_\tau, \\ \omega \neq 0}} \omega^{-2k}.$$

We use the notations $\wp(z)$ and G_{2k} instead of $\wp(z; \Lambda_\tau)$ and $G_{2k}(\Lambda_\tau)$, respectively, when the lattice Λ_τ has been fixed. Now, Laurent series for $\wp(z)$ about $z = 0$ is given by

$$\wp(z) = z^{-2} + \sum_{k=1}^{\infty} (2k+1)G_{2k+2}z^{2k}.$$

As is customary, by setting

$$g_2(\tau) = g_2(\Lambda_\tau) = 60G_4 \quad \text{and} \quad g_3(\tau) = g_3(\Lambda_\tau) = 140G_6,$$

the algebraic relation between $\wp(z)$ and $\wp'(z)$ becomes

$$\wp'(z)^2 = 4\wp(z)^3 - g_2(\tau)\wp(z) - g_3(\tau).$$

Corollary 4.5. Let $\chi(n) = \begin{cases} 0, & \text{for odd } n, \\ 1, & \text{for even } n. \end{cases}$ Then we have

$$\sum_{n=1}^{\infty} \chi(n) D(n) q^{\frac{n}{2}} \equiv \frac{\wp''(\frac{\tau}{2}, \tau)}{32\pi^4} \pmod{8}.$$

Proof. From [17, (28)], we see that

$$\begin{aligned} \wp''(\frac{\tau}{2}, \tau) &= 32\pi^4 q \frac{(q^2; q^2)_{\infty}^{16}}{(q)_{\infty}^8} \\ (14) \quad &= 32\pi^4 \left(\sum_{n=1}^{\infty} \sigma_3(n) q^n - \sum_{n=1}^{\infty} \sigma_3(n) q^{2n} \right) \\ &= 32\pi^4 \sum_{n=1}^{\infty} \sigma_3^*(n) q^n. \end{aligned}$$

Therefore, from (i) of Theorem 4.4, we obtain

$$\frac{\wp''(\frac{\tau}{2}, \tau)}{32\pi^4} = q \frac{(q^2; q^2)_{\infty}^{16}}{(q)_{\infty}^8} = \sum_{n=1}^{\infty} \sigma_3^*(n) q^n \equiv \sum_{n=1}^{\infty} \chi(n) D(n) q^{\frac{n}{2}} \pmod{8}. \quad \square$$

4.3. Application to the Function $r_8(4n)$:

The s squares problem is to count the number $r_s(n)$ of integer solutions $(x_1, \dots, x_s)$ of the Diophantine equation

$$(15) \quad x_1^2 + \dots + x_s^2 = n,$$

where changing the sign of the order of the x_i 's gives distinct solutions. Using the theory of theta functions, Jacobi studied the number $r_s(n)$

of integer solutions of Eq. (15) for some important cases. In fact, he considered the equation

$$(16) \quad \vartheta_3(0, -q)^s := 1 + \sum_{n=1}^{\infty} (-1)^n r_s(n) q^n,$$

where

$$\vartheta_3(z, q) := 1 + 2 \sum_{n=1}^{\infty} q^{n^2} \cos(2nz)$$

(See [30, p. 258]) and thus

$$\vartheta_3(0, q) = \sum_{j=-\infty}^{\infty} q^{j^2}.$$

Jacobi [15] also established that

$$(17) \quad \begin{aligned} \vartheta_3(0, -q)^4 &= 1 - 8 \sum_{r=1}^{\infty} (-1)^{r-1} \frac{rq^r}{1+q^r} \\ &= 1 + 8 \sum_{n=1}^{\infty} (-1)^n \left(\sum_{\substack{d|n, d>0, \\ 4 \nmid d}} d \right) q^n. \end{aligned}$$

Remark 4.6. From (16) and (17) we see that

$$(18) \quad r_4(n) = 8 \sum_{\substack{d|n, \\ 4 \nmid d}} d = 8 \left\{ \sum_{d|n} d - \sum_{\substack{d|n, \\ 4|d}} d \right\} = 8 \left\{ \sigma_1(n) - 4\sigma_1\left(\frac{n}{4}\right) \right\}.$$

Theorem 4.7. For $n \in \mathbb{N}$, we have

$$r_8(4n) = 16 \{56\sigma_3^*(n) + 15\sigma_3(n)\}.$$

Proof. By the property of $r_8(n)$ and (18), we can write

$$(19) \quad \begin{aligned} r_8(4n) &= \sum_{k=0}^{4n} r_4(k) r_4(4n-k) \\ &= 2r_4(4n) + \sum_{k=1}^{4n-1} r_4(k) r_4(4n-k) \\ &= 16 \{ \sigma_1(4n) - 4\sigma_1(n) \} \\ &\quad + 64 \sum_{k=1}^{4n-1} \left\{ \sigma_1(k) - 4\sigma_1\left(\frac{k}{4}\right) \right\} \left\{ \sigma_1(4n-k) - 4\sigma_1\left(\frac{4n-k}{4}\right) \right\}. \end{aligned}$$

Note that

$$\begin{aligned}
& \sum_{k=1}^{4n-1} \left\{ \sigma_1(k) - 4\sigma_1\left(\frac{k}{4}\right) \right\} \left\{ \sigma_1(4n-k) - 4\sigma_1\left(\frac{4n-k}{4}\right) \right\} \\
&= 4 \sum_{\substack{(a,b,x,y) \in \mathbb{N}^4, \\ ax+by=n, \\ x \not\equiv 0 \pmod{4}, \\ y \not\equiv 0 \pmod{4}}} ab - 3 \sum_{k=1}^{4n-1} \sigma_1(k) \sigma_1(4n-k) + 12 \sum_{k=1}^{n-1} \sigma_1(k) \sigma_1(n-k) \\
&= 4 \sum_{\substack{(a,b,x,y) \in \mathbb{N}^4, \\ ax+by=n, \\ x \not\equiv 0 \pmod{4}, \\ y \not\equiv 0 \pmod{4}}} ab - 3A_1(4n) + 12A_1(n) \\
&= \frac{1}{4} \{ \sigma_3(4n) - 2\sigma_3(2n) + 16\sigma_3(n) - \sigma_1(4n) + 4\sigma_1(n) \}.
\end{aligned}$$

Note that we use $A_1(n)$ as in Section 1 and we also use $\sigma_i^*(n) = \sigma_i(n) - \sigma_i(\frac{n}{2})$ along with (iv) of Theorem 1.2. Finally, to replace $4n$ and $2n$ into n in the last step, we use the equation $\sigma_k(pn) - (p^k + 1)\sigma_k(n) + p^k\sigma_k(\frac{n}{p}) = 0$ for any prime p and $k, n \in \mathbb{N}$. $\square$

4.4. Explicit Formulae for several Twisted Convolution Sums :

Theorem 4.8. Let $n \in \mathbb{N}$ and let $\sigma_1(n; u; r) := \sum_{\substack{d|n, \\ \frac{n}{d} \equiv r \pmod{u}}} d$ where u is the modulo and r is residue of modulo u . Note that $\sigma_1(n; 2; 1) = \sigma_1^*(n; 2) = \sigma_1^*(n)$. Then, we have

Convolution sums	Convolution formula
$\sum_{k=1}^{n-1} \sigma_1(k; 2; 0) \sigma_1(n-k; 2; 0)$	$\frac{1}{12} (5\sigma_3(\frac{n}{2}) + (1-3n)\sigma_1(\frac{n}{2}))$
$\sum_{k=1}^{n-1} \sigma_1^*(k; 2) \sigma_1(n-k; 2; 0)$	$\frac{1}{24} (2\sigma_3^*(n) + \sigma_1^*(n) - 3n\sigma_1(n))$
$\sum_{k=1}^{n-1} \sigma_1^*(k; 2) \sigma_1^*(n-k)$	$\frac{1}{4} (\sigma_3^*(n) - n\sigma_1^*(n))$
$\sum_{k=1}^{n-1} \sigma_1(k; 4; 0) \sigma_1(n-k; 4; 0)$	$\frac{1}{12} (5\sigma_3(\frac{n}{4}) + (1-\frac{3}{2}n)\sigma_1(\frac{n}{4}))$
$\sum_{k=1}^{n-1} \sigma_1(k; 4; 2) \sigma_1(n-k; 4; 2)$	$\frac{1}{8} (2\sigma_3^*(\frac{n}{2}) - n\sigma_1^*(\frac{n}{2}))$
$\sum_{k=1}^{n-1} \sigma_1(k; 4; 0) \sigma_1(n-k; 4; 2)$	$\frac{1}{24} \{2\sigma_3^*(\frac{n}{2}) + \sigma_1^*(\frac{n}{2}) - \frac{3}{2}n\sigma_1(\frac{n}{2})\}$
$\sum_{k=1}^{n-1} \sigma_1^*(k; 4) \sigma_1^*(n-k; 4)$	$\frac{1}{8} [3\sigma_3^*(n) + 2\sigma_3^*(\frac{n}{2}) - 3n\{\sigma_1^*(n) + \sigma_1^*(\frac{n}{2})\}]$
$\sum_{k=1}^{n-1} \sigma_1(k; 4; 0) \sigma_1^*(n-k; 4)$	$\frac{1}{48} [\sigma_3^*(n) + 4\sigma_3^*(\frac{n}{2}) - (3n-2)\{\sigma_1^*(n) + \sigma_1^*(\frac{n}{2})\} - 9n\sigma_1(\frac{n}{4})]$
$\sum_{k=1}^{n-1} \sigma_1(k; 4; 0) \sigma_1^*(n-k)$	$\frac{1}{48} [\sigma_3^*(n) - (3n-2)\sigma_1^*(n) - 6n\sigma_1(\frac{n}{4})]$
$\sum_{k=1}^{n-1} \sigma_1(k; 2; 0) \sigma_1(n-k; 4; 2)$	$\frac{1}{48} (16\sigma_3^*(\frac{n}{2}) - 2(3n-1)\sigma_1^*(\frac{n}{2}) - 3n\sigma_1(\frac{n}{2}))$
$\sum_{k=1}^{n-1} \sigma_1(k; 4; 0) \sigma_1(n-k; 2; 0)$	$\frac{1}{24} (2\sigma_3(\frac{n}{2}) + 8\sigma_3(\frac{n}{4}) + \frac{1}{2}(2-3n)\sigma_1(\frac{n}{2}) + (1-3n)\sigma_1(\frac{n}{4}))$
$\sum_{k=1}^{n-1} \sigma_1(k; 8; 0) \sigma_1(n-k; 4; 0)$	$\frac{1}{96} [8\{\sigma_3(\frac{n}{4}) + 4\sigma_3(\frac{n}{8})\} - (3n-4)\sigma_1(\frac{n}{4}) - 2(3n-2)\sigma_1(\frac{n}{8})]$
$\sum_{k=1}^{n-1} \sigma_1(k; 8; 4) \sigma_1(n-k; 4; 0)$	$\frac{1}{96} \{32\sigma_3^*(\frac{n}{4}) - (9n-4)\sigma_1^*(\frac{n}{4}) - 3n\sigma_1(\frac{n}{8})\}$
$\sum_{k=1}^{n-1} \sigma_1(k; 8; 0) \sigma_1(n-k; 2; 0)$	$\frac{1}{48} (\sigma_3(\frac{n}{2}) + (2-\frac{3n}{2})\sigma_1(\frac{n}{2}) + 3\sigma_3(\frac{n}{4}) + 16\sigma_3(\frac{n}{8}) + (2-6n)\sigma_1(\frac{n}{8}))$
$\sum_{k=1}^{n-1} \sigma_1(k; 8; 0) \sigma_1(n-k; 4; 2)$	$\frac{1}{96} \{2\sigma_3^*(\frac{n}{2}) - (3n-4)\sigma_1^*(\frac{n}{2}) - 6n\sigma_1(\frac{n}{8})\}$
$\sum_{k=1}^{n-1} \sigma_1(k; 8; 4) \sigma_1(n-k; 2; 0)$	$\frac{1}{96} \{6\sigma_3^*(\frac{n}{2}) + 32\sigma_3^*(\frac{n}{4}) - 3n\sigma_1(\frac{n}{2}) - 4(3n-1)\sigma_1^*(\frac{n}{4})\}$
$\sum_{k=1}^{n-1} \sigma_1(k; 8; 4) \sigma_1(n-k; 4; 2)$	$\frac{1}{32} \{2\sigma_3^*(\frac{n}{2}) - n\sigma_1^*(\frac{n}{2}) - 2n\sigma_1^*(\frac{n}{4})\}$
$\sum_{k=1}^{n-1} \sigma_1(k; 8; 0) \sigma_1(n-k; 8; 0)$	$\frac{1}{48} \{20\sigma_3(\frac{n}{8}) - (3n-4)\sigma_1(\frac{n}{8})\}$
$\sum_{k=1}^{n-1} \sigma_1(k; 8; 4) \sigma_1(n-k; 8; 0)$	$\frac{1}{96} \{8\sigma_3^*(\frac{n}{4}) + 4\sigma_1^*(\frac{n}{4}) - 3n\sigma_1(\frac{n}{4})\}$
$\sum_{k=1}^{n-1} \sigma_1(k; 8; 4) \sigma_1(n-k; 8; 4)$	$\frac{1}{16} \{4\sigma_3^*(\frac{n}{4}) - n\sigma_1^*(\frac{n}{4})\}$
$\sum_{k=1}^{n-1} \sigma_1^*(k; 8) \sigma_1^*(n-k; 8)$	$\frac{1}{32} [13\sigma_3^*(n) + 12\sigma_3^*(\frac{n}{2}) + 8\sigma_3^*(\frac{n}{4}) - 14n\{\sigma_1^*(n) + \sigma_1^*(\frac{n}{2}) + \sigma_1^*(\frac{n}{4})\} + c_8(n)]$
$\sum_{k=1}^{n-1} \sigma_1(k; 8; 0) \sigma_1^*(n-k; 8)$	$\frac{1}{192} \{\sigma_3^*(n) + 4\sigma_3^*(\frac{n}{2}) + 16\sigma_3^*(\frac{n}{4}) - 2(3n-4)\sigma_1(n) - 4(9n+2)\sigma_1(\frac{n}{8}) - 3c_8(n)\}$
$\sum_{k=1}^{n-1} \sigma_1(k; 8; 0) \sigma_1^*(n-k)$	$\frac{1}{192} \{\sigma_3^*(n) - 2(3n-4)\sigma_1^*(n) - 24n\sigma_1(\frac{n}{8}) - 3c_8(n)\}$

TABLE 5. Convolution sums

Proof. The proof follows from Theorems 1.1–1.6. $\square$

Corollary 4.9. For $n \in \mathbb{N}$, we have

$$\sigma_3^*(n) \equiv (3n-2)\sigma_1^*(n) \pmod{24}.$$

In particular, if $n \not\equiv 0 \pmod{4}$, then $\sigma_3^*(n) \equiv (3n-2)\sigma_1^*(n) \pmod{48}$.

Proof. From Table 5, we observe that

$$\sum_{k=1}^{n-1} \sigma_1(k; 4; 0) \sigma_1^*(n-k) = \frac{1}{48} \left[\sigma_3^*(n) - (3n-2)\sigma_1^*(n) - 6n\sigma_1(\frac{n}{4}) \right].$$

If $n \equiv 0 \pmod{4}$, i.e., $n = 4l$, then we have

$$48 \sum_{k=1}^{4l-1} \sigma_1(k; 4; 0) \sigma_1^*(4l - k) = \sigma_3^*(4l) - (12l - 2) \sigma_1^*(4l) - 24l \sigma_1(l).$$

Thus $\sigma_3^*(n) \equiv (3n - 2) \sigma_1^*(n) \pmod{24}$. If $n \not\equiv 0 \pmod{4}$ then we obtain

$$48 \sum_{k=1}^{n-1} \sigma_1(k; 4; 0) \sigma_1^*(n - k) = \sigma_3^*(n) - (3n - 2) \sigma_1^*(n).$$

So $\sigma_3^*(n) \equiv (3n - 2) \sigma_1^*(n) \pmod{48}$. $\square$

4.5. Convolution Sums and Congruences Involving the Arithmetic Function $c(n)$:

We denote by $\mathcal{H} = \{\tau \in \mathbb{C} \mid \text{Im}(\tau) > 0\}$ the complex upper half-plane and for $\tau \in \mathcal{H}$ write $q = e^{2\pi i \tau}$ and $Y = 4\pi \text{Im}(\tau)$, while D denotes the differential operator $D = \frac{1}{2\pi i} \frac{d}{d\tau} = q \frac{d}{dq}$. Recall that a modular form of weight k on a subgroup Γ of finite index of Γ_1 is a holomorphic function f on $\mathcal{H}$ satisfying

$$f\left(\frac{a\tau + b}{c\tau + d}\right) = (c\tau + d)^k f(\tau), \quad \forall \tau \in \mathcal{H}, \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$$

and growing at most polynomially in $1/Y$ as $Y \rightarrow 0$. If $\begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix} \in \Gamma$, then these conditions imply that f has a convergent Fourier series expansion namely, $f(\tau) = \sum_{n=0}^{\infty} a(n) q^{n/\lambda}$ at infinity. The space of holomorphic modular forms of weight k on Γ is denoted by $M_k(\Gamma)$ and the graded ring $\oplus_k M_k(\Gamma)$ by $M_*(\Gamma)$.

Resembling the holomorphic modular forms, there are also functions $F(\tau)$ which satisfy the same transformation properties and growth conditions as before but which belong to $\mathbb{C}[[q^{1/\lambda}]] [Y^{-1}]$ instead of $\mathbb{C}[[q^{1/\lambda}]]$, i.e. which have the form

$$(20) \quad F(\tau) = \sum_{m=0}^M f_m(\tau) Y^{-m} \quad (f_m(\tau) \text{ holomorphic for } m = 0, 1, \dots, M)$$

for some integer $M \geq 0$ (and necessarily $\leq k/2$). We call such a function an almost-holomorphic modular form of weight k and denote the vector space of them by $\widehat{M}_k(\Gamma)$, while the holomorphic function $f_0(\tau)$ obtained formally as the “constant term with respect to $1/Y$ ” of f will

be called a quasi-modular form of weight k and the space of such functions denoted by $\widetilde{M}_k(\Gamma)$. It is clear that the spaces $\widehat{M}_*(\Gamma) = \oplus \widehat{M}_k(\Gamma)$ and $\widetilde{M}_*(\Gamma) = \oplus \widetilde{M}_k(\Gamma)$ are graded rings and the map $\widehat{M}_*(\Gamma) \rightarrow \widetilde{M}_*(\Gamma)$ is a ring homomorphism (see [16]).

Let $\widehat{M}_*(\Gamma_1)$ denote the graded ring of quasimodular forms on the full modular group $\Gamma_1 = PSL(2, \mathbb{Z})$. This is the ring generated by G_2, G_4, G_6 and graded by assigning to each G_k the weight k , where

$$G_k(\tau) = -\frac{B_k}{2k} + \sum_{n=1}^{\infty} \left(\sum_{d|n} d^{k-1} \right) q^n,$$

$$(k = 2, 4, 6, \dots, \quad B_k = \text{k-th Bernoulli numbers}),$$

and let

$$G_k^*(\tau) = G_k(\tau) - G_k(2\tau)$$

are the Eisenstein series, all of which except G_2 are modular. If we consider the coefficient of q^n in G_k and G_k^* , then we determine the values of $\sigma_{k-1}(n)$.

Using the theory of quasi-modular forms stated above, we get two identities below in Lemma 4.11 of convolution sums involving divisor functions.

Proposition 4.10. *Let $\Gamma \subset SL_2(\mathbb{Z})$ be a subgroup of finite index of the full modular group. Let $\widetilde{M}_k(\Gamma)$ be the space of quasi modular forms of weight k on Γ . Then*

- (1) *For $f_1 \in \widetilde{M}_{k_1}(\Gamma)$ and $f_2 \in \widetilde{M}_{k_2}(\Gamma)$, $f_1 f_2 \in \widetilde{M}_{k_1+k_2}(\Gamma)$.*
- (2) *For even $k > 0$, $\widetilde{M}_k(\Gamma) = \oplus_{0 \leq i \leq k/2} D^i M_{k-2i}(\Gamma) \oplus < D^{k/2-1} G_2 >.$*

Proof. The first assertion is easy and for second one see Proposition 1 in [16]. $\square$

Lemma 4.11. *For any integer $n \geq 3$, we have*

(i)

$$\sum_{1 \leq k < n/2} k^2 \sigma_1(k) \sigma_1(n-2k) = \frac{1}{960} \{ 6n^2 \sigma_3(n) + 24n^2 \sigma_3\left(\frac{n}{2}\right) - 5n^3 \sigma_1(n) \\ - 30n^3 \sigma_1\left(\frac{n}{2}\right) + 10n^2 \sigma_1\left(\frac{n}{2}\right) - c(n) \},$$

where

$$\sum_{n=1}^{\infty} c(n) q^n := q \prod_{m=1}^{\infty} (1 - q^m)^8 (1 - q^{2m})^8$$

and
(ii)

$$\begin{aligned} & \sum_{1 \leq k < n/2} k\sigma_1(k)\sigma_1(n-2k) \\ &= \frac{1}{48}n \left\{ \sigma_3(n) + 4\sigma_3\left(\frac{n}{2}\right) - n\sigma_1(n) - 4n\sigma_1\left(\frac{n}{2}\right) + \sigma_1\left(\frac{n}{2}\right) \right\}. \end{aligned}$$

Proof. (i) Let

$$\eta(\tau) := q^{\frac{1}{24}} \prod_{m=1}^{\infty} (1 - q^m)$$

with $q = e^{2\pi i\tau}$, $\tau \in \mathcal{H}$ and

$$\Gamma_0(2) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma(1) : c \equiv 0 \pmod{2} \right\}, \quad \Gamma(1) := SL(2, \mathbb{Z})/\{\pm 1\}.$$

We note that

$$\eta^8(\tau)\eta^8(2\tau) = q \prod_{m=1}^{\infty} (1 - q^m)^8 (1 - q^{2m})^8 = \sum_{n=1}^{\infty} c(n)q^n$$

is a unique normalized newform in $S_8(\Gamma_0(2))$ (the space of cusp forms of weight 8 associated to the congruence group $\Gamma_0(2)$). So, we take $\Gamma = \Gamma_0(2)$ in Proposition 4.10 and observe that with $k = 8$,

$$\begin{aligned} \widetilde{M}_8(\Gamma_0(2)) &= M_8(\Gamma_0(2)) \oplus DM_6(\Gamma_0(2)) \oplus D^2M_4(\Gamma_0(2)) \\ &\quad \oplus D^3M_2(\Gamma_0(2)) \oplus \langle D^3G_2 \rangle \end{aligned}$$

since $M_0(\Gamma_0(2)) = \{0\}$. We also note that with $k_1 = 6$ and $k_2 = 2$ in the Proposition 4.10, the functions

$$f_1(\tau) = D^2G_2(2\tau) \in \langle D^{3-1}G_2(2\tau) \rangle \subset \widetilde{M}_6(\Gamma_0(2))$$

and

$$f_2(\tau) = G_2(\tau) \in \langle G_2(\tau) \rangle \subset \widetilde{M}_2(\Gamma_0(2))$$

are quasi-modular forms and the product

$$f_1(\tau)f_2(\tau) = \left(4 \sum_{m=1}^{\infty} m^2 \sigma_1(m) q^{2m} \right) \left(-\frac{1}{24} + \sum_{n=1}^{\infty} \sigma_1(n) q^n \right) \in \widetilde{M}_8(\Gamma_0(2)).$$

Hence there exist a unique tuple $(a_1, a_2, \dots, a_{11})$ of rational numbers a_j such that

$$\begin{aligned} \sum_{1 \leq k < n/2} k^2 \sigma_1(k) \sigma_1(n-2k) &= a_1 n^2 \sigma_3(n) + a_2 n^2 \sigma_3\left(\frac{n}{2}\right) + a_3 n^3 \sigma_1(n) \\ &+ a_4 n^3 \sigma_1\left(\frac{n}{2}\right) + a_5 n^2 \sigma_1(n) + a_6 n^2 \sigma_1\left(\frac{n}{2}\right) + a_7 n \sigma_1(n) \\ &+ a_8 n \sigma_1\left(\frac{n}{2}\right) + a_9 \sigma_1(n) + a_{10} \sigma_1\left(\frac{n}{2}\right) + a_{11} c(n) \end{aligned}$$

holds for every integer $n \geq 1$. Therefore, we get 11 equations in 11 variables by taking $n = 1, \dots, 11$. Using Mathematica 8.0, we solve these simultaneous equations to obtain

$$\begin{aligned} a_1 &= \frac{1}{160}, & a_2 &= \frac{1}{40}, & a_3 &= -\frac{1}{192}, & a_4 &= -\frac{1}{32}, & a_5 &= 0, \\ a_6 &= \frac{1}{96}, & a_7 &= 0, & a_8 &= 0, & a_9 &= 0, & a_{10} &= 0, & a_{11} &= -\frac{1}{960}. \end{aligned}$$

Therefore,

$$\begin{aligned} \sum_{1 \leq k < n/2} k^2 \sigma_1(k) \sigma_1(n-2k) &= \frac{1}{160} n^2 \sigma_3(n) + \frac{1}{40} n^2 \sigma_3\left(\frac{n}{2}\right) - \frac{1}{192} n^3 \sigma_1(n) \\ &- \frac{1}{32} n^3 \sigma_1\left(\frac{n}{2}\right) + \frac{1}{96} n^2 \sigma_1\left(\frac{n}{2}\right) - \frac{1}{960} c(n). \end{aligned}$$

(ii) We take $\Gamma = \Gamma_0(2)$, $k_1 = k_2 = 4$, $k = k_1 + k_2 = 8$ in Proposition 4.10. We note that

$$\begin{aligned} f_1(\tau) &= DG_2(2\tau) \in \langle D^{2-1}G_2(2\tau) \rangle \subset \widetilde{M}_4(\Gamma_0(2)), \\ f_2(\tau) &= G_4(\tau) \in \langle M_4(\Gamma_0(2)) \rangle \subset \widetilde{M}_4(\Gamma_0(2)) \end{aligned}$$

with $f_1 f_2 \in \widetilde{M}_8(\Gamma_0(2))$ by Proposition 4.10. We also note that

$$f_1(\tau) f_2(\tau) = 2 \left(\sum_{m=1}^{\infty} m \sigma_1(m) q^{2m} \right) \left(\frac{1}{240} + \sum_{n=1}^{\infty} \sigma_3(n) q^n \right) \in \widetilde{M}_8(\Gamma_0(2)).$$

Now the proof of (ii) follows in a similar fashion as of (i). $\square$

Lemma 4.12. *For any integer $n \geq 2$, we have*

$$\sum_{k=1}^{n-1} k^2 \sigma_1^*(k) \sigma_1^*(n-k) = \frac{1}{120} \{ 9n^2 \sigma_3^*(n) - 10n^3 \sigma_1^*(n) + c(n) \}.$$

Proof. From the definition of $\sigma_1^*(n)$ and by suitable change of variables, we deduce that

$$\begin{aligned}
& \sum_{k=1}^{n-1} k^2 \sigma_1^*(k) \sigma_1^*(n-k) \\
&= \sum_{k=1}^{n-1} k^2 \left\{ \sigma_1(k) - \sigma_1\left(\frac{k}{2}\right) \right\} \left\{ \sigma_1(n-k) - \sigma_1\left(\frac{n-k}{2}\right) \right\} \\
&= \sum_{k=1}^{n-1} k^2 \sigma_1(k) \sigma_1(n-k) - \sum_{k < \frac{n}{2}} (n-2k)^2 \sigma_1(n-2k) \sigma_1(k) \\
&\quad + \sum_{k < \frac{n}{2}} (2k)^2 \sigma_1(k) \sigma_1(n-2k) + \sum_{k < \frac{n}{2}} (2k)^2 \sigma_1(k) \sigma_1\left(\frac{n}{2} - k\right).
\end{aligned}$$

Thus the above equation can be written as

$$\begin{aligned}
&= \sum_{k=1}^{n-1} k^2 \sigma_1(k) \sigma_1(n-k) - n^2 \sum_{k < \frac{n}{2}} \sigma_1(n-2k) \sigma_1(k) \\
&\quad + 4n \sum_{k < \frac{n}{2}} k \sigma_1(n-2k) \sigma_1(k) - 8 \sum_{k < \frac{n}{2}} k^2 \sigma_1(k) \sigma_1(n-2k) \\
&\quad + 4 \sum_{k < \frac{n}{2}} k^2 \sigma_1(k) \sigma_1\left(\frac{n}{2} - k\right).
\end{aligned}$$

Therefore the Lemma follows from Lemma 4.11 and the result

$$\sum_{k=1}^{n-1} k^2 \sigma_1(k) \sigma_1(n-k) = \frac{1}{8} n^2 \sigma_3(n) + \left(\frac{1}{24} n^2 - \frac{1}{6} n^3 \right) \sigma_1(n)$$

of [31, p. 155]. $\square$

Corollary 4.13. *For any integer $M \geq 2$ with $M = p_1^{e_1} \cdots p_r^{e_r}$, we define $\text{ord}_{p_j} M := e_j$.*

(i) *Let n be an odd integer. Then we have*

$$c(n) \equiv 6n^2 \sigma_3(n) - 5n^3 \sigma_1(n) \pmod{960}.$$

(ii) *Let $n \in \mathbb{N}$. Then we have*

$$c(n) \equiv -9n^2 \sigma_3^*(n) + 10n^3 \sigma_1^*(n) \pmod{120}.$$

In particular, if n is odd or $\text{ord}_p n$ is odd for an odd prime p , then

$$c(n) \equiv -9n^2 \sigma_3^*(n) + 10n^3 \sigma_1^*(n) \pmod{240}$$

and if $4|n$ then

$$c(n) \equiv -9n^2\sigma_3^*(n) + 10n^3\sigma_1^*(n) \pmod{960}.$$

Proof. (i) Since $c(1) = 1$, we have

$$c(1) = 6 \cdot 1^2\sigma_3(1) - 5 \cdot 1^3\sigma_1(1) \equiv 1 \pmod{960}.$$

For an odd integer $n \geq 3$, it is obvious from (i) of Lemma 4.11.

(ii) By Lemma 4.12, we have

$$(21) \quad 120 \sum_{k=1}^{n-1} k^2 \sigma_1^*(k) \sigma_1^*(n-k) = 9n^2\sigma_3^*(n) - 10n^3\sigma_1^*(n) + c(n).$$

Thus

$$c(n) \equiv -9n^2\sigma_3^*(n) + 10n^3\sigma_1^*(n) \pmod{120}.$$

If n is odd then the left hand side of (21) is

$$\begin{aligned} & 120 \left\{ \sum_{\substack{k=1 \\ 2|k}}^{n-1} k^2 \sigma_1^*(k) \sigma_1^*(n-k) + \sum_{\substack{k=1 \\ 2 \nmid k}}^{n-1} k^2 \sigma_1^*(k) \sigma_1^*(n-k) \right\} \\ (22) \quad &= 120 \left\{ \sum_{\substack{k=1 \\ 2|k}}^{n-1} k^2 \sigma_1^*(k) \sigma_1^*(n-k) + \sum_{\substack{k=1 \\ 2|(n-k)}}^{n-1} k^2 \sigma_1^*(k) \sigma_1^*(n-k) \right\} \\ &= 240 \left\{ \sum_{\substack{k=1 \\ 2|k}}^{n-1} k^2 \sigma_1^*\left(\frac{k}{2}\right) \sigma_1^*(n-k) + \sum_{\substack{k=1 \\ 2|(n-k)}}^{n-1} k^2 \sigma_1^*(k) \sigma_1^*\left(\frac{n-k}{2}\right) \right\}. \end{aligned}$$

Note that $\sigma_1^*(k) = 2\sigma_1^*\left(\frac{k}{2}\right)$ for $2|k$. Therefore, for odd n , we have

$$(23) \quad c(n) \equiv -9n^2\sigma_3^*(n) + 10n^3\sigma_1^*(n) \pmod{240}.$$

Let $\text{ord}_p n$ be odd. For odd n , we have already considered in (23). Thus let $n = 2^m p^s p_1^{e_1} \cdots p_r^{e_r}$ with odd s for an odd prime p and $m \geq 1$. Then the left hand side of (21) is

$$\begin{aligned}
(24) \quad & 120 \sum_{k=1}^{n-1} k^2 \sigma_1^*(k) \sigma_1^*(n-k) \\
&= 120 \left\{ \sum_{k < \frac{n}{2}} (n^2 + 2k^2 - 2kn) \sigma_1^*(k) \sigma_1^*(n-k) + \left(\frac{n}{2}\right)^2 \sigma_1^*\left(\frac{n}{2}\right) \sigma_1^*\left(\frac{n}{2}\right) \right\} \\
&= 120 \left[2 \sum_{k < \frac{n}{2}} \left(\frac{n^2}{2} + k^2 - kn\right) \sigma_1^*(k) \sigma_1^*(n-k) + \left(\frac{n}{2}\right)^2 \{\sigma_1^*(2^{m-1} p^s p_1^{e_1} \cdots p_r^{e_r})\}^2 \right].
\end{aligned}$$

Using $\sigma_1^*(2^{m-1} p^s p_1^{e_1} \cdots p_r^{e_r}) = \sigma_1^*(p^s) \sigma_1^*(2^{m-1} p_1^{e_1} \cdots p_r^{e_r})$ and

$$\begin{aligned}
\sigma_1^*(p^s) &= 1 + p + p^2 + \cdots + p^s \\
&\equiv 0 \pmod{2},
\end{aligned}$$

since p odd and s odd. Therefore, we can write $\sigma_1^*\left(\frac{n}{2}\right) = 2l$ for some $l \in \mathbb{N}$. So (24) is

$$240 \left[\sum_{k < \frac{n}{2}} \left(\frac{n^2}{2} + k^2 - kn\right) \sigma_1^*(k) \sigma_1^*(n-k) + 2l^2 \left(\frac{n}{2}\right)^2 \right].$$

Therefore

$$c(n) \equiv -9n^2 \sigma_3^*(n) + 10n^3 \sigma_1^*(n) \pmod{240}.$$

Let us consider the case $4|n$, i.e., $n = 4l$. Then the left hand side of (21) is

$$\begin{aligned}
(25) \quad & 120 \sum_{k=1}^{n-1} k^2 \sigma_1^*(k) \sigma_1^*(n-k) \\
&= 120 \left\{ \sum_{k < \frac{n}{2}} k^2 \sigma_1^*(k) \sigma_1^*(n-k) + \sum_{\frac{n}{2} < k < n} k^2 \sigma_1^*(k) \sigma_1^*(n-k) + \left(\frac{n}{2}\right)^2 \sigma_1^*\left(\frac{n}{2}\right) \sigma_1^*\left(\frac{n}{2}\right) \right\} \\
&= 120 \left\{ \sum_{k < \frac{n}{2}} k^2 \sigma_1^*(k) \sigma_1^*(n-k) + \sum_{k < \frac{n}{2}} (n-k)^2 \sigma_1^*(n-k) \sigma_1^*(k) + \left(\frac{n}{2}\right)^2 \sigma_1^*\left(\frac{n}{2}\right) \sigma_1^*\left(\frac{n}{2}\right) \right\} \\
&= 120 \left\{ \sum_{k < \frac{n}{2}} (n^2 + 2k^2 - 2kn) \sigma_1^*(k) \sigma_1^*(n-k) + n^2 \sigma_1^*\left(\frac{n}{4}\right) \sigma_1^*\left(\frac{n}{4}\right) \right\} \\
&= 240 \left\{ \sum_{k < 2l} (8l^2 + k^2 - 4kl) \sigma_1^*(k) \sigma_1^*(4l-k) + 8l^2 \sigma_1^*(l) \sigma_1^*(l) \right\}.
\end{aligned}$$

Thus if we can show $\sum_{k < 2l} k^2 \sigma_1^*(k) \sigma_1^*(4l - k) \equiv 0 \pmod{4}$ in (25), then we can conclude that

$$c(n) \equiv -9n^2 \sigma_3^*(n) + 10n^3 \sigma_1^*(n) \pmod{960}.$$

So we investigate the sum $\sum_{k < 2l} k^2 \sigma_1^*(k) \sigma_1^*(4l - k)$. If $k \equiv 0 \pmod{2}$, then $k^2 \sigma_1^*(k) \sigma_1^*(4l - k) \equiv 0 \pmod{4}$. Next suppose that $k \equiv 1 \pmod{2}$. If $k \equiv 1 \pmod{4}$ then $4l - k \equiv 3 \pmod{4}$ and if $k \equiv 3 \pmod{4}$ then $4l - k \equiv 1 \pmod{4}$. For integers $n \geq 3$ with $n \equiv 3 \pmod{4}$, we observe the fact that there exists a prime p satisfying $p|n$, $p \equiv 3 \pmod{4}$, and $\text{ord}_p n = e_i$ with e_i is odd. It is easily checked that

$$\sigma_1^*(p^{e_i}) \equiv 1 + p + p^2 + \cdots + p^{e_i} \equiv 0 \pmod{4}.$$

Since $\sigma_1^*(p^{e_i}) | \sigma_1^*(n)$, we have $\sigma_1^*(n) \equiv 0 \pmod{4}$. Applying this property, we infer that $\sigma_1^*(4l - k) \equiv 0 \pmod{4}$ or $\sigma_1^*(k) \equiv 0 \pmod{4}$. Thus we have $k^2 \sigma_1^*(k) \sigma_1^*(4l - k) \equiv 0 \pmod{4}$ for any positive integer k satisfying $k < 2l$. Thus the claim

$$c(n) \equiv -9n^2 \sigma_3^*(n) + 10n^3 \sigma_1^*(n) \pmod{960}$$

follows. □

Theorem 4.14. *Let $n \in \mathbb{N}$. Then we have*

$$\sum_{k=1}^{4n-1} \sigma_3^*(k) \sigma_3^*(4n - k) \equiv 8(n \sigma_1^*(n) - \sigma_3^*(n)) \pmod{24}.$$

Proof. By Corollary 4.9, we obtain

$$\begin{aligned} & \sum_{k=1}^{4n-1} \sigma_3^*(k) \sigma_3^*(4n - k) \\ (26) \quad & \equiv \sum_{k=1}^{4n-1} (3k - 2) \sigma_1^*(k) [\{3(4n - k) - 2\} \sigma_1^*(4n - k)] \pmod{24} \\ & \equiv \sum_{k=1}^{4n-1} (36nk - 9k^2 + 4) \sigma_1^*(k) \sigma_1^*(4n - k) \pmod{24}. \end{aligned}$$

Note that

$$\sum_{k=1}^{4n-1} k \sigma_1^*(k) \sigma_1^*(4n - k) = \sum_{k=1}^{4n-1} (4n - k) \sigma_1^*(4n - k) \sigma_1^*(k)$$

and hence

$$\sum_{k=1}^{4n-1} k\sigma_1^*(k)\sigma_1^*(4n-k) = 2n \sum_{k=1}^{4n-1} \sigma_1^*(k)\sigma_1^*(4n-k).$$

Thus we conclude that

$$36n \sum_{k=1}^{4n-1} k\sigma_1^*(k)\sigma_1^*(4n-k) = 72n^2 \sum_{k=1}^{4n-1} \sigma_1^*(k)\sigma_1^*(4n-k) \equiv 0 \pmod{24}.$$

Therefore,

$$\begin{aligned} & \sum_{k=1}^{4n-1} \sigma_3^*(k)\sigma_3^*(4n-k) \\ & \equiv -9 \sum_{k=1}^{4n-1} k^2 \sigma_1^*(k)\sigma_1^*(4n-k) + 4 \sum_{k=1}^{4n-1} \sigma_1^*(k)\sigma_1^*(4n-k) \pmod{24} \\ & \equiv -\frac{9}{120} \{9(4n)^2 \sigma_3^*(4n) - 10(4n)^3 \sigma_1^*(4n) + c(4n)\} \\ & \quad + 4 \cdot \frac{1}{4} \{\sigma_3^*(4n) - 4n\sigma_1^*(4n)\} \pmod{24} \end{aligned}$$

using Lemma 4.12 and Table 5. By (ii) of Corollary 4.13, we find that if $4|n$ then $c(n) = -9n^2\sigma_3^*(n) + 10n^3\sigma_1^*(n) + 960L$ for some $L \in \mathbb{Z}$. Thus, we get

$$\begin{aligned} \sum_{k=1}^{4n-1} \sigma_3^*(k)\sigma_3^*(4n-k) & \equiv \sigma_3^*(4n) - 4n\sigma_1^*(4n) \pmod{24} \\ & \equiv 64\sigma_3^*(n) - 16n\sigma_1^*(n) \pmod{24}. \end{aligned}$$

Therefore, the theorem follows. $\square$

From Theorem 4.14 and Corollary 4.9, it is easy to obtain :

Corollary 4.15. *If n is any prime other than 2 and 3 or any integer congruent to 1 (mod 3), then we have*

$$\sum_{k=1}^{4n-1} \sigma_3^*(k)\sigma_3^*(4n-k) \equiv 0 \pmod{24}.$$

5. Appendix

The table below gives values of $c_8(n)$ for $1 \leq n \leq 30$ and $n \in \mathbb{N}$.

n	$c_8(n)$	n	$c_8(n)$	n	$c_8(n)$
1	1	11	-44	21	-96
2	0	12	0	22	0
3	-4	13	22	23	-56
4	0	14	0	24	0
5	-2	15	8	25	-121
6	0	16	0	26	0
7	24	17	50	27	152
8	0	18	0	28	0
9	-11	19	44	29	198
10	0	20	0	30	0

TABLE 6. Examples $c_8(n)$ ($1 \leq n \leq 30$)

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