TOPOLOGICAL PROPERTIES OF GRAPHICAL ARRANGEMENTS

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Abstract. We show that for any graph G , the proper part of the intersection poset of the corresponding graphical arrangement $\mathcal{A}_G$ has the homotopy type of a wedge of spheres. Furthermore, we also indicate the number of spheres in the wedge, based on the number of spanning forests of G and other graphs that are obtained from G .

1. Introduction

A *hyperplane arrangement* $\mathcal{A}$ is a finite set of affine hyperplanes in some vector space $V \cong K^n$, where K is a field. Graphical arrangements are special hyperplane arrangements obtained from finite simple undirected graphs. We know that the graphical arrangement of a graph G on vertex set $[n] := \{1, 2, \dots, n\}$ is a subarrangement of the braid arrangement B_n that is the graphical arrangement of the complete graph K_n . Graphical arrangements are studied in the connection with graph theory [3, 6]. Many of the invariants associated with a graphical arrangement can be computed directly from the corresponding graph such as the characteristic polynomial ([9]).

One of the important invariants of a hyperplane arrangement $\mathcal{A}$ is the intersection poset $L(\mathcal{A})$, that is, the collection of all nonempty intersections of hyperplanes in $\mathcal{A}$ ordered by reverse inclusion. The number of regions and bounded regions of a real hyperplane arrangement is determined by its intersection poset [10] and the Betti numbers of

Received April 3, 2014. Accepted May 28, 2014.

2010 Mathematics Subject Classification. 05C88, 05C89.

Key words and phrases. graphical arrangement, EL-shallability.

This research was supported by Basic Science Research Program through the National Research Foundation of Korea(NRF) funded by the Ministry of Education, Science and Technology(NRF-2012R1A1A1013149).

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the complement of a complex hyperplane arrangement can be expressed using the intersection poset [4].

In this paper, we study the topological properties of the intersection poset of a graphical arrangement. Björner and Wach [2] show that if a bounded poset P is EL-shellable, then the order complex $\Delta(\overline{P})$ has the homotopy type of a wedge of spheres, where $\overline{P}$ is the proper part of P and the *order complex* $\Delta(\overline{P})$ is the abstract simplicial complex whose vertices are all elements of $\overline{P}$ and whose simplices are all finite chains of $\overline{P}$, including the empty chain. They also show that the number of spheres in the wedge can be obtained from decreasing maximal chains. A well-known example of an EL-shellable poset is $\overline{\Pi}_n$, the proper part of the partition lattice Π_n of the n -element set ordered by refinement. The poset $\overline{\Pi}_n$ has the homotopy type of a wedge of spheres, and the number of spheres is computed by counting decreasing maximal chains (see [5, 11, 12]).

The main purpose of this paper is to show that the intersection poset of a graphical arrangement is EL-shellable. We also find the number of spheres in the wedge by using the number of spanning forests of G and other graphs that are obtained from G .

The rest of the paper is organized as follows. Section 2 provides the basic notions and definitions which will be used throughout the paper. In Section 3, an edge-labeling is defined and shown to be an EL-labeling. Section 4 describes the number of spheres in the wedge using the number of decreasing chains. The number of regions of a graphical arrangement and the enumerative results for special cases will be given in the last section.

2. Basic notions and definitions

Let G be a simple undirected graph on a vertex set $[n] = \{1, 2, \dots, n\}$ and $E(G) = \{e_1, e_2, \dots, e_m\}$ denote the set of edges of G . We also use the notation ij for an edge of G with two vertices i and j . Let $V = K^n$ be a vector space over a field K with coordinates $x_1, x_2, \dots, x_n$.

Definition 2.1. The *graphical arrangement* $\mathcal{A}_G$ in a vector space V is the hyperplane arrangement

$$\{x_i - x_j = 0 \mid ij \in E(G)\}.$$

In the intersection poset of the graphical arrangement $\mathcal{A}_G$, we also label each hyperplane $x_i - x_j = 0$ by ij . There are two cases to label an intersection of two hyperplanes. If two hyperplanes are ij and kl , where

FIGURE 1. A graph and the intersection poset of its graphical arrangement with labeling values

$\{i, j\} \cap \{k, l\} = \emptyset$, then the common subspace is labeled by $ij - kl$. Otherwise, we may assume that $l = j$. Then the common subspace will be labeled by ijk . A general subspace in the intersection poset is denoted by $A = A_1 - A_2 - \cdots - A_r$, where A_i , $i = 1, 2, \dots, r$, are the sets of indices of coordinates. We call A_i *blocks* of A , and those indices of coordinates *indices* of A . When we get the intersection of two subspaces $H = H_1 - H_2 - \cdots - H_p$ and $K = K_1 - K_2 - \cdots - K_q$, where H_i and K_i are blocks of H and K respectively, we also have two cases to label their common subspace. It is similar to labeling common subspace of two hyperplanes. Furthermore, $K < H$ if and only if each block K_j of K is contained in some block H_i of H .

Example 2.2. Consider the graph G_1 shown in Figure 1. The graphical arrangement $\mathcal{A}_{G_1}$ of G_1 consists of hyperplanes $x_1 - x_2 = 0$, $x_2 - x_3 = 0$, $x_3 - x_4 = 0$, and $x_1 - x_4 = 0$. In the intersection poset P_{G_1} of $\mathcal{A}_{G_1}$, we label hyperplane $x_1 - x_2 = 0$ by 12, $x_2 - x_3 = 0$ by 23, $x_3 - x_4 = 0$ by 34, and $x_1 - x_4 = 0$ by 14. Figure 1 also shows the intersection poset P_{G_1} .

Definition 2.3. An *edge-labeling* of a poset P is a map $\lambda : \mathcal{E}(P) \rightarrow \Lambda$, where $\mathcal{E}(P)$ is the set of edges of the Hasse diagram of poset P and Λ is an ordered set. For an edge $\{s, t\} \in \mathcal{E}(P)$, we denote $\lambda(s, t)$

for $\lambda(\{s, t\})$. A maximal chain $m : x_0 \leq x_1 \leq x_2 \leq \cdots \leq x_t$ of P is *increasing* if $\lambda(x_0, x_1) < \lambda(x_1, x_2) < \cdots < \lambda(x_{t-1}, x_t)$. Similarly, m is *decreasing* if $\lambda(x_0, x_1) > \lambda(x_1, x_2) > \cdots > \lambda(x_{t-1}, x_t)$. The maximal chain $m' := y_0 \leq y_1 \leq y_2 \leq \cdots \leq y_s$ is less than m in *lexicographic partial order* if

- $\lambda(x_i, x_{i+1}) = \lambda(y_i, y_{i+1})$ for $i = 0, \dots, s-1$ and $s < t$, or
- $\lambda(x_i, x_{i+1}) \neq \lambda(y_i, y_{i+1})$ for some i and $\lambda(x_i, x_{i+1}) < \lambda(y_i, y_{i+1})$ for the least such i .

Definition 2.4. An edge-labeling is called an *EL-labeling* (edge lexicographical labeling) if for every interval $[x, y]$ in poset P , there is a unique increasing maximal chain, which lexicographically precedes all other maximal chains of $[x, y]$.

Björner and Wachs [2] showed the following fundamental result about the EL-labeling.

Theorem 2.5. *If a bounded poset P admits an EL-labeling, then the order complex $\Delta(\bar{P})$ has the homotopy type of a wedge of spheres. Furthermore, the number of i -spheres is the number of decreasing maximal $(i+2)$ -chains of P .*

3. The edge-labeling

In this section, we will define an edge-labeling of the intersection poset of the graphical arrangement for graph G , and show that it is an EL-labeling.

Let P_G be the intersection poset of the graphical arrangement $\mathcal{A}_G$ of G and H and K be two elements in P_G where H covers K . We label an edge $\{K, H\}$ by

$$\lambda(K, H) = \min\{i : K \cap e_i = H\}.$$

Theorem 3.1. *The labeling λ is an EL-labeling.*

Proof. Let M and N be two elements in the poset P_G such that $N < M$. We construct a maximal chain

$$\mathcal{C} = \{N = N_0 \leq N_1 \leq N_2 \leq \cdots \leq N_l = M\}$$

where N_{i+1} is an element covering N_i satisfying

$$\lambda(N_i, N_{i+1}) = \min\{\lambda(N_i, N') : N_i \leq N' < M\}.$$

We claim that $\mathcal{C}$ is the only increasing maximal chain from N to M , which lexicographically precedes all other maximal chains of $[N, M]$.

Let A, B and C be three elements in $\mathcal{C}$ such that $A \leq B \leq C$. If $\lambda(A, B) = x$ and $\lambda(B, C) = y$, then $A \cap e_x = B$ and $B \cap e_y = C$. Since the hyperplane e_y doesn't contain the subspace A , $A \cap e_y$ is another covering of A . By the choice of B , $x = \lambda(A, B) < \lambda(A, A \cap e_y) \leq y$. Thus $\mathcal{C}$ is increasing.

Let $\mathcal{D} = \{N = M_0 \leq M_1 \leq M_2 \leq \dots \leq M_l = M\}$ be another maximal chain in $[N, M]$ and let k be the smallest index such that $N_k \neq M_k$. Let $\lambda(N_{k-1}, N_k) = z_1$ and $\lambda(M_{k-1}, M_k) = z_2$. If $z_1 = z_2$, then

$$N_k = N_{k-1} \cap e_{z_1} = N_{k-1} \cap e_{z_2} = M_k$$

which is impossible. Thus $z_1 < z_2$. Since $e_{z_1} < M$ and $e_{z_1} \not\leq M_k$ in P_G , we have $M_k < M_t \cap e_{z_1} = M_{t+1} < M$ for some t . But then $\lambda(M_t, M_{t+1}) \leq z_1 < z_2 = \lambda(M_{k-1}, M_k)$. Therefore the maximal chain $\mathcal{D}$ is not increasing.

By the construction of the maximal chain $\mathcal{C}$, it is obvious that $\mathcal{C}$ lexicographically precedes all other maximal chains of $[N, M]$. $\square$

Remark 3.2. Theorem 3.1 can be obtained from [7] since the intersection poset of a graphical arrangement is a geometric lattice. We provide an elementary proof for completeness.

Example 3.3. Figure 1 shows a graph G_1 and its intersection poset P_{G_1} with the labeling values.

Since P_G is the intersection poset of a hyperplane arrangement, all maximal chains in P_G have the same length $n - c(G)$, where $c(G)$ is the number of connected components in G . Thus the order complex $\Delta(\overline{P_G})$ has the homotopy type of a wedge of spheres of the same dimension $n - c(G) - 2$.

Remark 3.4. In [6], Stanley gives a labeling rule for the graphical arrangement of a chordal graph, and the labeling values are from the set $[l]$ where l is the rank of the intersection poset. In our paper, we also give a labeling rule for graphical arrangements, and the labeling values are from the set $[n]$ where n is the number of hyperplanes in the arrangement. In general, the two labeling rules are different. For example, see the chordal graph G_2 in Figure 2. By the labeling rule of this paper, the unique increasing maximal chain in $[\hat{0}, \hat{1}]$ of the intersection poset is $\hat{0} < 12 < 123 < 1234$. If we choose that maximal chain for the M-chain in the labeling of Stanley, then we get $\hat{0} < 34 < 134 < 1234$ for a decreasing maximal chain while it is not decreasing in the labeling of this paper.

FIGURE 2. A chordal graph G_2

4. The number of decreasing maximal chains

In this section, we will find the number of decreasing maximal chains in the poset P_G .

Let $\mathcal{M} := \hat{0} < X_1 < X_2 < X_3 < \cdots < X_l < \hat{1}$ be a maximal chain in $[\hat{0}, \hat{1}]$. We associate $\mathcal{M}$ with the ordered string of labeling values:

$$\lambda(\mathcal{M}) = [\lambda(\hat{0}, X_1), \lambda(X_1, X_2), \dots, \lambda(X_l, \hat{1})].$$

Since P_G is the intersection poset of a hyperplane arrangement, the labeling values $\lambda(\hat{0}, X_1), \lambda(X_1, X_2), \dots, \lambda(X_l, \hat{1})$ are corresponding to hyperplanes or edges of G . For example, we can associate the maximal chain $\mathcal{M} := \hat{0} < 12 < 123 < 1234$ of Figure 1 with the string of edges $[12, 23, 34]$.

Definition 4.1. A string X of $l(P_G)$ edges in G is *associated with a maximal chain* if all the edges of X are used to find labeling values of a maximal chain. Furthermore, X is *associated with a decreasing maximal chain* if the labeling values corresponding to the edges are decreasing.

Definition 4.2. A *spanning tree* of a connected graph G is a subgraph that connects all the vertices of G and has no cycle. For a general graph G , a subgraph of G formed by the union of spanning trees in all connected components is called a *spanning forest* of G .

The following lemma describes the condition when a set corresponds to a decreasing maximal chain.

Lemma 4.3. Let A be a set of edges in G . Then we can arrange edges in A to get a string X that can be associated with a decreasing maximal chain if and only if

- The edges in A form a spanning forest in G .
- A doesn't contain $C \setminus e_j$, the set of all edges of a cycle C except the edge e_j with the smallest label in C .

Proof. If we can arrange edges in A to get a string X associated with a decreasing maximal chain $\mathcal{M}$, then it is easy to see that A must contain edges that can make a spanning forest in G .

Assume that A contains $C \setminus e_j$ where C is a cycle of G and j is the smallest label of edges in C . Let S be an element of the poset P_G , and S is the intersection of some hyperplanes (or edges) in A such that the set of edges making S contains all edges in cycle C except e_j and another edge $e_k \in C$. By the condition of j , we have $j < k$.

Let $v_1, v_2, \dots, v_m$ be all vertices of C .

If e_j and e_k share one vertex v_t , then $v_1 v_2 \cdots \hat{v}_t \cdots v_m < S$ in P_G , where $\hat{v}_t$ means that we don't have v_t in that subspace. Let T be the smallest element in $\mathcal{M}$ such that $T > S$ and $v_1 v_2 \cdots v_m < T$. If R is an element in $\mathcal{M}$ covered by T , then $R \cap e_k = R \cap e_j = T$. By the labeling rule, $\lambda(R, T) \leq j < k$. Therefore, e_k is not used to find a labeling value. So X is not associated with a maximal chain.

If e_j and e_k share no vertex, then $A_1 - A_2 < S$, where $A_1 \cup A_2 = C$ and e_j, e_k can connect these two blocks. Similar to the previous case, e_k can not be used to find a labeling value.

On the other side, if the edges in A form a spanning forest in G and A doesn't contain $C \setminus e_j$, then we arrange edges of A in decreasing order of labels to get a string X . We claim that X can be associated with a decreasing maximal chain. Since edges in X are arranged in decreasing order of labels, we only need to prove that all edges in X are used to find labeling values.

Let e_w be an edge of X , and $P = e_w \cap Q$ where Q is the subspace formed by getting the intersection of hyperplanes standing before e_w in X . If e_w doesn't belong to any cycle of G , then it is obvious that $\lambda(Q, P) = w$. If e_w belongs to a cycle of G , then e_w is not used to find a labeling value when there is another edge e_z such that $Q \cap e_z = P$ and $z < w$. That means e_z, e_w and some other edges appearing before e_w in X form a cycle of G . Since X doesn't contain a cycle, e_z doesn't belong to X . If there is an edge $e_{z'}$ appearing before e_w in X and $z' < z < w$, we get a contradiction because all edges in X are in decreasing order of labels. So there doesn't exist such edge $e_{z'}$. Then X contains $C \setminus e_z$ where C is the cycle above and z is the smallest label of edges in C . It is impossible. Therefore, e_w is used to find a labeling value. $\square$

Example 4.4. For $A = \{e_4 = 14, e_2 = 23, e_3 = 34\}$ in Figure 1, we cannot get a string that is associated with a decreasing maximal chain from A .

By Lemma 4.3, the number of decreasing maximal chains of P_G is the same as the the number of sets A satisfying conditions of Lemma 4.3.

Let $\mathcal{B}_G$ be the set of all spanning forests in G . For each cycle C of graph G . Let $x_C = \min\{j : e_j \text{ is an edge of } C\}$, and $\mathcal{B}_{x_C}$ be the set of spanning forests containing $C \setminus e_{x_C}$. Then the following theorem follows immediately.

Theorem 4.5. *The number of decreasing maximal chains of the poset P_G is*

$$|\mathcal{B}_G| - |\cup_{C: \text{cycle of } G} \mathcal{B}_{x_C}|.$$

Furthermore, the order complex $\Delta(\overline{P_G})$ has the homotopy type of a wedge of $|\mathcal{B}_G| - |\cup_{C: \text{cycle of } G} \mathcal{B}_{x_C}|$ spheres of dimension $n - c(G) - 2$ where $c(G)$ is the number of connected components of G .

Example 4.6. Let G_1 be the graph shown in Figure 1. There are three decreasing maximal chains $\hat{0} < 34 < 234 < 1234$, $\hat{0} < 14 < 23 - 14 < 1234$ and $\hat{0} < 14 < 134 < 1234$. This coincides with the result of Theorem 4.5.

Lemma 4.7. *Let $C_1, C_2, \dots, C_k$ be cycles of G . If there is no $e_{x_{C_i}}$ that lies on C_j where $i \neq j$, then*

$$|\mathcal{B}_{x_{C_1}} \cap \mathcal{B}_{x_{C_2}} \cap \dots \cap \mathcal{B}_{x_{C_k}}| = |\mathcal{B}_{G/C_1, C_2, \dots, C_k}|,$$

where $G/C_1, C_2, \dots, C_k$ is the graph obtained from G by contracting all edges in $C_1, C_2, \dots, C_k$. Otherwise, $|\mathcal{B}_{x_{C_1}} \cap \mathcal{B}_{x_{C_2}} \cap \dots \cap \mathcal{B}_{x_{C_k}}| = 0$.

Proof. The first statement follows from the definition of $\mathcal{B}_{x_C}$.

For the second statement, assume that e_{C_i} lies on C_j where $i \neq j$. If $e_{C_i} \neq e_{C_j}$, then it is obvious that $|\mathcal{B}_{x_{C_i}} \cap \mathcal{B}_{x_{C_j}}| = 0$. If $e_{C_i} = e_{C_j}$, then any subgraph of G that contains $C_i \setminus e_{C_i}$ and $C_j \setminus e_{C_j}$ always has a cycle $(C_i \cup C_j) \setminus e_{C_i}$ and so $|\mathcal{B}_{x_{C_i}} \cap \mathcal{B}_{x_{C_j}}| = 0$. Thus the result follows. $\square$

Example 4.8. Let G_3 be the graph shown in Figure 3(a). Let C be the cycle of G_3 with vertices 1, 2, 3, 4. See Figure 3(c) for the spanning trees of G_3 that doesn't contain $e_{x_C} = e_1$ and contains all other edges of C . Those spanning trees are corresponding to spanning trees of graph G_3/C shown in Figure 3(d).

Theorem 4.5 can be rephrased in the following form using Lemma 4.7 and Principle of inclusion and exclusion.

Theorem 4.9. *The number of decreasing maximal chains of the poset P_G is*

$$|\mathcal{B}_G| - \sum_{j=1}^c (-1)^{j-1} \sum_{i_1, i_2, \dots, i_j} |\mathcal{B}_{G/C_{i_1}, C_{i_2}, \dots, C_{i_j}}|,$$

FIGURE 3. $G_3, G_3/C$ and their spanning trees

where c is the number of cycles of G , and $\{i_1, i_2, \dots, i_j\}$ are the sets of labels of cycles such that there is no $e_{x_{C_a}}$ that lies on C_b where $a \neq b$ and $a, b \in \{i_1, i_2, \dots, i_j\}$.

In the formula above, we need to find the number of spanning forests of a graph. The Matrix tree theorem will help us to find the number of spanning trees of a connected graph, and we can use it to find the number of spanning forests.

Definition 4.10. Given an undirected graph G with the vertices set $[n] = \{1, 2, \dots, n\}$, the *Laplacian matrix* $L(G)$ is defined to be $n \times n$ matrix with the following entries:

$$l_{i,j} = \begin{cases} \deg(i) & \text{if } i = j, \\ -m & \text{where } m \text{ is the number of edges between } i \text{ and } j, \\ 0 & \text{otherwise.} \end{cases}$$

When counting the degree of a vertex, all loops are excluded for multi-graph.

Theorem 4.11 (The Matrix Tree Theorem, [8]). For a given undirected connected graph G with n vertices, let $\lambda_1, \dots, \lambda_{n-1}$ be the non-zero eigenvalues of $L(G)$. Then the number of distinct spanning trees of G is equal to

$$t(G) = \frac{1}{n} \prod_{i=1}^{n-1} \lambda_i.$$

If G is not connected with connected components $G_1, G_2, \dots, G_d$, then it is not difficult to see that

$$|\mathcal{B}_G| = \prod_{i=1}^d t(G_i).$$

Example 4.12. We will use Theorem 4.9 to find the number of decreasing maximal chains in P_{G_3} . We have

$$L(G_3) = \begin{pmatrix} 2 & -1 & -1 & 0 & 0 & 0 \\ -1 & 2 & 0 & -1 & 0 & 0 \\ -1 & 0 & 3 & -1 & -1 & 0 \\ 0 & -1 & -1 & 3 & 0 & -1 \\ 0 & 0 & -1 & 0 & 2 & -1 \\ 0 & 0 & 0 & -1 & -1 & 2 \end{pmatrix}$$

The 5 non-zero eigenvalues of $L(G_3)$ are 1, 2, 3, 3, 5. So the number of spanning trees of G_3 is $\frac{1 \cdot 2 \cdot 3 \cdot 3 \cdot 5}{6} = 15$. Let C_1 be the cycle with vertices 1, 2, 3, 4, and C_2 be the cycle with vertices 2, 3, 5, 6, and C_3 be the cycle with vertices 1, 2, 5, 6, 3, 4. Since G_3/C_1 and G_3/C_3 are triangles, they both have 3 spanning trees. Since G_3/C_2 is a loop, it has 1 spanning tree. Since e_4 lies on C_1 , $|\mathcal{B}_{G/C_1, C_2}| = 0$. Similarly, $|\mathcal{B}_{G/C_1, C_3}| = |\mathcal{B}_{G/C_1, C_2, C_3}| = 0$. It is obvious that $|\mathcal{B}_{G/C_2, C_3}| = 1$. Theorem 4.9 shows that the number of decreasing maximal chains in P_{G_3} is

$$15 - (3 + 3 + 1 - 1 - 0 - 0 + 0) = 9.$$

Now we provide several enumerative results for some classes of graphs.

Proposition 4.13. *Let G be a graph such that cycles in G do not share edges. Then the number of decreasing maximal chains in P_G is*

$$\prod_k (k-1)^{\alpha_k}$$

where α_k is the number of k -cycles, $k \geq 3$.

Proof. Let G' be a spanning forest of G that satisfies the conditions of Lemma 4.3. Then G' must contain e_{x_C} for all cycles of G . Since G' is a spanning forest, it doesn't contain only one edge in $C \setminus e_{x_C}$ for every C . If C is a cycle of length k , then we have $k-1$ choices for that edge. Therefore, the number of spanning forests satisfying the conditions of Lemma 4.3 is

$$\prod_k (k-1)^{\alpha_k}.$$

□

FIGURE 4. An example of G

Definition 4.14. A cycle C is called a *minimal cycle* if C doesn't contain the whole vertices of another cycle.

Proposition 4.15. Let G be a graph such that all minimal cycles in G can form a chain $C_1, C_2, \dots, C_s$ where

$$C_i \cap C_j = \begin{cases} \emptyset, \text{ a vertex, or an edge} & \text{if } j = i + 1, \\ \emptyset & \text{if } j \neq i + 1. \end{cases}$$

Then the number of decreasing maximal chains in P_G is

$$\prod_k (k - 1)^{\alpha_k}$$

where α_k is the number of minimal cycles of length k .

Proof. Let G be a graph satisfying the conditions above. Since G does not contain a subdivision of K_5 or $K_{3,3}$, G is a planar graph. Thus the chain of minimal cycles can be labeled from left to right as Figure 4.

We label the edges of G in increasing order from the left to the right. If we meet a cycle, we label all edges of that cycle first before we move to other edges. This rule guarantees that every common edge of consecutive minimal cycles is the smallest edge of latter cycle.

Let G' be a spanning forest that satisfies the condition of Lemma 4.3. If an edge e doesn't belong to any cycle, then G' must contain e .

Let C be a minimal cycle of G of length k . Since G' is a spanning forest, G' can contain at most $k - 1$ edges of C . Since G' doesn't contain $C \setminus \{e_{x_C}\}$, there are $k - 1$ ways to choose $(k - 1)$ -subsets of edges in C in which we can choose some edges to put into G' . Those subsets contain the smallest edge of C .

We construct a multiset M that will contain edges of G as follows. Firstly, we add all edges that are not in any cycle into M . In the next step, we choose a $(k - 1)$ -subset for every minimal cycle C as described

above to add into M . By the labeling rule of edges, every common edge of consecutive cycles must appear in M at least once. Furthermore, each common edge can appear in M at most twice because each edge can be a common edge of at most two cycles.

Now we prove that if we remove every common edge one time from the multiset M , we will get a set of edges that can form a spanning forest of G satisfying the condition of Lemma 4.3. And also, whenever we change the subset of edges of a cycle, we will get another spanning forest.

By the construction, M can not contain $C \setminus \{e_{x_C}\}$ for all C . Consider a maximal chain of minimal cycles $C_j, C_{j+1}, \dots, C_{j+t}$ of G such that C_{j+i} and $C_{j+(i+1)}$ have a common edge or a common vertex for all $0 \leq i \leq t-1$. By the rule of labeling edges, each common edge of the consecutive cycles are the smallest edges of the latter cycle and it will appear in M . If C_{j+i} has length k_{j+i} , M contains a subset of $k_{j+i} - 1$ edges of C_{j+i} . Let $\cup_i C_{j+i}$ be the union of C_{j+i} for $0 \leq i \leq t$. If there are x common vertices and y common edges of minimal cycles in $\cup_i C_{j+i}$, then $\cup_i C_{j+i}$ has $(\sum_i k_{j+i}) - x - 2y$ vertices. The subsets of edges of C_{j+i} form a multiset N with $(\sum_i k_{j+i}) - t$ elements including edges of $\cup_i C_{j+i}$. When we remove the common edges of consecutive cycles in N one time, we get a set N' with $(\sum_i k_{j+i}) - t - y$ edges. Since $t = x + y + 1$, the set N' has $(\sum_i k_{j+i}) - x - 2y - 1$ edges. Those edges form a subgraph of $\cup_i C_{j+i}$ including $(\sum_i k_{j+i}) - x - 2y$ vertices and $(\sum_i k_{j+i}) - x - 2y - 1$ edges with no isolated vertex. Thus N' form a spanning tree of $\cup_i C_{j+i}$. Therefore we can conclude that when we remove every common edge one time in the multiset M , we will get a set of edges that can form a spanning forest of G satisfying the condition of Lemma 4.3.

For each multiset M , the edges removed are the same, and they are all common edges in G . Since each edge except the common edges will appear in the subsets of edges of only one cycle, it will appear only once in M . Therefore, whenever we change the subset of edges of a minimal cycle, we will get another spanning forest.

On the other hand, let T be a spanning tree of a connected component of G . We add common edges of cycles in that component into T one time to get a multiset N'' . Then we can partition N'' into a set of edges that are not in any cycle and $(k_j - 1)$ -subsets of cycles C_j in the component.

Therefore, the number of spanning forests satisfying the condition of Lemma 4.3 is

$$\prod_k (k-1)^{\alpha_k}$$

multisets M	decreasing maximal chains
$\{1, 10, 2, 3, 3, 5, 5, 7, 8\}$	$[10, 8, 7, 5, 3, 2, 1]$
$\{1, 10, 2, 3, 3, 5, 5, 7, 9\}$	$[10, 9, 7, 5, 3, 2, 1]$
$\{1, 10, 2, 3, 3, 5, 5, 8, 9\}$	$[10, 9, 8, 5, 3, 2, 1]$
$\{1, 10, 2, 3, 3, 6, 5, 7, 8\}$	$[10, 8, 7, 6, 3, 2, 1]$
$\{1, 10, 2, 3, 3, 6, 5, 7, 9\}$	$[10, 9, 7, 6, 3, 2, 1]$
$\{1, 10, 2, 3, 3, 6, 5, 8, 9\}$	$[10, 9, 8, 6, 3, 2, 1]$
$\{1, 10, 2, 4, 3, 5, 5, 7, 8\}$	$[10, 8, 7, 5, 4, 2, 1]$
$\{1, 10, 2, 4, 3, 5, 5, 7, 9\}$	$[10, 9, 7, 5, 4, 2, 1]$
$\{1, 10, 2, 4, 3, 5, 5, 8, 9\}$	$[10, 9, 8, 5, 4, 2, 1]$
$\{1, 10, 2, 4, 3, 6, 5, 7, 8\}$	$[10, 8, 7, 6, 4, 2, 1]$
$\{1, 10, 2, 4, 3, 6, 5, 7, 9\}$	$[10, 9, 7, 6, 4, 2, 1]$
$\{1, 10, 2, 4, 3, 6, 5, 8, 9\}$	$[10, 9, 8, 6, 4, 2, 1]$

TABLE 1. Multisets and decreasing maximal chains

where α_k is the number of minimal cycles of length k . $\square$

Example 4.16. Let G be the graph with edge labeled shown in Figure 4. The subsets of edges of C_1 are $\{2, 3\}$ and $\{2, 4\}$. The subsets of edges of C_2 are $\{3, 5\}$ and $\{3, 6\}$. The subset of edges of C_3 are $\{5, 8, 9\}$, $\{5, 7, 9\}$ and $\{5, 7, 8\}$. The common edges are 3, 5. Table 1 shows multisets and corresponding decreasing maximal chains. Thus there are 12 different decreasing maximal chains.

Proposition 4.17. Let G be a graph such that we can label minimal cycles in G to form a chain of cycles $C_1, C_2, \dots, C_s$ of length $k_1, k_2, \dots, k_s$ respectively. If

$$C_i \cap C_j = \begin{cases} \emptyset, \text{ a vertex, or an edge} & \text{if } i \neq t \text{ and } j = i + 1, \\ \alpha \text{ edges } (\alpha > 1) & \text{if } i = t \text{ and } j = t + 1, \\ \emptyset & \text{if } j \neq i + 1, \end{cases}$$

then the number of decreasing maximal chains in P_G is

$$[(k_t - 1)(k_{t+1} - 1) - \alpha(\alpha - 1)] \prod_{i \neq t, t+1} (k_i - 1).$$

Proof. Since G is a planar graph, we can label cycles and edges in G as in the proof of Proposition 4.15. Similar to Proposition 4.15, we add each $(k_i - 1)$ -subset of each C_i into a multiset M , and remove common edges one time.

If a connected component that doesn't contain C_t and C_{t+1} , the number of spanning trees is counted by Proposition 4.15.

Consider the connected component that contains C_t and C_{t+1} . Assume that this connected component of G contains cycles

$$C_{t-v}, C_{t-v+1}, \dots, C_{t-1}, C_t, C_{t+1}, C_{t+2}, \dots, C_{t+u}.$$

Consider the case when $C_{t-1} \cap C_t, C_{t+1} \cap C_{t+2} \in \{\emptyset, \text{a vertex}\}$. By the labeling of edges, the smallest edge of C_{t+1} is a common edge of C_t and C_{t+1} . Since edges are labeled from the left to the right, the smallest edge of C_t can not be a common edge of the two cycles. There are $k_t - \alpha$ and $k_{t+1} - \alpha$ edges of C_t and C_{t+1} respectively that are not common edges of C_t, C_{t+1} . If $(k_t - 1)$ -subset of C_t and $(k_{t+1} - 1)$ -subset of C_{t+1} added into M contain all those edges, then after removing common edges we get a cycle. There are α ways to choose $(k_t - 1)$ -subsets of C_t that contain all non-common edges of C_t, C_{t+1} . Since the smallest edge of C_{t+1} is a common edge, there are $\alpha - 1$ ways to choose $(k_{t+1} - 1)$ -subsets of C_{t+1} that contain all non-common edges of the two cycles. So we have $\alpha(\alpha - 1)$ ways of combination of subsets of C_t and C_{t+1} in which we have a cycle. Then the number of spanning trees of this component is

$$[(k_t - 1)(k_{t+1} - 1) - \alpha(\alpha - 1)]k_{t-v} \cdots k_{t-1}k_{t+2} \cdots k_{t+u}.$$

The proof is similar to the proof of Proposition 4.15.

When $C_{t-1} \cap C_t = \{\text{an edge}\}$, $C_{t+1} \cap C_{t+2} \in \{\emptyset, \text{a vertex}\}$, the common edge of C_{t-1} and C_t must be the smallest edge of C_t . If M contains the $(k_{t-1} - 1)$ -subset of C_{t-1} that does not contain the common edge of C_{t-1} and C_t , and $(k_t - 1)$ -subset, $(k_{t+1} - 1)$ -subset of C_t and C_{t+1} respectively such that they contain all non-common edges of C_t, C_{t+1} , then we have a cycle. We also have $\alpha(\alpha - 1)$ ways of combination of subsets of C_t and C_{t+1} as the above situation. Therefore, the number of spanning trees is

$$[(k_t - 1)(k_{t+1} - 1) - \alpha(\alpha - 1)]k_{t-v} \cdots k_{t-1}k_{t+2} \cdots k_{t+u}.$$

When $C_{t-1} \cap C_t \in \{\emptyset, \text{a vertex}\}$ and $C_{t+1} \cap C_{t+2} = \{\text{an edge}\}$, that edge must be the smallest edge of C_{t+2} . Every $(k_{t+2} - 1)$ -subset of C_{t+2} contains that common edge. Similarly, we also have

$$[(k_t - 1)(k_{t+1} - 1) - \alpha(\alpha - 1)]k_{t-v} \cdots k_{t-1}k_{t+2} \cdots k_{t+u}$$

spanning trees.

When $C_{t+1} \cap C_{t+2} = \{\text{an edge}\}$ and $C_{t-1} \cap C_t = \{\text{an edge}\}$, the common edge of C_{t+1} and C_{t+2} must be the smallest edge of C_{t+2} . If M contains the $(k_{t-1} - 1)$ -subset of C_{t-1} that does not contain the common edge of C_{t-1} and C_t , and $(k_t - 1)$ -subset, $(k_{t+1} - 1)$ -subset of C_t and C_{t+1} respectively such that they contain all non-common edges of C_t, C_{t+1} ,

then after removing the common edges we also have a cycle. So the number of spanning trees of this component is

$$[(k_t - 1)(k_{t+1} - 1) - \alpha(\alpha - 1)]k_{t-v} \cdots k_{t-1}k_{t+2} \cdots k_{t+u}.$$

Consequently, the number of spanning forests that satisfy the condition of Lemma 4.3 is

$$[(k_t - 1)(k_{t+1} - 1) - \alpha(\alpha - 1)] \prod_{k_i \neq t, t+1} (k_i - 1).$$

□

Proposition 4.18. *Let G be a graph such that in each connected component all minimal cycles share only one common edge. Then the number of decreasing maximal chains of the intersection poset of graphical arrangement of G is equal to*

$$\prod_k (k - 1)^{\alpha_k}$$

where α_k is the number of minimal cycles of length k .

Proof. In each connected component, we label edges so that the common edge is the smallest edge of all minimal cycles in that component. We construct a multi set M by adding all edges that do not belong to any cycle and adding one $(k - 1)$ -subset of each minimal cycle C of length k . If e is a common edge of x cycles, then e will appear x times in M . We remove e $x - 1$ times to get a set N of edges in G . We prove that edges in N form a spanning forest that satisfies the condition of Lemma 4.3.

It is obvious that N doesn't contain $C \setminus \{e_{x_C}\}$ for all C .

Let $C_1, C_2, \dots, C_t$ be all minimal cycles of a connected component. Let $\cup_i C_i$ be the union of $C_1, C_2, \dots, C_t$. Assume that $C_1, C_2, \dots, C_t$ have length $k_1, k_2, \dots, k_t$ respectively. Then the number of vertices in $\cup_i C_i$ is $[\sum_{i=1}^t k_i] - 2(t - 1)$. After removing edges, the number of edges of $\cup_i C_i$ in N is $(\sum_{i=1}^t (k_i - 1)) - (t - 1) = (\sum_{i=1}^t k_i) - 2t + 1$. Since the subgraph formed by edges in N doesn't contain isolated vertices, the edges in N form a spanning tree of $\cup_i C_i$.

Whenever we change a $(k_i - 1)$ -subset of C_i in M , we get a new spanning tree of $\cup_i C_i$.

On the other hand, let T be a spanning tree of a connected component of G which has t cycles. We add the common edge of those cycles $t - 1$ times into T to get a multiset N' . Then we can partition N' into a set

of edges that are not in any cycle and $(k_i - 1)$ -subsets of cycles C_i in the component.

Therefore, the number of spanning forests of G satisfying condition of Lemma 4.3 is

$$\prod_k (k - 1)^{\alpha_k}$$

where α_k is the number of minimal cycles of length k . $\square$

5. The number of regions

An application of finding the number of decreasing maximal chains is counting regions of an arrangement. Recall that a *region* of an arrangement $\mathcal{A}$ is a connected component of the complement X of the hyperplanes,

$$X = \mathbb{R}^n - \cup_{H \in \mathcal{A}} H.$$

By Zaslavsky, the number of regions of an arrangement is found by the formula.

Theorem 5.1 (see [10]). *Let $\mathcal{A}$ be a hyperplane arrangement in $\mathbb{R}^n$. Then*

$$\#\{\text{regions}\} = \sum_{x \in P} |\mu(\hat{0}, x)|,$$

where P is the intersection poset of arrangement $\mathcal{A}$ and μ is the Möbius function of P .

The following theorem describes the number of regions of graphical arrangement.

Theorem 5.2. *The number of regions of the graphical arrangement of a graph G is equal to*

$$\sum_H d_H$$

where the sum is over all subgraphs H of G whose connected components are induced subgraphs with at least two vertices and d_H is the number of decreasing maximal chains in the intersection poset P_H .

Proof. Let $A = A_1 - A_2 - \cdots - A_p$ be an element of the intersection poset P_G where A_i , $i = 1, 2, \dots, p$, are blocks of A . Each A_i contains indices or vertices of G . Let G_i be the subgraph induced by A_i . Let G_A be the union of subgraphs G_i , $i = 1, 2, \dots, p$. Then for each element A of P_G , we get a subgraph G_A of G . It is obvious that the intersection poset P_{G_A} is isomorphic to the interval $[\hat{0}, A]$ of P_G (see [5, Lemma 2.11]). By

[9, Theorem 4.11], $|\mu(\hat{0}, A)|$ is equal to the number d_{G_A} of decreasing maximal chains of P_{G_A} . By Theorem 5.1, the number of regions of $\mathcal{A}_G$ is

$$\sum_{A \in P_G} |\mu(\hat{0}, A)| = \sum_{A \in P_G} d_{G_A}.$$

Since each block of an element A in P_G can be associated with an induced subgraph of G with at least two vertices, the result follows. $\square$

Now we provide some enumerative results.

Proposition 5.3. *The number of regions of graphical arrangement of a forest with m edges is 2^m*

Proof. It is easy to see that the intersection poset of $\mathcal{A}_G$, where G is a forest, has only one decreasing maximal chain, and the intersection poset of arrangement of every its subgraph also has one decreasing maximal chain. To find the number of regions, we need to find the number of subgraphs of G satisfying the condition of Theorem 5.2. Since the union of any edges can give a subgraph satisfying the condition of Theorem 5.2, the number of regions is the number of subsets of edges, and equal to 2^m . $\square$

Remark 5.4. In general, a graphical arrangement is not in general position, but the graphical arrangement for a forest is in general position. It is known that if an arrangement of m hyperplanes in $\mathbb{R}^n$ is in general position, then the number of regions is found by the formula (see [9, Proposition 2.4])

$$1 + m + \binom{m}{2} + \cdots + \binom{m}{n}.$$

So we can also use this formula to show Proposition 5.3.

Proposition 5.5. *Let G be a graph with m edges and a unique cycle C of length k . Then the number of regions of graphical arrangement of G is $2^m - 2^{m-k+1}$.*

Proof. We would like to count the number of subgraphs of G satisfying the conditions of Theorem 5.2.

First, consider the case when the subgraph H does not contain C . If H contains all but one edge of C , the connected component of H containing those edges is not an induced subgraph. Since there are 2^{m-k} subgraphs of G containing C and $k2^{m-k}$ subgraphs of G containing all but one edge of C , there are $2^m - (k+1)2^{m-k}$ subgraphs of G with no cycle which satisfy the condition of Theorem 5.2.

Since any subgraph of G containing C satisfies the condition of Theorem 5.2, there are 2^{m-k} subgraphs of G containing C .

By Proposition 4.13, the intersection poset of graphical arrangement of graph with no cycle has 1 decreasing maximal chain, and the number of decreasing maximal chains is $k - 1$ when the graph has one cycle of length k . Theorem 5.2 implies the number of regions is

$$2^m - (k + 1)2^{m-k} + (k - 1)2^{m-k} = 2^m - 2^{m-k+1}.$$

□

Proposition 5.6. *Let G be a graph with m edges and s cycles $C_1, C_2, \dots, C_s$ of length $k_1, k_2, \dots, k_s$ respectively such that they do not share edges. Then the number of regions of graphical arrangement of G is*

$$\sum_{Y \subseteq [s]} [S_Y \prod_{j \in Y} (k_j - 1)],$$

where

$$S_Y = 2^{m - \sum_{j \in Y} k_j} - \sum_{\emptyset \neq X \subseteq [s] \setminus Y} [(\prod_{j \in X} k_j + 1) 2^{m - \sum_{j \in Y} k_j - \sum_{j \in X} k_j}]$$

for subsets X, Y of $[s]$.

Proof. We would like to count the number of subgraphs of G satisfying the conditions of Theorem 5.2.

Consider the case when the subgraph H does not contain any cycle. If H contains all but one edge of any cycle, the connected component of H containing those edges is not an induced subgraph. For r cycles $C_{x_1}, C_{x_2}, \dots, C_{x_r}$, there are $2^{m-k_{x_1}-\dots-k_{x_r}}$ subgraphs containing $C_{x_1}, C_{x_2}, \dots, C_{x_r}$ and $k_{x_1}k_{x_2} \dots k_{x_r} 2^{m-k_{x_1}-\dots-k_{x_r}}$ subgraphs containing all but one edge in each cycle $C_{x_1}, C_{x_2}, \dots, C_{x_r}$. Thus the number of subgraphs of G with no cycle which satisfy the condition of Theorem 5.2 is

$$\begin{aligned} S &= 2^m - \sum_{r=1}^s [\sum_{x_1, x_2, \dots, x_r} [(\prod_{q=1}^r k_{x_q} + 1) 2^{m - \sum_{q=1}^r k_{x_q}}]] \\ &= 2^m - \sum_{\emptyset \neq X \subseteq [s]} [(\prod_{j \in X} k_j + 1) 2^{m - \sum_{j \in X} k_j}] \end{aligned}$$

When H contains cycles $C_{y_1}, C_{y_2}, \dots, C_{y_u}$ for $y_1, y_2, \dots, y_u \in Y \subseteq [s]$, we have

$$S_Y = 2^{m - \sum_{j \in Y} k_j} - \sum_{\emptyset \neq X \subseteq [s] \setminus Y} [(\prod_{j \in X} k_j + 1) 2^{m - \sum_{j \in Y} k_j - \sum_{j \in X} k_j}]$$

subgraphs of G containing $C_{y_1}, C_{y_2}, \dots, C_{y_u}$ which satisfy the condition of Theorem 5.2.

By Proposition 4.15 and Theorem 5.2, the number of regions is

$$S + \sum_{\emptyset \neq Y \subseteq [s]} [S_Y \prod_{j \in Y} (k_j - 1)].$$

Since $S = S_Y$ when $Y = \emptyset$, the result follows. $\square$

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