DERIVATIONS OF A COMBINATORIAL LIE ALGEBRA

SEUL HEE CHOI

Abstract. We consider the simple antisymmetrized algebra $N(e^{\mathfrak{A}_P}, n, t)_1^-$. The simple non-associative algebra and its simple subalgebras are defined in the papers [1], [3], [4], [5], [6], [8], [13]. Some authors found all the derivations of an associative algebra, a Lie algebra, and a non-associative algebra in their papers [2], [3], [5], [7], [9], [10], [13], [15], [16]. We find all the derivations of the Lie subalgebra $N(e^{\pm x_1 x_2 x_3}, 0, 3)_{[1]}^-$ of $N(e^{\mathfrak{A}_P}, n, t)_k^-$ in this paper.

1. Preliminaries

Let $\mathbb{N}$ be the set of all non-negative integers and $\mathbb{Z}$ be the set of all integers. Let $\mathbb{N}^+$ be the set of all positive integers. Let $\mathbb{F}$ be a field of characteristic zero and $\mathbb{F}^\bullet$ the set of all non-zero elements in $\mathbb{F}$. For fixed integers $i_1, \dots, i_m$ and for given irreducible polynomials $f_1, \dots, f_m \in \mathbb{F}[x_1, \dots, x_{n+t}]$, we define P_m as the set $\{f_1^{i_1} \dots f_m^{i_m}, f_1^{i_1} \dots f_{m-1}^{i_{m-1}}, \dots, f_2^{i_2} \dots f_m^{i_m}, \dots, f_1^{i_1}, \dots, f_m^{i_m}\}$. For any subset P of P_m , we define the $\mathbb{F}$ -algebra $\mathbb{F}[e^{\pm P}, n, t] := \mathbb{F}[e^{\pm P}, x_1^{\pm 1}, \dots, x_n^{\pm 1}, x_{n+1}, \dots, x_{n+t}]$ such that its basis is

$$\mathbf{B} = \{e^{a_1 g_1} \dots e^{a_r g_r} x_1^{j_1} \dots x_{n+t}^{j_{n+t}} \mid g_1, \dots, g_r \in P, a_1, \dots, a_r \in \mathbb{Z}, \\ j_1, \dots, j_n \in \mathbb{Z}, j_{n+1}, \dots, j_{n+t} \in \mathbb{N}\}$$

(see [11], [12], [14], [17]). We denote $\partial_{\gamma_1}^{s_1} \dots \partial_{\gamma_r}^{s_r}$ as the composition of the partial derivatives $\partial_{\gamma_1}, \dots, \partial_{\gamma_r}$ on $\mathbb{F}[e^{\pm P}, n, t]$, $1 \leq \gamma_1, \dots, \gamma_r \leq n+t$, and we denote ∂_h^0 , $1 \leq h \leq n+t$, as the identity map on $\mathbb{F}[e^{\pm P}, n, t]$ where $s_1, \dots, s_r \in \mathbb{N}$. For any $u \in P \subset P_m$, let $\mathfrak{A}_u$ be an additive subgroup of $\mathbb{F}$ such that $\mathbb{Z}$ is a subgroup of $\mathfrak{A}_u$. For a given $k \in \mathbb{N}$

Received April 30, 2014. Accepted July 1, 2014.

2010 Mathematics Subject Classification. Primary 17B40, 17B56.

Key words and phrases. simple, combinatorial algebra, antisymmetrized algebra, derivation, lexicographic order.

(resp. $k \in \mathbb{N}^+$), we define the (free) $\mathbb{F}$ -vector space $N(e^{\mathfrak{A}_P}, n, t)_k$ (resp. $N(e^{\mathfrak{A}_P}, n, t)_{k+}$) spanned by the set

$$(1) \quad \mathbf{B}_1 = \{e^{a_1 g_1} \cdots e^{a_r g_r} x_1^{j_1} \cdots x_{n+t}^{j_{n+t}} \partial_{\gamma_1}^{s_1} \cdots \partial_{\gamma_r}^{s_r} \mid a_1 \in \mathfrak{A}_1, \dots, a_r \in \mathfrak{A}_r, \\ g_1, \dots, g_r \in P, \gamma_1, \dots, \gamma_r \leq n+t, s_1, \dots, s_r \leq k\}.$$

For any $g_1 \partial_{\gamma_1}^{s_1} \cdots \partial_{\gamma_r}^{s_r}, g_2 \partial_{u_1}^{v_1} \cdots \partial_{u_q}^{v_q} \in N(e^{\mathfrak{A}_P}, n, t)_k$, if we define the multiplication $*$ on $N(e^{\mathfrak{A}_P}, n, t)_k$ as follows:

$$(2) \quad g_1 \partial_{\gamma_1}^{s_1} \cdots \partial_{\gamma_r}^{s_r} * g_2 \partial_{u_1}^{v_1} \cdots \partial_{u_q}^{v_q} = g_1 (\partial_{\gamma_1}^{s_1} \cdots \partial_{\gamma_r}^{s_r} (g_2)) \partial_{u_1}^{v_1} \cdots \partial_{u_q}^{v_q},$$

then we define the combinatorial algebra $N(e^{\mathfrak{A}_P}, n, t)_k$ whose underlying vector space is $N(e^{\mathfrak{A}_P}, n, t)_k$ and whose multiplication is $*$ in (2) (see [3], [5], [13] and [15]). For a given positive integer k , we define the non-associative subalgebra $N(e^{\mathfrak{A}_P}, n, t)_{<k>}$ of the algebra $N(e^{\mathfrak{A}_P}, n, t)_k$ such that its set is

$$(3) \quad \{g \partial_{\gamma_1}^{s_1} \cdots \partial_{\gamma_r}^{s_r} \mid g \in \mathbf{B}, 1 \leq \gamma_1, \dots, \gamma_r \leq n+t, s_1 + \cdots + s_r = k\}.$$

Similarly for a given positive integer k , we define the non-associative subalgebra $N(e^{\mathfrak{A}_P}, n, t)_{[k]}$ of the algebra $N(e^{\mathfrak{A}_P}, n, t)_k$ such that $N(e^{\mathfrak{A}_P}, n, t)_{[k]}$ is the set

$$(4) \quad \{f \partial_h^k \mid f \in \mathbf{B}, 1 \leq h \leq n\}.$$

For an algebra A and $l \in A$, an element $l_1 \in A$ is a right (resp. left) identity of l , if $l * l_1 = l$ (resp. $l_1 * l = l$) holds. The set of all right identities of $N(e^{\mathfrak{A}_P}, n, t)_{[1]}$ is $\{\sum_{1 \leq u \leq n+t} x_u \partial_u + \sum_{1 \leq u \leq n+t} c_u \partial_u \mid c_u \in \mathbb{F}\}$. There is no left identity of $N(e^{\mathfrak{A}_P}, n, t)_{k+}$ (see [6], [7], and [13]). The algebra $N(e^{\mathfrak{A}_P}, n, t)_k$ has the left identity 1. If A is an associative $\mathbb{F}$ -algebra, then the antisymmetrized algebra A^- of A is a Lie algebra relative to the commutator $[x, y] := xy - yx$. So we define the antisymmetrized algebra $N(e^{\mathfrak{A}_P}, n, t)_k^-$ and its subalgebras obviously. For a general nonassociative $\mathbb{F}$ -algebra N we define in the same way its antisymmetrized algebra N^- . In case N^- is a Lie algebra we shall say that N is Lie admissible. For a subset S of N^- , an element l is an ad-diagonal element with respect to S , if for any $l_1 \in S$, $[l, l_1] = cl_1$ for $c \in \mathbb{F}$. For any element l of an algebra A , the abelian hull $A\gamma_l$ of l is the maximal abelian subalgebra of the antisymmetrized algebra A^- such that it contains l . The algebra $N(e^{\mathfrak{A}_P}, n, t)_{[1]}$ is Lie admissible (see [1], [13] and [19]), i.e., $N(e^{\mathfrak{A}_P}, n, t)_{[1]}^-$ is a Lie algebra (see [18]). Since the cardinality $|P|$ of

P is $2^m - 1$, then the algebra $N(e^{\mathfrak{A}_{P_m}}, n, t)_k$ is $\mathbb{Z}^{2^m}$ -graded as follows:

$$(5) \quad N(e^{\mathfrak{A}_{P_m}}, n, t)_k = \bigoplus_{(a_1, \dots, a_{m^2})} N_{(a_1, \dots, a_{m^2})}$$

where $N_{(a_1, \dots, a_{m^2})}$ is the vector subspace of $N(e^{\mathfrak{A}_{P_m}}, n, t)_k$ spanned by

$$(6) \quad \{e^{a_1 f_1} \dots e^{a_r f_r} x_1^{j_1} \dots x_{n+t}^{j_{n+t}} \partial_{\gamma_1}^{s_1} \dots \partial_{\gamma_r}^{s_r} \mid j_1, \dots, j_n \in \mathbb{Z}, \\ j_{n+1}, \dots, j_{n+t} \in \mathbb{N}, s_1, \dots, s_r \leq k\}.$$

This implies that $N(e^{\mathfrak{A}_P}, n, t)_k$ and $N(e^{\mathfrak{A}_P}, n, t)_{k+}$ are appropriate graded algebras as (5). Throughout the paper, the $(0, \dots, 0)$ -homogeneous component N_0 of $N(e^{\mathfrak{A}_P}, n, t)_k$ is the subalgebra $N(0, n, t)_k$ of $N(e^{\mathfrak{A}_P}, n, t)_k$. For any standard basis element $e^{a_1 f_1} \dots e^{a_r f_r} x_1^{j_1} \dots x_{n+t}^{j_{n+t}} \partial_{\gamma_1}^{s_1} \dots \partial_{\gamma_r}^{s_r}$ of $N(e^{\mathfrak{A}_{P_m}}, n, t)_k$, we define the homogeneous degree of the element as follows:

$$hd(e^{a_1 f_1} \dots e^{a_r f_r} x_1^{j_1} \dots x_{n+t}^{j_{n+t}} \partial_{\gamma_1}^{s_1} \dots \partial_{\gamma_r}^{s_r}) = \sum_{u=1}^{n+t} |j_u|$$

where $|j_u|$ is the absolute value of j_u for $1 \leq u \leq n+t$ (see [7]). For any element $l \in N(e^{\mathfrak{A}_P}, n, t)_k$, we define $hd(l)$ as the highest homogeneous degree of each monomial of l . Note that the set of all right annihilators of $N(e^{\mathfrak{A}_P}, n, t)_k$ is the subalgebra T_{n+t} of $N(e^{\mathfrak{A}_P}, n, t)_k$ which is spanned by $\{\partial_{\gamma_1}^{s_1} \dots \partial_{\gamma_r}^{s_r} \mid 1 \leq \gamma_1, \dots, \gamma_r \leq n+t, s_1, \dots, s_r \in k\}$.

2. Derivations of the Lie algebra $N(e^{\pm x_1 x_2 x_3}, 0, 3)_{[1]}^-$

Since $N(e^{\mathfrak{A}_P}, n, t)_k$, $N(e^{\mathfrak{A}_P}, n, t)_{<k>}$, $N(e^{\mathfrak{A}_P}, n, t)_{[k]}$, $N(e^{\mathfrak{A}_P}, n, t)_k^-$, $N(e^{\mathfrak{A}_P}, n, t)_{<k>}$, $N(e^{\mathfrak{A}_P}, n, t)_{[k]}^-$ are simple, we have the following.

Corollary 2.1. *The non-associative algebra $N(e^{\pm x_1 x_2 x_3}, 0, 3)_{[1]}$ and the Lie algebra $N(e^{\pm x_1 x_2 x_3}, 0, 3)_{[1]}^-$ are simple (see [8]).*

For a standard base $e^{p x_1 x_2 x_3} x_1^i x_2^j x_3^k \partial_u$, $1 \leq u \leq 3$, of the algebra $N(e^{\pm x_1 x_2 x_3}, 0, 3)_{[1]}^-$, we define the total degree $t_d(e^{p x_1 x_2 x_3} x_1^i x_2^j x_3^k \partial_u)$ as $i + j + k$. Note that $N(e^{\pm x_1 x_2 x_3}, 0, 3)_{[1]}^-$ is not a self-centralized Lie algebra (see [12]). For any two bases elements $e^{p_1 x_1 x_2 x_3} x_1^{i_1} x_2^{j_1} x_3^{k_1} \partial_u$ and $e^{p_2 x_1 x_2 x_3} x_1^{i_2} x_2^{j_2} x_3^{k_2} \partial_v$ of the algebra $N(e^{\pm x_1 x_2 x_3}, 0, 3)_{[1]}^-$, we define the lexicographic order $>$ as follows:
 $e^{p_1 x_1 x_2 x_3} x_1^{i_1} x_2^{j_1} x_3^{k_1} \partial_u > e^{p_2 x_1 x_2 x_3} x_1^{i_2} x_2^{j_2} x_3^{k_2} \partial_v$ if and only if $p_1 > p_2$ or $p_1 = p_2$ and $i_1 + j_1 + k_1 > i_2 + j_2 + k_2$, or $p_1 = p_2$, $i_1 + j_1 + k_1 = i_2 + j_2 + k_2$,

and $i_1 > i_2$ or $p_1 = p_2$, $i_1 + j_1 + k_1 = i_2 + j_2 + k_2$, $i_1 = i_2$ and $j_1 > j_2$. The maximal total degree of a $f(x_1, x_2, x_3)\partial_u \in N(e^{\pm x_1 x_2 x_3}, 0, 3)_{[1]}^-$, $1 \leq u \leq 3$, is the highest total degree of each monomial of $f(x_1, x_2, x_3)$.

Lemma 2.1. *For any derivation D of the Lie algebra $N(e^{\pm x_1 x_2 x_3}, 0, 3)_{[1]}^-$, if $D(\partial_u) = 0$, $1 \leq u \leq 3$, then we have that*

$$D(x_1^i x_2^j x_3^k \partial_u) = \sum_{1 \leq v \leq 3} ad_{\partial_v}(x_1^i x_2^j x_3^k \partial_u)$$

where ad_{∂_u} is an inner derivation and $1 \leq u \leq 3$.

Proof. Since $D([\partial_1, x_1 \partial_1]) = 0$ holds, we are able to prove that $D(x_1 \partial_1) = \sum_{j=1}^3 c_{1,j} \partial_j$, where $c_{1,j} \in \mathbb{F}$. Similarly we prove that $D(x_v \partial_1) = \sum_{j=1}^3 c_{v,j} \partial_j$, where $c_{v,j} \in \mathbb{F}$, $2 \leq v \leq 3$. Since $D([\partial_1, x_1^2 \partial_1]) = 2D(x_1 \partial_1) = 2 \sum_{j=1}^3 c_{1,j} \partial_j$ and $D([\partial_1, x_1^2 \partial_1]) = D(x_1^2 \partial_1)$ hold, we also prove that $D(x_1^2 \partial_1) = 2 \sum_{j=1}^3 c_{1,j} x_1 \partial_j$. Since $D([x_2 \partial_1, x_1 \partial_1]) = D(x_2 \partial_1)$ holds, we can prove that $c_{1,2} = c_{2,2} = c_{2,3} = 0$. Similarly, $D([x_3 \partial_1, x_1 \partial_1]) = D(x_3 \partial_1)$, we can also prove that $c_{1,3} = c_{3,2} = c_{3,3} = 0$. This implies that $D(x_v \partial_1) = c_{v,1} \partial_1$, $1 \leq v \leq 3$ and $D(x_1^2 \partial_1) = 2c_{1,1} x_1 \partial_1$. Since $D([x_2 \partial_1, x_1^2 \partial_1]) = 2D(x_1 x_2 \partial_1)$ holds, we can also prove that $D(x_1 x_2 \partial_1) = c_{2,1} x_1 \partial_1 + c_{1,1} x_2 \partial_1$. Similarly, we prove that $D(x_i x_3 \partial_1) = c_{3,1} x_i \partial_1 + c_{i,1} x_3 \partial_1$, where $1 \leq i \leq 2$. Since $D([x_2 \partial_1, x_1 x_2 \partial_1]) = D(x_2^2 \partial_1)$ holds, we are able to prove that $D(x_2^2 \partial_1) = 2c_{2,1} x_2 \partial_1$. We are able to prove that $D(x_3^2 \partial_1) = 2c_{3,1} x_3 \partial_1$. So we have the following:

$$(7) \quad \begin{aligned} D(x_i^v \partial_1) &= v c_{i,1} x_i^{v-1} \partial_1, \\ D(x_i x_j \partial_1) &= c_{j,1} x_i \partial_1 + c_{i,1} x_j \partial_1, \end{aligned}$$

where $1 \leq i, j \leq 3$ and $1 \leq v \leq 2$.

Assume that $D(x_1^{i-1} \partial_1) = (i-1)c_{1,1}x_1^{i-2} \partial_1$ holds. Since $D([x_1^2 \partial_1, x_1^{i-1} \partial_1]) = (i-3)D(x_1^i \partial_1)$ holds, we also have that $D(x_1^i \partial_1) = i c_{1,1} x_1^{i-1} \partial_1$. Similarly, we also have that $D(x_j^i \partial_1) = i c_{j,1} x_j^{i-1} \partial_1$, $2 \leq j \leq 3$ and for all $i \in \mathbb{N}$. So, we are able to also prove that

$$D(x_j^i \partial_u) = i c_{j,u} x_j^{i-1} \partial_u = ad_{\partial_i}(x_j^i \partial_u),$$

where $1 \leq j, u \leq 3$. Thus, by $D([x_2^j \partial_1, x_1^{i+1} \partial_1]) = (i+1)D(x_1^i x_2^j \partial_1)$, we are able to prove that $D(x_1^i x_2^j \partial_1) = i c_{1,1} x_1^{i-1} x_2^j \partial_1 + j c_{2,1} x_1^i x_2^{j-1} \partial_1$. By $D([x_3^k \partial_1, x_1^{i+1} x_2^j \partial_1]) = (i+1)D(x_1^i x_2^j x_3^k \partial_1)$, we are also able to prove that $D(x_1^i x_2^j x_3^k \partial_1) = \sum_{1 \leq v \leq 3} (i \delta_{1,v} + j \delta_{2,v} + k \delta_{3,v}) c_{v,1} x_1^{i-\delta_{1,v}} x_2^{j-\delta_{2,v}} x_3^{k-\delta_{3,v}}$, where

$\delta_{a,v}$ is the Kronecker delta. Similarly, we can prove that

$$\begin{aligned} D(x_1^i x_2^j x_3^k \partial_u) &= \sum_{1 \leq v \leq 3} (i\delta_{1,v} + j\delta_{2,v} + k\delta_{3,v}) c_{v,u} x_1^{i-\delta_{1,v}} x_2^{j-\delta_{2,v}} x_3^{k-\delta_{3,v}} \partial_u \\ &= \sum_{1 \leq v \leq 3} ad_{\partial_v} (x_1^i x_2^j x_3^k \partial_u) \end{aligned}$$

where $2 \leq u \leq 3$. This is the required form of the lemma. Thus we have proven the lemma. $\square$

Lemma 2.2. *For any derivation D of the Lie algebra $N(e^{\pm x_1 x_2 x_3}, 0, 3)_{[1]}^-$, if $D(\partial_u) = 0$ and $D(x_1^i x_2^j x_3^k \partial_u) = 0$, $i, j, k \in \mathbb{N}$, then we have that*

$$D(e^{p x_1 x_2 x_3} x_1^i x_2^j x_3^k \partial_u) = p t_u e^{p x_1 x_2 x_3} x_1^i x_2^j x_3^k \partial_u$$

where $p \in \mathbb{Z}$, $t_u \in \mathbb{F}$, and $1 \leq u \leq 3$.

Proof. Since the algebra $N(e^{\pm x_1 x_2 x_3}, 0, 3)_{[1]}^-$ is $\mathbb{Z}$ -graded, it is enough to consider the following case:

$$(8) \quad D(e^{x_1 x_2 x_3} \partial_1) = \sum_{1 \leq v \leq 3} \sum_{i,j,k \geq 0} t_{i,j,k,v} e^{a x_1 x_2 x_3} x_1^i x_2^j x_3^k \partial_v$$

where $a \neq 0$. Since

$$(9) \quad \begin{aligned} D([x_1 \partial_1, e^{x_1 x_2 x_3} \partial_1]) &= D(e^{x_1 x_2 x_3} x_1 x_2 x_3 \partial_1) - D(e^{x_1 x_2 x_3} \partial_1), \\ D([x_v \partial_v, e^{x_1 x_2 x_3} \partial_1]) &= D(e^{x_1 x_2 x_3} x_1 x_2 x_3 \partial_1), \quad 2 \leq v \leq 3 \end{aligned}$$

hold, we have that

$$\begin{aligned} D(e^{x_1 x_2 x_3} \partial_1) &= D([x_2 \partial_2, e^{x_1 x_2 x_3} \partial_1]) - D([x_1 \partial_1, e^{x_1 x_2 x_3} \partial_1]) \\ &= D([x_3 \partial_3, e^{x_1 x_2 x_3} \partial_1]) - D([x_1 \partial_1, e^{x_1 x_2 x_3} \partial_1]). \end{aligned}$$

By $D(e^{x_1 x_2 x_3} \partial_1) = D([x_2 \partial_2, e^{x_1 x_2 x_3} \partial_1]) - D([x_1 \partial_1, e^{x_1 x_2 x_3} \partial_1])$, we can prove that

$$(10) \quad \begin{aligned} D(e^{x_1 x_2 x_3} \partial_1) &= \sum_{1 \leq v \leq 3} \sum_{i,j,k \geq 0} (j-i) t_{i,j,k,v} e^{a x_1 x_2 x_3} x_1^i x_2^j x_3^k \partial_v \\ &+ \sum_{i,j,k \geq 0} (t_{i,j,k,1} e^{a x_1 x_2 x_3} x_1^i x_2^j x_3^k \partial_1 - t_{i,j,k,2} e^{a x_1 x_2 x_3} x_1^i x_2^j x_3^k \partial_2). \end{aligned}$$

By $D(e^{x_1 x_2 x_3} \partial_1) = D([x_3 \partial_3, e^{x_1 x_2 x_3} \partial_1]) - D([x_1 \partial_1, e^{x_1 x_2 x_3} \partial_1])$, we also prove that

$$(11) \quad \begin{aligned} D(e^{x_1 x_2 x_3} \partial_1) &= \sum_{1 \leq v \leq 3} \sum_{i,j,k \geq 0} (k-i) t_{i,j,k,v} e^{a x_1 x_2 x_3} x_1^i x_2^j x_3^k \partial_v \\ &+ \sum_{i,j,k \geq 0} (t_{i,j,k,1} e^{a x_1 x_2 x_3} x_1^i x_2^j x_3^k \partial_1 - t_{i,j,k,3} e^{a x_1 x_2 x_3} x_1^i x_2^j x_3^k \partial_3). \end{aligned}$$

By comparing (10) and (11), we have that $t_{i,j,k,2} = t_{i,j,k,3} = 0$, for all $i, j, k \in \mathbb{N}$ and $t_{i,j,k,1} = 0$, for all $i, j, k \geq 1$. By putting $t_1 = t_{0,0,0,1}$, we have that

$$(12) \quad D(e^{x_1 x_2 x_3} \partial_1) = t_1 e^{a x_1 x_2 x_3} \partial_1.$$

Since $D([x_1 \partial_1, e^{x_1 x_2 x_3} \partial_1]) = D(e^{x_1 x_2 x_3} x_1 x_2 x_3 \partial_1) - D(e^{x_1 x_2 x_3} \partial_1)$ holds, we prove that

$$(13) \quad D(e^{x_1 x_2 x_3} x_1 x_2 x_3 \partial_1) = a t_1 e^{a x_1 x_2 x_3} x_1 x_2 x_3 \partial_1.$$

By $D([\partial_3, e^{x_1 x_2 x_3} \partial_1]) = D(e^{x_1 x_2 x_3} x_1 x_2 \partial_1)$, we prove that

$$(14) \quad D(e^{x_1 x_2 x_3} x_1 x_2 \partial_1) = a t_1 e^{a x_1 x_2 x_3} x_1 x_2 \partial_1.$$

Since

$$\begin{aligned} D([\partial_3, e^{x_1 x_2 x_3} x_1 x_2 x_3 \partial_1]) &= D(e^{x_1 x_2 x_3} x_1^2 x_2^2 x_3 \partial_1) + D(e^{x_1 x_2 x_3} x_1 x_2 \partial_1) \\ &= a^2 t_1 e^{a x_1 x_2 x_3} x_1^2 x_2^2 x_3 \partial_1 + a t_1 e^{a x_1 x_2 x_3} x_1 x_2 \partial_1 \end{aligned}$$

holds, we have that

$$(15) \quad D(e^{x_1 x_2 x_3} x_1^2 x_2^2 x_3 \partial_1) = a^2 t_1 e^{a x_1 x_2 x_3} x_1^2 x_2^2 x_3 \partial_1.$$

Since

$$\begin{aligned} (16) \quad D([x_1^2 x_2 \partial_1, e^{x_1 x_2 x_3} \partial_1]) &= D(e^{x_1 x_2 x_3} x_1^2 x_2^2 x_3 \partial_1) - 2D(e^{x_1 x_2 x_3} x_1 x_2 \partial_1) \\ &= a t_1 e^{a x_1 x_2 x_3} x_1^2 x_2^2 x_3 \partial_1 - 2 t_1 e^{a x_1 x_2 x_3} x_1 x_2 \partial_1 \end{aligned}$$

holds, by comparing (14) and (15) with (16), we are able to derive $a = 1$. So we have that

$$\begin{aligned} (17) \quad D(e^{x_1 x_2 x_3} \partial_1) &= t_1 e^{x_1 x_2 x_3} \partial_1, \\ D(e^{x_1 x_2 x_3} x_1 x_2 \partial_1) &= t_1 e^{x_1 x_2 x_3} x_1 x_2 \partial_1, \\ D(e^{x_1 x_2 x_3} x_1 x_2 x_3 \partial_1) &= t_1 e^{x_1 x_2 x_3} x_1 x_2 x_3 \partial_1. \end{aligned}$$

Since

$$\begin{aligned} D([\partial_2, e^{x_1 x_2 x_3} x_1 x_2 \partial_1]) &= D(e^{x_1 x_2 x_3} x_1^2 x_2 x_3 \partial_1) + D(e^{x_1 x_2 x_3} x_1 \partial_1) \\ &= t_1 e^{x_1 x_2 x_3} x_1^2 x_2 x_3 \partial_1 + t_1 e^{x_1 x_2 x_3} x_1 \partial_1 \end{aligned}$$

and

$$\begin{aligned} D([x_1^2 \partial_1, e^{x_1 x_2 x_3} \partial_1]) &= D(e^{x_1 x_2 x_3} x_1^2 x_2 x_3 \partial_1) - 2D(e^{x_1 x_2 x_3} x_1 \partial_1) \\ &= t_1 e^{x_1 x_2 x_3} x_1^2 x_2 x_3 \partial_1 - 2 t_1 e^{x_1 x_2 x_3} x_1 \partial_1 \end{aligned}$$

hold, we have that $D(e^{x_1 x_2 x_3} x_1 \partial_1) = t_1 e^{x_1 x_2 x_3} x_1 \partial_1$. So, by

$$D([e^{x_1 x_2 x_3} \partial_1, e^{x_1 x_2 x_3} x_1 \partial_1]) = D(e^{2x_1 x_2 x_3} \partial_1),$$

we are able to prove $D(e^{2x_1x_2x_3}\partial_1) = 2t_1e^{2x_1x_2x_3}\partial_1$. Inductively assume that

$$D(e^{(p-1)x_1x_2x_3}x_1^i\partial_1) = (p-1)t_1e^{(p-1)x_1x_2x_3}x_1^i\partial_1,$$

where $0 \leq i \leq 1$. From the following identities

$$D([e^{x_1x_2x_3}\partial_1, e^{(p-1)x_1x_2x_3}x_1\partial_1]) = (p-2)D(e^{px_1x_2x_3}x_1x_2x_3\partial_1) + D(e^{px_1x_2x_3}\partial_1)$$

and

$$D([e^{x_1x_2x_3}x_1\partial_1, e^{(p-1)x_1x_2x_3}\partial_1]) = (p-2)D(e^{px_1x_2x_3}x_1x_2x_3\partial_1) - D(e^{px_1x_2x_3}\partial_1),$$

we have that

$$\begin{aligned} 2D(e^{x_1x_2x_3}\partial_1) &= D([e^{x_1x_2x_3}\partial_1, e^{(p-1)x_1x_2x_3}x_1\partial_1]) \\ &\quad - D([e^{x_1x_2x_3}x_1\partial_1, e^{(p-1)x_1x_2x_3}\partial_1]). \end{aligned}$$

So we are able to prove that $D(e^{px_1x_2x_3}\partial_1) = pt_1e^{px_1x_2x_3}\partial_1$. Similarly, we also prove that

$$D(e^{px_1x_2x_3}\partial_u) = pt_ue^{px_1x_2x_3}\partial_u,$$

where $2 \leq u \leq 3$. Inductively let us assume that $D(e^{px_1x_2x_3}x^{i-1}\partial_1) = pt_1e^{px_1x_2x_3}x^{i-1}\partial_1$ holds. Since

$$D([x_1^2\partial_1, e^{px_1x_2x_3}x_1^{i-1}\partial_1]) - D(x_1^{i+1}\partial_1, e^{px_1x_2x_3}\partial_1) = (2i-2)D(e^{px_1x_2x_3}x_1^i\partial_1)$$

holds, we have that

$$D(e^{px_1x_2x_3}x_1^i\partial_1) = pt_1e^{px_1x_2x_3}x_1^i\partial_1.$$

Similarly, we can prove that

$$D(e^{px_1x_2x_3}x_v^{i_v}\partial_u) = pt_1e^{px_1x_2x_3}x_v^{i_v}\partial_u,$$

where $1 \leq u, v \leq 3$. Since $D([x_1^2x_2^j\partial_1, e^{ax_1x_2x_3}x_1^{i-1}\partial_1]) - D([x_1^{i+1}x_2^j\partial_1, e^{ax_1x_2x_3}\partial_1]) = (2i-2)D(e^{ax_1x_2x_3}x_1^ix_2^j\partial_1)$ holds, we prove that

$$D(e^{px_1x_2x_3}x_1^ix_2^j\partial_1) = pt_1e^{px_1x_2x_3}x_1^ix_2^j\partial_1.$$

Since $D([x_1^2x_2^jx_3^k\partial_1, e^{ax_1x_2x_3}x_1^{i-1}\partial_1]) - D([x_1^{i+1}x_2^jx_3^k\partial_1, e^{ax_1x_2x_3}\partial_1]) = (2i-2)D(e^{px_1x_2x_3}x_1^ix_2^jx_3^k\partial_1)$ holds, we also prove that

$$D(e^{px_1x_2x_3}x_1^ix_2^jx_3^k\partial_1) = pt_1e^{px_1x_2x_3}x_1^ix_2^jx_3^k\partial_1.$$

Similarly, we can have that

$$D(e^{px_1x_2x_3}x_1^ix_2^jx_3^k\partial_u) = pt_1e^{px_1x_2x_3}x_1^ix_2^jx_3^k\partial_u,$$

$2 \leq u \leq 3$. This is the required form of the lemma. Thus we have proven the lemma. $\square$

Lemma 2.3. *There is no derivation D of the Lie algebra $N(e^{\pm x_1 x_2 x_3}, 0, 3)_{[1]}^-$ such that*

$$(18) \quad D(\partial_u) = \sum_{1 \leq v \leq 3} \alpha_{u,v} e^{p x_1 x_2 x_3} x_{i_v}^{j_v} \partial_v$$

where at least one of scalars $\alpha_{u,v}$ is not zero, $1 \leq i_v \leq 3$, $j_v \geq 0$, $\alpha_{u,v} \in \mathbb{F}$, $1 \leq u, v \leq 3$, and $p \in \mathbb{Z}$.

Proof. Since ∂_1 centralizes ∂_2 and ∂_3 , all of $\alpha_{u,1}$, $\alpha_{u,2}$, and $\alpha_{u,3}$ are zero. This is a contradiction. So there is no derivation D of $N(e^{\pm x_1 x_2 x_3}, 0, 3)_{[1]}^-$ such that $D(\partial_u) = \sum_{1 \leq v \leq 3} \alpha_{u,v} e^{p x_1 x_2 x_3} x_{i_v}^{j_v} \partial_v$ where at least one of scalars $\alpha_{u,v}$ is not zero. This completes the proof of the lemma. $\square$

Note 1. For any basis element $e^{p x_1 x_2 x_3} x_1^i x_2^j x_3^k \partial_u$, $1 \leq u \leq 3$, of $N(e^{\pm x_1 x_2 x_3}, 0, 3)_{[1]}^-$, if we define an $\mathbb{F}$ -map S from $N(e^{\pm x_1 x_2 x_3}, 0, 3)_{[1]}^-$ to itself as follows:

$$S(e^{p x_1 x_2 x_3} x_1^i x_2^j x_3^k \partial_u) = c p e^{p x_1 x_2 x_3} x_1^i x_2^j x_3^k \partial_u,$$

then S can be linearly extended to a derivation of $N(e^{\pm x_1 x_2 x_3}, 0, 3)_{[1]}^-$ where $c \in \mathbb{F}$ and S is called a scalar derivation.

Theorem 2.1. *For any derivation D of the Lie algebra $N(e^{\pm x_1 x_2 x_3}, 0, 3)_{[1]}^-$, D is the sum of inner and scalar derivations S which are shown in Note 2 (see [6], [9], [10]).*

Proof. Since the algebra $N(e^{\pm x_1 x_2 x_3}, 0, 3)_{[1]}^-$ is $\mathbb{Z}$ -graded, by Lemma 2.1 and Lemma 2.2, if $D(\partial_u)$, $1 \leq u \leq 3$, is in the 0-homogeneous component, then it is easy to prove that D is the sum of inner derivations and a scalar derivation. Let us assume that $D(\partial_1)$ is the sum of terms in different homogeneous components. Since D is a linear map, without loss of generality, we can assume that it is in the p -homogeneous component where $p \neq 0$. We may put $D(\partial_1)$ as follows:

$$(19) \quad \begin{aligned} D(\partial_1) = & b_{i,j,k,1} e^{p x_1 x_2 x_3} x_1^i x_2^j x_3^k \partial_1 + b_{i,j,k,2} e^{p x_1 x_2 x_3} x_1^i x_2^j x_3^k \partial_2 \\ & + b_{i,j,k,3} e^{p x_1 x_2 x_3} x_1^i x_2^j x_3^k \partial_3 + \#_1 \end{aligned}$$

where $e^{p x_1 x_2 x_3} x_1^i x_2^j x_3^k \partial_1$, $e^{p x_1 x_2 x_3} x_1^i x_2^j x_3^k \partial_2$ or $e^{p x_1 x_2 x_3} x_1^i x_2^j x_3^k \partial_3$ is the maximal term of $D(\partial_1)$ depending on their coefficients and $\#_1$ is the sum of the remaining terms of $D(\partial_1)$. Let us show that D is the sum of inner derivations and scalar derivations. Let us prove the statement by induction on i , j and k of $e^{p x_1 x_2 x_3} x_1^i x_2^j x_3^k \partial_u$, $1 \leq u \leq 3$. By

(19), we have that

$$(20) \quad \begin{aligned} D(\partial_1) = & [\partial_1, \frac{b_{i,j,k,1}}{p} e^{px_1x_2x_3} x_1^i x_2^{j-1} x_3^{k-1} \partial_1] \\ & + [\partial_1, \frac{b_{i,j,k,1}}{p} e^{px_1x_2x_3} x_1^i x_2^{j-1} x_3^{k-1} \partial_2] + \#_2 \end{aligned}$$

where $\#_2$ is the sum of the remaining terms of $D(\partial_1)$. By induction, we have that

$$(21) \quad \begin{aligned} D(\partial_1) = & [\partial_1, \frac{b_{i,j,k,1}}{p} e^{px_1x_2x_3} x_1^i x_2^{j-1} x_3^{k-1} \partial_1] \\ & + [\partial_1, \frac{b_{i,j,k,1}}{p} e^{px_1x_2x_3} x_1^i x_2^{j-1} x_3^{k-1} \partial_2] + \#_2 \\ = & ad_{\frac{b_{i,j,k,1}}{p} e^{px_1x_2x_3} x_1^i x_2^{j-1} \partial_1}(\partial_1) + ad_{\frac{b_{i,j,k,1}}{p} e^{px_1x_2x_3} x_1^i x_2^{j-1} x_3^{k-1} \partial_2}(\partial_1) \\ & + \cdots + ad_{\frac{b'_{0,0,0,1}}{p} e^{px_1x_2x_3} \partial_1}(\partial_1) + ad_{\frac{u'_{0,0,0,2}}{p} e^{px_1x_2x_3} \partial_2}(\partial_1) \\ & + ad_{\frac{u'_{0,0,0,3}}{p} e^{px_1x_2x_3} \partial_3}(\partial_1) + \cdots \\ & + \alpha_{1,1,i_1} e^{px_1x_2x_3} x_1^{i_1} \partial_1 \cdots \alpha_{1,1,0} e^{px_1x_2x_3} \partial_1 + \cdots \\ & + \alpha_{3,3,i_3} e^{px_1x_2x_3} x_3^{i_3} \partial_3 + \cdots + \alpha_{3,3,0} e^{px_1x_2x_3} \partial_3 \end{aligned}$$

where $\alpha_{1,1,0}, \dots, \alpha_{3,3,0} \in \mathbb{F}$ with appropriate scalars. By Lemma 2.3, $\alpha_{1,1,0}, \dots, \alpha_{3,3,0}$ are zeros. Inductively, by Lemma 2.2, we are able to prove that $D = S + \sum_l ad_l$, where S is shown in Note 1. Thus we have proven the lemma. $\square$

Corollary 2.2. *Let A be a Lie algebra. If every derivation of A is inner, then the algebras A and $N(e^{\pm x_1x_2x_3}, 0, 3)_{[1]}^-$ are not isomorphic.*

Proof. By Theorem 2.1, every derivation of $N(e^{\pm x_1x_2x_3}, 0, 3)_{[1]}^-$ is the sum of inner derivations and scalar derivations. Since every derivation of A is inner, A and $N(e^{\pm x_1x_2x_3}, 0, 3)_{[1]}^-$ are not isomorphic. Thus we have proven the corollary. $\square$

Proposition 2.1. *The maximal associative algebra of the $N(e^{\pm x_1x_2x_3}, 0, 3)_{[1]}$ is the matrix ring $M_3(\mathbb{F})$ and the maximal simple Lie subalgebra of $N(e^{\pm x_1x_2x_3}, 0, 3)_{[1]}^-$ is $sl_3(\mathbb{F})$.*

Proof. Since the algebra $N(e^{\pm x_1x_2x_3}, 0, 3)_{[1]}$ is Z -graded, $\sum_{1 \leq u \leq 3} (x_u \partial_u + c_u \partial_u)$ is a right identity of the algebra, where $c_u \in \mathbb{F}$. This implies that $\{(x_u + c_u) \partial_u, x_v \partial_v \mid 1 \leq u \leq 3, 1 \leq v \neq u \leq 3\}$ is a basis of subalgebra A of $N(e^{\pm x_1x_2x_3}, 0, 3)_{[1]}$ which is isomorphic to $M_3(\mathbb{F})$. Obviously, A is maximal simple subalgebra of $N(e^{\pm x_1x_2x_3}, 0, 3)_{[1]}$. This also implies that

$sl_3(\mathbb{F})$ is the 3-dimensional maximal Lie subalgebra of the Lie algebra $N(e^{\pm x_1 x_2 x_3}, 0, 3)_{[1]}^-$. This completes the proof of the proposition. $\square$

References

- [1] Mohammad H. Ahmadi, Ki-Bong Nam, and Jonathan Pakianathan, *Lie admissible non-associative algebras*, Algebra Colloquium, **12(1)**, World Scientific, (March 2005), 113-120.
- [2] Matej Brešar, Mikhail A. Chebotar, Wallace S. Martindale, *Function Identities*, Frontiers in Mathematics, Birkhauser, 2007.
- [3] Seul Hee Choi and Ki-Bong Nam, *The Derivation of a Restricted Weyl Type Non-Associative Algebra*, Hadronic Journal, **28(3)** (2005), 287-295.
- [4] Seul Hee Choi, *An algebra with right identities and its antisymmetrized algebra*, Honam Mathematical Journal, **29(2)** (2007), 213-222.
- [5] Seul Hee Choi and Ki-Bong Nam, *Weyl type non-associative algebra using additive groups I*, Algebra Colloquium, **14(3)** (2007), 479-488.
- [6] Seul Hee Choi and Ki-Bong Nam, *Derivations of a restricted Weyl Type Algebra I*, Rocky Mountain Math. Journals, **37(6)** (2007), 67-84.
- [7] Seul Hee Choi, Jongwoo Lee, and Ki-Bong Nam, *Derivations of a restricted Weyl type algebra containing the polynomial ring*, Communications in Algebra, **36(9)** (September 2008), 3435-3446.
- [8] Seul Hee Choi, Hong Gu Park, Moon-Ok Wang, and Ki-Bong Nam, *Combinatorial Algebra and Its Antisymmetrized Algebra I*, Algebra Colloquium, Accepted, 2012.
- [9] James E. Humphreys, *Introduction to Lie Algebras and Representation Theory*, Springer-Verlag, New York, (1987), 7-21.
- [10] Tsshiharū Ikeda, Naoki Kawamoto and Ki-Bong Nam, *A class of simple subalgebras of Generalized W algebras*, Proceedings of the International Conference in 1998 at Pusan (Eds. A. C. Kim), Walter de Gruyter GmbH Co. KG, (2000), 189-202.
- [11] Victor G. Kac, *Description of Filtered Lie Algebra with which Graded Lie algebras of Cartan type are Associated*, Izv. Akad. Nauk SSSR, Ser. Mat. Tom, **38** (1974), 832-834.
- [12] Naoki Kawamoto, Atsushi Mitsukawa, Ki-Bong Nam, and Moon-Ok Wang, *The automorphisms of generalized Witt type Lie algebras*, Journal of Lie Theory, **13(2)**, Heldermann Verlag, (2003), 571-576.
- [13] Jongwoo Lee and Ki-bong Nam, *Non-Associative Algebras containing the Matrix Ring*, Linear Algebra and its Applications **429(1)** (1 July 2008), 72-78.
- [14] Ki-Bong Nam, *Generalized W and H Type Lie Algebras*, Algebra Colloquium **6(3)** (1999), 329-340.
- [15] Ki-Bong Nam, *On Some Non-Associative Algebras Using Additive Groups*, Southeast Asian Bulletin of Mathematics, **27**, Springer Verlag, (2003), 493-500.
- [16] Ki-Bong Nam and Moon-Ok Wang, *Notes on Some Non-Associative Algebras*, Journal of Applied Algebra and Discrete Structured, **1(3)**, 159-164.
- [17] Donald P. Passman, *Simple Lie algebras of Witt type*, J. Algebra **206** (1998).
- [18] Alexei N. Rudakov, *Groups of Automorphisms of Infinite-Dimensional Simple Lie Algebras*, Math. USSR-Izvestija, **3** (1969), 707-722.

- [19] Richard D. Schafer, *Introduction to nonassociative algebras*, Dover, (1995), 128-138.

Seul Hee Choi
 Department of Mathematics, Jeonju University,
 Chon-ju 560-759, Korea.
 E-mail: chois@jj.ac.kr