THE INCOMPLETE LAURICELLA AND FIRST APPELL FUNCTIONS AND ASSOCIATED PROPERTIES

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Abstract. Recently, Srivastava *et al.* [18] introduced the incomplete Pochhammer symbol and studied some fundamental properties and characteristics of a family of potentially useful incomplete hypergeometric functions. Here we introduce the incomplete Lauricella function $\gamma_D^{(n)}$ and $\Gamma_D^{(n)}$ of n variables, and investigate certain properties of the incomplete Lauricella functions, for example, their various integral representations, differential formula and recurrence relation, in a rather systematic manner. Some interesting special cases of our main results are also considered.

1. Introduction, definitions and preliminaries

Throughout this paper, $\mathbb{N}$, $\mathbb{Z}^-$, and $\mathbb{C}$ denote the sets of positive integers, negative integers, complex numbers, respectively, $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$ and $\mathbb{Z}_0^- := \mathbb{Z}^- \cup \{0\}$.

The familiar *incomplete Gamma functions* $\gamma(s, x)$ and $\Gamma(s, x)$ defined by

$$(1.1) \quad \gamma(s, x) := \int_0^x t^{s-1} e^{-t} dt \quad (\Re(s) > 0; x \geq 0)$$

and

$$(1.2) \quad \Gamma(s, x) := \int_x^\infty t^{s-1} e^{-t} dt \quad (x \geq 0; \Re(s) > 0 \text{ when } x = 0)$$

Received May 19, 2014. Accepted July 7, 2014.

2010 Mathematics Subject Classification. Primary 33B15, 33B20, 33C05, 33C15, 33C20; Secondary 33B99, 33C99, 60B99.

Key words and phrases. Gamma functions, incomplete gamma function, Pochhammer symbol, incomplete Pochhammer symbol, incomplete generalized hypergeometric functions, Lauricella functions, Appell function, Bessel and modified Bessel functions, incomplete first Appell function, incomplete Lauricella function of n variables.

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respectively, satisfy the following decomposition formula:

$$(1.3) \quad \gamma(s, x) + \Gamma(s, x) := \Gamma(s) \quad (\Re(s) > 0),$$

where $\Gamma(s)$ is the familiar Gamma function defined by

$$(1.4) \quad \Gamma(s) = \int_0^\infty t^{s-1} e^{-t} dt \quad (\Re(s) > 0).$$

Each of these functions plays an important role in the study of the analytic solutions of a variety of problems in diverse areas of science and engineering (see, *e.g.*, [1, 2, 6, 7, 9, 11, 12, 13, 19, 20, 21, 23, 24]).

Very recently, Srivastava *et al.* [18] introduced and studied in a rather systematic manner the following two families of generalized incomplete hypergeometric functions:

$$(1.5) \quad {}_p\gamma_q \left[\begin{matrix} (\alpha_1, x), \alpha_2, \dots, \alpha_p; \\ \beta_1, \dots, \beta_q; \end{matrix} z \right] = \sum_{n=0}^{\infty} \frac{(\alpha_1; x)_n (\alpha_2)_n \cdots (\alpha_p)_n}{(\beta_1)_n \cdots (\beta_q)_n} \frac{z^n}{n!}$$

and

$$(1.6) \quad {}_p\Gamma_q \left[\begin{matrix} (\alpha_1, x), \alpha_2, \dots, \alpha_p; \\ \beta_1, \dots, \beta_q; \end{matrix} z \right] = \sum_{n=0}^{\infty} \frac{[\alpha_1; x]_n (\alpha_2)_n \cdots (\alpha_p)_n}{(\beta_1)_n \cdots (\beta_q)_n} \frac{z^n}{n!}$$

where, in terms of the incomplete Gamma functions $\gamma(s, x)$ and $\Gamma(s, x)$ defined by (1.1) and (1.2), the *incomplete* Pochhammer symbols $(\lambda; x)_\nu$ and $[\lambda; x]_\nu$ ($\lambda, \nu \in \mathbb{C}$; $x \geq 0$) are defined as follows:

$$(1.7) \quad (\lambda; x)_\nu := \frac{\gamma(\lambda + \nu, x)}{\Gamma(\lambda)} \quad (\lambda, \nu \in \mathbb{C}; x \geq 0)$$

and

$$(1.8) \quad [\lambda; x]_\nu := \frac{\Gamma(\lambda + \nu, x)}{\Gamma(\lambda)} \quad (\lambda, \nu \in \mathbb{C}; x \geq 0)$$

so that, obviously, these incomplete Pochhammer symbols $(\lambda; x)_\nu$ and $[\lambda; x]_\nu$ satisfy the following decomposition relation:

$$(1.9) \quad (\lambda; x)_\nu + [\lambda; x]_\nu := (\lambda)_\nu \quad (\lambda, \nu \in \mathbb{C}; x \geq 0),$$

where $(\lambda)_n$ is the Pochhammer symbol defined (for $\lambda \in \mathbb{C}$) by (see [20, p. 2 and p. 5]):

$$(1.10) \quad (\lambda)_n := \begin{cases} 1 & (n = 0) \\ \lambda(\lambda + 1) \cdots (\lambda + n - 1) & (n \in \mathbb{N}) \end{cases} \\ = \frac{\Gamma(\lambda + n)}{\Gamma(\lambda)} \quad (\lambda \in \mathbb{C} \setminus \mathbb{Z}_0^-).$$

As already pointed out by Srivastava *et al.* [18, Remark 7], since

$$|(\lambda; x)_n| \leq |(\lambda)_n| \quad \text{and} \quad |[\lambda; x]_n| \leq |(\lambda)_n| \quad (\lambda \in \mathbb{C}; n \in \mathbb{N}_0; x \geq 0)$$

the precise *sufficient* conditions under which the infinite series in definitions (1.5) and (1.6) would converge absolutely can be derived from those that are well-documented in the case of the generalized hypergeometric function ${}_pF_q$ ($p, q \in \mathbb{N}_0$) (see [15, p. 72-73] and [22, p. 20]); see also, [3, 4, 10] and [17]). Indeed, in their special case when $x = 0$, both ${}_p\gamma_q$ ($p, q \in \mathbb{N}_0$) and ${}_p\Gamma_q$ ($p, q \in \mathbb{N}_0$) would reduce immediately to the extensively-investigated generalized hypergeometric function ${}_pF_q$ ($p, q \in \mathbb{N}_0$).

Furthermore, it is easy to see from the definitions (1.5) and (1.6), we have the following decomposition formula:

$$(1.11) \quad \begin{aligned} & {}_p\gamma_q \left[\begin{matrix} (\alpha_1, x), \alpha_2, \dots, \alpha_p; \\ \beta_1, \dots, \beta_q; \end{matrix} z \right] + {}_p\Gamma_q \left[\begin{matrix} (\alpha_1, x), \alpha_2, \dots, \alpha_p; \\ \beta_1, \dots, \beta_q; \end{matrix} z \right] \\ & = {}_pF_q \left[\begin{matrix} \alpha_1, \alpha_2, \dots, \alpha_p; \\ \beta_1, \dots, \beta_q; \end{matrix} z \right] \end{aligned}$$

in terms of the familiar generalized hypergeometric function ${}_pF_q$.

Motivated essentially by the demonstrated potential for applications of these incomplete hypergeometric functions ${}_p\gamma_q$ and ${}_p\Gamma_q$, in many diverse areas of mathematical, physical, engineering and statistical sciences (see, for details, [5, 18] and the references cited therein), here, we aim at investigating, in a rather systematic manner, the incomplete family of Lauricella's functions $\gamma_D^{(n)}$ and $\Gamma_D^{(n)}$ of n variables and investigate certain properties of the incomplete Lauricella functions, for example, their various integral representations, differential formula and recurrence relations. Some integral representations involving Bessel and modified Bessel functions are also presented. In particular, for $n = 2$, we obtain similar results for the first Appell functions γ_1 and Γ_1 .

2. Incomplete Lauricella and first Appell functions

In terms of the incomplete Pochhammer symbol $(\lambda; x)_v$ and $[\lambda; x]_v$ defined by (1.7) and (1.8), we introduce the incomplete families of Lauricella hypergeometric functions $\gamma_D^{(n)}$ and $\Gamma_D^{(n)}$ of n variables as follows: For $\alpha, \beta_1, \dots, \beta_n \in \mathbb{C}$ and $\gamma_1, \dots, \gamma_n \in \mathbb{C} \setminus \mathbb{Z}_0^-$,

$$\begin{aligned}
 & \gamma_D^{(n)}[(\alpha, x), \beta_1, \dots, \beta_n; \gamma; x_1, \dots, x_n] \\
 (2.1) \quad &= \sum_{m_1, \dots, m_n=0}^{\infty} \frac{(\alpha; x)_{m_1+\dots+m_n} (\beta_1)_{m_1} \cdots (\beta_n)_{m_n}}{(\gamma)_{m_1+\dots+m_n}} \frac{x_1^{m_1}}{m_1!} \cdots \frac{x_n^{m_n}}{m_n!}
 \end{aligned}$$

and

$$\begin{aligned}
 & \Gamma_D^{(n)}[(\alpha, x), \beta_1, \dots, \beta_n; \gamma; x_1, \dots, x_n] \\
 (2.2) \quad &= \sum_{m_1, \dots, m_n=0}^{\infty} \frac{[\alpha; x]_{m_1+\dots+m_n} (\beta_1)_{m_1} \cdots (\beta_n)_{m_n}}{(\gamma)_{m_1+\dots+m_n}} \frac{x_1^{m_1}}{m_1!} \cdots \frac{x_n^{m_n}}{m_n!}.
 \end{aligned}$$

In view of (1.9), these incomplete families of Lauricella functions satisfy the following decomposition formula:

$$(2.3) \quad \gamma_D^{(n)} + \Gamma_D^{(n)} = F_D^{(n)},$$

where $F_D^{(n)}$ is the familiar Lauricella function (see [21, 22]).

Theorem 1. *The incomplete Lauricella's function $\gamma_D^{(n)}$ and $\Gamma_D^{(n)}$ satisfy the following partial differential equation:*

$$\begin{aligned}
 (2.4) \quad & x_j(1-x_j) \frac{\partial^2 u}{\partial x_j^2} + (1-x_j) \sum_{k=1, k \neq j}^n x_k \frac{\partial^2 u}{\partial x_k \partial x_j} \\
 & + [\gamma_j - (\alpha + \beta_j + 1)x_j] \frac{\partial u}{\partial x_j} - \beta_j \sum_{k=1, k \neq j}^n x_k \frac{\partial u}{\partial x_k} - \alpha \beta_j u = 0,
 \end{aligned}$$

where $j = 1, \dots, n$ and

$$\begin{aligned}
 u = u(x_1, \dots, x_n) := & \gamma_D^{(n)}[(\alpha, x), \beta_1, \dots, \beta_n; \gamma; x_1, \dots, x_n] \\
 & + \Gamma_D^{(n)}[(\alpha, x), \beta_1, \dots, \beta_n; \gamma; x_1, \dots, x_n].
 \end{aligned}$$

Proof. In the light of (2.3), it is easy to prove (2.4) since $F_D^{(n)}$ satisfies the same system of partial differential equations as in (2.4). $\square$

In particular, for $n = 2$, the corresponding incomplete families of first Appell hypergeometric functions

$$(2.5) \quad \gamma_D^{(2)} := \gamma_1 \quad \text{and} \quad \Gamma_D^{(2)} := \Gamma_1$$

are given by

$$\begin{aligned} \gamma_1[(\alpha, x), \beta_1, \beta_2; \gamma; x_1, x_2] &= \sum_{m,n=0}^{\infty} \frac{(\alpha; x)_{m+n} (\beta_1)_m (\beta_2)_n}{(\gamma)_{m+n}} \frac{x_1^m x_2^n}{m! n!} \\ (2.6) \quad &= \sum_{m=0}^{\infty} \frac{(\alpha)_m (\beta_1)_m}{(\gamma)_m} {}_2\gamma_1 \left[\begin{matrix} (\alpha + m, x), \beta_2; \\ \gamma + m; \end{matrix} x_2 \right] \frac{x_1^m}{m!} \end{aligned}$$

and

$$\begin{aligned} \Gamma_1[(\alpha, x), \beta_1, \beta_2; \gamma; x_1, x_2] &= \sum_{m,n=0}^{\infty} \frac{[\alpha; x]_{m+n} (\beta_1)_m (\beta_2)_n}{(\gamma)_{m+n}} \frac{x_1^m x_2^n}{m! n!} \\ (2.7) \quad &= \sum_{m=0}^{\infty} \frac{(\alpha)_m (\beta_1)_m}{(\gamma)_m} {}_2\Gamma_1 \left[\begin{matrix} (\alpha + m, x), \beta_2; \\ \gamma + m; \end{matrix} x_2 \right] \frac{x_1^m}{m!}, \end{aligned}$$

respectively.

Remark 1. The special cases of (2.1) and (2.2) when $n = 1$ are seen to yield the known incomplete families of Gauss hypergeometric functions [18]:

$$\gamma_D^{(1)} = {}_2\gamma_1[(\alpha, x); \beta_1; \gamma; x_1] \quad \text{and} \quad \Gamma_D^{(1)} = {}_2\Gamma_1[(\alpha, x); \beta_1; \gamma; x_1],$$

respectively.

It is noted in passing that, in view of the formula (2.3), it is sufficient to discuss the properties and characteristics of the incomplete Lauricella and first Appell functions $\Gamma_D^{(n)}$ and Γ_1 .

3. Integral Representations of Incomplete Lauricella and first Appell functions

Here, we present certain integral representations of incomplete Lauricella function by applying (1.2) and (1.8). We also obtain some integral representations involving Bessel and modified Bessel functions. Further, we give corresponding results for incomplete first Appell functions for $n = 2$.

Theorem 2. *The following integral representation for incomplete Lauricella's function $\Gamma_D^{(n)}$ in (2.2) holds true:*

$$\begin{aligned}
& \Gamma_D^{(n)}[(\alpha, x), \beta_1, \dots, \beta_n; \gamma; x_1, \dots, x_n] \\
(3.1) \quad &= \frac{1}{\Gamma(\alpha)} \int_x^\infty e^{-t} t^{\alpha-1} \Phi_2^{(n)}[\beta_1, \dots, \beta_n; \gamma; x_1 t, \dots, x_n t] dt \\
& (x \geq 0; \max\{\Re(x_1), \dots, \Re(x_n)\} < 1; \Re(\alpha) > 0 \text{ when } x = 0),
\end{aligned}$$

where $\Phi_2^{(n)}$ is one of the confluent forms of Lauricella series in n variables defined by (see, e.g., [21, p. 34, Eq. (8)])

$$\begin{aligned}
& \Phi_2^{(n)}[b_1, \dots, b_n; c; x_1, \dots, x_n] \\
(3.2) \quad &= \sum_{m_1, \dots, m_n=0}^{\infty} \frac{(b_1)_{m_1} \cdots (b_n)_{m_n}}{(c)_{m_1+\dots+m_n}} \frac{x_1^{m_1}}{m_1!} \cdots \frac{x_n^{m_n}}{m_n!}.
\end{aligned}$$

Proof. Using the definition of the incomplete Pochhammer symbol $[\alpha; x]_{m_1+\dots+m_n}$ in (2.2) by considering the integral representation from (1.2) and (1.8) and using the definition [21, p. 34(8)], we are led to the desired result. $\square$

Theorem 3. The following n -multiple integral representation for $\Gamma_D^{(n)}$ in (2.2) holds true:

$$\begin{aligned}
& \Gamma_D^{(n)}[(\alpha, x), \beta_1, \dots, \beta_n; \gamma; x_1, \dots, x_n] \\
(3.3) \quad &= \frac{1}{\Gamma(\beta_1) \cdots \Gamma(\beta_n)} \int_0^\infty \cdots \int_0^\infty e^{-s_1-\dots-s_n} s_1^{\beta_1-1} \cdots s_n^{\beta_n-1} \\
& \quad \times {}_1\Gamma_1((\alpha, x); \gamma; x_1 s_1 + \cdots + x_n s_n) ds_1 \cdots ds_n \\
& (x \geq 0; \Re(\beta_1), \dots, \Re(\beta_n) > 0 \text{ when } x = 0).
\end{aligned}$$

Proof. Using the integral representation of the classical Pochhammer symbol for $(\beta)_{m_1}, \dots, (\beta)_{m_n}$ in the definition (2.2) and using the definition of incomplete confluent hypergeometric function for the case $p = 1$ and $q = 1$ in [18, p. 675, Eq.(4.1)], we are led to the desired result. $\square$

Theorem 4. The following $n + 1$ -multiple integral representation for $\Gamma_D^{(n)}$ in (2.2) holds true:

$$\begin{aligned}
 & \Gamma_D^{(n)}[(\alpha, x), \beta_1, \dots, \beta_n; \gamma; x_1, \dots, x_n] \\
 (3.4) \quad &= \frac{1}{\Gamma(\alpha)\Gamma(\beta_1)\dots\Gamma(\beta_n)} \int_0^\infty \dots \int_0^\infty \int_x^\infty e^{-t-s_1-\dots-s_n} t^{\alpha-1} \\
 & \quad \times s_1^{\beta_1-1} \dots s_n^{\beta_n-1} {}_0F_1(-; \gamma; x_1 s_1 t + \dots + x_n s_n t) ds_1 \dots ds_n dt \\
 & \quad (x \geq 0; \Re(\alpha) > 0, \Re(\beta_1), \dots, \Re(\beta_n) > 0 \text{ when } x = 0).
 \end{aligned}$$

Proof. Using the integral representation of incomplete confluent hypergeometric function [18, p. 678, Eq.(5.2)] in the (3.3), we are led to the desired result. $\square$

In particular, for $n = 2$, the corresponding integral representations for incomplete first Appell functions are presented, without proof, as asserted by Theorems 5, 6 and 7 below.

Theorem 5. *The following integral representation for incomplete first Appell function Γ_1 in (2.7) holds true:*

$$\begin{aligned}
 & \Gamma_1[(\alpha, x), \beta_1, \beta_2; \gamma; x_1, x_2] \\
 (3.5) \quad &= \frac{1}{\Gamma(\alpha)} \int_x^\infty e^{-t} t^{\alpha-1} \Phi_2[\beta_1, \beta_2; \gamma; x_1 t, x_2 t] dt \\
 & \quad (x \geq 0; \max\{\Re(x_1), \Re(x_2)\} < 1; \Re(\alpha) > 0 \text{ when } x = 0).
 \end{aligned}$$

Theorem 6. *The following double integral representation for incomplete first Appell function Γ_1 in (2.7) holds true:*

$$\begin{aligned}
 & \Gamma_1[(\alpha, x), \beta_1, \beta_2; \gamma; x_1, x_2] = \frac{1}{\Gamma(\beta_1)\Gamma(\beta_2)} \int_0^\infty \int_0^\infty e^{-s_1-s_2} \\
 (3.6) \quad & \quad \times s_1^{\beta_1-1} s_2^{\beta_2-1} {}_1F_1((\alpha, x); \gamma; x_1 s_1 + x_2 s_2) ds_1 ds_2 \\
 & \quad (x \geq 0; \Re(\beta_1), \Re(\beta_2) > 0 \text{ when } x = 0).
 \end{aligned}$$

Theorem 7. *The following triple integral representation for incomplete first Appell function Γ_1 in (2.7) holds true:*

$$\begin{aligned}
 & \Gamma_1[(\alpha, x), \beta_1, \beta_2; \gamma; x_1, x_2] \\
 (3.7) \quad &= \frac{1}{\Gamma(\alpha)\Gamma(\beta_1)\Gamma(\beta_2)} \int_0^\infty \int_0^\infty \int_x^\infty e^{-t-s_1-s_2} t^{\alpha-1} s_1^{\beta_1-1} s_2^{\beta_2-1} \\
 & \quad \times {}_0F_1(-; \gamma; x_1 s_1 t + x_2 s_2 t) ds_1 ds_2 dt \\
 & \quad (x \geq 0; \Re(\alpha) > 0, \Re(\beta_1), \Re(\beta_2) > 0 \text{ when } x = 0).
 \end{aligned}$$

Remark 2. The Bessel function $J_\nu(z)$ and the modified Bessel function $I_\nu(z)$ are expressible in terms of hypergeometric functions as follows (see, e.g., [15]; see also [4, 10, 11, 13, 24, 25]):

$$(3.8) \quad J_\nu(z) = \frac{\left(\frac{z}{2}\right)^\nu}{\Gamma(\nu+1)} {}_0F_1\left(-; \nu+1; -\frac{1}{4}z^2\right) \quad (\nu \in \mathbb{C} \setminus \mathbb{Z}^-)$$

and

$$(3.9) \quad I_\nu(z) = \frac{\left(\frac{z}{2}\right)^\nu}{\Gamma(\nu+1)} {}_0F_1\left(-; \nu+1; \frac{1}{4}z^2\right) \quad (\nu \in \mathbb{C} \setminus \mathbb{Z}^-).$$

Now, applying the relationships (3.8) and (3.9) to (3.4) and (3.7), we can deduce certain interesting integral representations for the incomplete Lauricella's and Appell hypergeometric functions in (2.2) and (2.7) asserted by Corollaries 1 and 2 below. We give them without proof.

Corollary 1. *Each of the following $n+1$ -multiple integral representations holds true:*

$$(3.10) \quad \begin{aligned} & \Gamma_D^{(n)}[(\alpha, x), \beta_1, \beta_2; \gamma+1; -x_1, \dots, -x_n] = \frac{\Gamma(\gamma+1)}{\Gamma(\alpha)\Gamma(\beta_1)\dots\Gamma(\beta_n)} \\ & \times \int_0^\infty \dots \int_0^\infty \int_x^\infty e^{-t-s_1-\dots-s_n} t^{\alpha-\frac{\gamma}{2}-1} s_1^{\beta_1-1} \dots s_n^{\beta_n-1} \\ & \times (x_1 s_1 + \dots + x_n s_n)^{-\frac{\gamma}{2}} J_\gamma(2\sqrt{x_1 s_1 t + \dots + x_n s_n t}) ds_1 \dots ds_n dt \end{aligned}$$

and

$$(3.11) \quad \begin{aligned} & \Gamma_D^{(n)}[(\alpha, x), \beta_1, \dots, \beta_n; \gamma+1; x_1, \dots, x_n] = \frac{\Gamma(\gamma+1)}{\Gamma(\alpha)\Gamma(\beta_1)\dots\Gamma(\beta_n)} \\ & \times \int_0^\infty \dots \int_0^\infty \int_x^\infty e^{-t-s_1-\dots-s_n} t^{\alpha-\frac{\gamma}{2}-1} s_1^{\beta_1-1} \dots s_n^{\beta_n-1} \\ & \times (x_1 s_1 + \dots + x_n s_n)^{-\frac{\gamma}{2}} I_\gamma(2\sqrt{x_1 s_1 t + \dots + x_n s_n t}) ds_1 \dots ds_n dt \end{aligned}$$

provided that the integrals involved are convergent.

Corollary 2. *Each of the following triple integral representations holds true:*

$$(3.12) \quad \begin{aligned} & \Gamma_1[(\alpha, x), \beta_1, \beta_2; \gamma+1; -x_1, -x_2] = \frac{\Gamma(\gamma+1)}{\Gamma(\alpha)\Gamma(\beta_1)\Gamma(\beta_2)} \\ & \times \int_0^\infty \int_0^\infty \int_x^\infty e^{-t-s_1-s_2} t^{\alpha-\frac{\gamma}{2}-1} s_1^{\beta_1-1} s_2^{\beta_2-1} (x_1 s_1 + x_2 s_2)^{-\frac{\gamma}{2}} \\ & \times J_\gamma(2\sqrt{x_1 s_1 t + x_2 s_2 t}) ds_1 ds_2 dt \end{aligned}$$

and

$$\begin{aligned}
 \Gamma_1[(\alpha, x), \beta_1, \beta_2; \gamma + 1; x_1, x_2] &= \frac{\Gamma(\gamma + 1)}{\Gamma(\alpha)\Gamma(\beta_1)\Gamma(\beta_2)} \\
 (3.13) \quad &\times \int_0^\infty \int_0^\infty \int_x^\infty e^{-t-s_1-s_2} t^{\alpha-\frac{\gamma}{2}-1} s_1^{\beta_1-1} s_2^{\beta_2-1} (x_1 s_1 + x_2 s_2)^{-\frac{\gamma}{2}} \\
 &\times I_\gamma(2\sqrt{x_1 s_1 t + x_2 s_2 t}) ds_1 ds_2 dt
 \end{aligned}$$

provided that the integrals involved are convergent.

4. Derivative formula

Differentiating both sides of (2.2) with respect to $x_1, \dots, x_n$ partially $m_1, \dots, m_n$ times, respectively, we obtain a derivative formula for the incomplete Lauricella's hypergeometric function $\Gamma_D^{(n)}$ given in the following theorem.

Theorem 8. *The following derivative formula for $\Gamma_D^{(n)}$ holds true:*

$$\begin{aligned}
 (4.1) \quad &\frac{\partial^{m_1+\dots+m_n}}{\partial x_1^{m_1} \dots \partial x_n^{m_n}} \Gamma_D^{(n)}[(\alpha, x), \beta_1, \dots, \beta_n; \gamma; x_1, \dots, x_n] \\
 &= \frac{(\alpha)_{m_1+\dots+m_n} (\beta_1)_{m_1} \dots (\beta_n)_{m_n}}{(\gamma)_{m_1+\dots+m_n}} \Gamma_D^{(n)}[(\alpha + m_1 + \dots + m_n, x), \\
 &\quad \beta_1 + m_1, \dots, \beta_n + m_n; \gamma + m_1 + \dots + m_n; x_1, \dots, x_n].
 \end{aligned}$$

It is noted that a similar argument will establish the corresponding derivative formula of incomplete first Appell function Γ_1 as a special case of (4.1) when $n = 2$.

5. Recurrence Relation

In this section, we present a recurrence relation for incomplete Lauricella functions.

Theorem 9. *The following recurrence relations for $\Gamma_D^{(n)}$ holds true:*

$$\begin{aligned}
& \Gamma_D^{(n)}[(\alpha, x), \beta_1, \dots, \beta_n; \gamma; x_1, \dots, x_n] \\
&= \Gamma_D^{(n)}[(\alpha, x), \beta_1, \dots, \beta_n; \gamma - 1; x_1, \dots, x_n] \\
(5.1) \quad &+ \frac{\alpha\beta_1x_1}{\gamma(1-\gamma)} \Gamma_D^{(n)}[(\alpha + 1, x), \beta_1 + 1, \beta_2, \dots, \beta_n; \gamma + 1; x_1, \dots, x_n] \cdots \\
&+ \frac{\alpha\beta_nx_n}{\gamma(1-\gamma)} \Gamma_D^{(n)}[(\alpha + 1, x), \beta_1, \beta_2, \dots, \beta_{n-1}, \beta_n + 1; \gamma + 1; x_1, \dots, x_n].
\end{aligned}$$

Proof. Using the well-known contiguous relation for the function ${}_0F_1$ (see [16, p. 12])

$${}_0F_1(-; \gamma - 1; x) - {}_0F_1(-; \gamma; x) - \frac{x}{\gamma(\gamma - 1)} {}_0F_1(-; \gamma + 1; x) = 0$$

in the integral representation (3.4), we are led to the desired result. $\square$

It is noted that a similar argument will establish the corresponding recurrence relation of incomplete first Appell function Γ_1 as a special case of (5.1) when $n = 2$.

Acknowledgements. The authors would like to express their deep gratitude for the reviewers's helpful comments to improve this paper in the present form.

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