FALLING FUZZY BCI-COMMUTATIVE IDEALS

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Abstract. On the basis of the theory of a falling shadow and fuzzy sets, the notion of a falling fuzzy BCI-commutative ideal of a BCI-algebra is introduced. Relations between falling fuzzy BCI-commutative ideals and falling fuzzy ideals are given. Relations between fuzzy BCI-commutative ideals and falling fuzzy BCI-commutative ideals are provided. Characterizations of a falling fuzzy BCI-commutative ideal are established, and conditions for a falling fuzzy (closed) ideal to be a falling fuzzy BCI-commutative ideal are considered.

1. Introduction

In the study of a unified treatment of uncertainty modelled by means of combining probability and fuzzy set theory, Goodman [1] pointed out the equivalence of a fuzzy set and a class of random sets. Wang and Sanchez [2] introduced the theory of falling shadows which directly relates probability concepts with the membership function of fuzzy sets. Falling shadow representation theory shows us the way of selection re-laid on the joint degrees distributions. It is reasonable and convenient approach for the theoretical development and the practical applications of fuzzy sets and fuzzy logics. The mathematical structure of the theory of falling shadows is formulated in [3]. Tan et al. [4, 5] established a theoretical approach to define a fuzzy inference relation and fuzzy set operations based on the theory of falling shadows. Jun and Park [6] discussed the notion of a falling fuzzy subalgebra/ideal of a

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BCK/BCI-algebra. Yuan and Lee [7] considered a fuzzy subgroup (subring, ideal) as the falling shadow of the cloud of the subgroup (subring, ideal). Jun et al. [8] applied fuzzy soft sets for dealing with several kinds of theories in BCK/BCI-algebras. Jun et al. [9] discussed the fuzzy structures of hyper-MV-deductive systems in hyper-MV-algebras. Zhan and Jun [10] studied soft BL-algebras based on fuzzy sets. Jun et al. [11] dealt with more general form of fuzzy ideals in BCK/BCI-algebras. Jun and Kang [12] studied the falling fuzzy positive implicative ideals of a BCK-algebra. They provided relations between falling fuzzy positive implicative ideals and falling fuzzy ideals. They also considered relations between fuzzy positive implicative ideals and falling fuzzy positive implicative ideals. Jun and Kang [13] considered the fuzzification of generalized Tarski filters of generalized Tarski algebras, and investigated related properties. They established characterizations of a fuzzy generalized Tarski filter, and introduced the notion of falling fuzzy generalized Tarski filters in generalized Tarski algebras based on the theory of falling shadows. They provided relations between fuzzy generalized Tarski filters and falling fuzzy generalized Tarski filters, and established a characterization of a falling fuzzy generalized Tarski filter. They showed that every falling fuzzy generalized Tarski filter is a T_m -fuzzy generalized Tarski filter.

In this paper, we discuss the fuzzification of BCI-commutative ideals in BCI-algebras, and investigate related properties. We establish a theoretical approach for defining a fuzzy BCI-commutative ideal in a BCI-algebra based on the theory of falling shadows. We provide relations between falling fuzzy BCI-commutative ideals and falling fuzzy ideals. We also consider relations between fuzzy BCI-commutative ideals and falling fuzzy BCI-commutative ideals. We consider characterizations of a falling fuzzy BCI-commutative ideal, and give conditions for a falling fuzzy (closed) ideal to be a falling fuzzy BCI-commutative ideal. We show that every falling fuzzy BCI-commutative ideal is a T_m -fuzzy BCI-commutative ideal.

2. Preliminaries

An algebra $(X; *, 0)$ of type $(2, 0)$ is called a *BCI-algebra* if it satisfies the following conditions:

- (I) $(\forall x, y, z \in X) (((x * y) * (x * z)) * (z * y) = 0),$
- (II) $(\forall x, y \in X) ((x * (x * y)) * y = 0),$
- (III) $(\forall x \in X) (x * x = 0),$

(IV) $(\forall x, y \in X) (x * y = 0, y * x = 0 \Rightarrow x = y)$.

Any BCI-algebra X satisfies the following conditions:

- (a1) $(\forall x \in X) (x * 0 = x)$,
- (a2) $(\forall x, y, z \in X) ((x * y) * z = (x * z) * y)$,
- (a3) $(\forall x, y, z \in X) (0 * (0 * ((x * z) * (y * z))) = (0 * y) * (0 * x))$,
- (a4) $(\forall x, y \in X) (0 * (0 * (x * y)) = (0 * y) * (0 * x))$.
- (a5) $(\forall x, y, z \in X) (x \leq y \Rightarrow x * z \leq y * z, z * y \leq z * x)$.
- (a6) $(\forall x, y \in X) (x * (x * (x * y)) = x * y)$.
- (a7) $(\forall x, y \in X) (0 * (x * y) = (0 * x) * (0 * y))$.

A subset I of a BCI-algebra X is called an ideal of X if it satisfies:

- (b1) $0 \in I$.
- (b2) $(\forall x \in X) (\forall y \in I) (x * y \in I \Rightarrow x \in I)$.

Every ideal I of a BCI-algebra X has the following assertion:

$$(2.1) \quad (\forall x \in X) (\forall y \in I) (x \leq y \Rightarrow x \in I).$$

A subset I of a BCI-algebra X is called a BCI-commutative ideal of X if it satisfies (b1) and for all $x, y, z \in X$,

$$(b3) \quad (x * y) * z \in I, z \in I \Rightarrow x * ((y * (y * x)) * (0 * (0 * (x * y)))) \in I.$$

A fuzzy set μ in a BCI-algebra X is called a fuzzy ideal of X if it satisfies:

- (c1) $(\forall x \in X) (\mu(0) \geq \mu(x))$.
- (c2) $(\forall x, y \in X) (\mu(x) \geq \min\{\mu(x * y), \mu(y)\})$.

A fuzzy set μ in a BCI-algebra X is called a fuzzy BCI-commutative ideal of X if it satisfies (c1) and for all $x, y, z \in X$,

$$(c3) \quad (\mu(x * ((y * (y * x)) * (0 * (0 * (x * y)))))) \geq \min\{\mu((x * y) * z), \mu(z)\}.$$

We now display the basic theory on falling shadows. We refer the reader to the papers [1, 2, 3, 4, 5] for further information regarding the theory of falling shadows.

Given a universe of discourse U , let $\mathbb{P}(U)$ denote the power set of U . For each $u \in U$, let

$$(2.2) \quad \dot{u} := \{E \mid u \in E \text{ and } E \subseteq U\},$$

and for each $E \in \mathbb{P}(U)$, let

$$(2.3) \quad \dot{E} := \{\dot{u} \mid u \in E\}.$$

An ordered pair $(\mathbb{P}(U), \mathbb{B})$ is said to be a hyper-measurable structure on U if $\mathbb{B}$ is a σ -field in $\mathbb{P}(U)$ and $\dot{U} \subseteq \mathbb{B}$. Given a probability space $(\Omega, \mathbb{A}, P)$ and a hyper-measurable structure $(\mathbb{P}(U), \mathbb{B})$ on U , a random set on U

is defined to be a mapping $\xi : \Omega \rightarrow \mathbb{P}(U)$ which is $\mathbb{A}$ - $\mathbb{B}$ measurable, that is,

$$(2.4) \quad (\forall C \in \mathbb{B}) (\xi^{-1}(C) = \{\omega \mid \omega \in \Omega \text{ and } \xi(\omega) \in C\} \in \mathbb{A}).$$

Suppose that ξ is a random set on U . Let

$$\tilde{H}(u) := P(\omega \mid u \in \xi(\omega)) \text{ for each } u \in U.$$

Then $\tilde{H}$ is a kind of fuzzy set in U . We call $\tilde{H}$ a falling shadow of the random set ξ , and ξ is called a cloud of $\tilde{H}$.

For example, $(\Omega, \mathbb{A}, P) = ([0, 1], \mathbb{A}, m)$, where $\mathbb{A}$ is a Borel field on $[0, 1]$ and m is the usual Lebesgue measure. Let $\tilde{H}$ be a fuzzy set in U and $\tilde{H}_t := \{u \in U \mid \tilde{H}(u) \geq t\}$ be a t -cut of $\tilde{H}$. Then

$$\xi : [0, 1] \rightarrow \mathbb{P}(U), \quad t \mapsto \tilde{H}_t$$

is a random set and ξ is a cloud of $\tilde{H}$. We shall call ξ defined above as the cut-cloud of $\tilde{H}$ (see [1]).

3. Falling fuzzy BCI-commutative ideals

In what follows let X denote a BCI-algebra unless otherwise specified.

Definition 3.1 ([6]). Let $(\Omega, \mathbb{A}, P)$ be a probability space, and let

$$\xi : \Omega \rightarrow \mathbb{P}(X)$$

be a random set. If $\xi(\omega)$ is an ideal (resp. a subalgebra) of X for any $\omega \in \Omega$ with $\xi(\omega) \neq \emptyset$, then the falling shadow $\tilde{H}$ of the random set ξ , i.e.,

$$(3.1) \quad \tilde{H}(x) = P(\omega \mid x \in \xi(\omega))$$

is called a falling fuzzy ideal (resp. falling fuzzy subalgebra) of X .

Definition 3.2. Let $(\Omega, \mathbb{A}, P)$ be a probability space and let

$$\xi : \Omega \rightarrow \mathbb{P}(X)$$

be a random set. If $\xi(\omega)$ is a BCI-commutative ideal of X for any $\omega \in \Omega$ with $\xi(\omega) \neq \emptyset$, then the falling shadow $\tilde{H}$ of the random set ξ , i.e.,

$$(3.2) \quad \tilde{H}(x) = P(\omega \mid x \in \xi(\omega))$$

is called a falling fuzzy BCI-commutative ideal of X .

Example 3.3. Consider a BCI-algebra $X = \{0, a, b, c\}$ with a Cayley table as follows:

$*$	0	a	b	c
0	0	a	b	c
a	a	0	c	b
b	b	c	0	a
c	c	b	a	0

Let $(\Omega, \mathbb{A}, P) = ([0, 1], \mathbb{A}, m)$ and define a random set $\xi : [0, 1] \rightarrow \mathbb{P}(X)$ as follows:

$$\xi(t) := \begin{cases} \{0, a\} & \text{if } t \in [0, 0.4), \\ \{0, b\} & \text{if } t \in [0.4, 0.5), \\ \{0, c\} & \text{if } t \in [0.5, 0.8), \\ \emptyset & \text{if } t \in [0.8, 1]. \end{cases}$$

Then the falling shadow $\tilde{H}$ of ξ is a falling fuzzy BCI-commutative ideal of X and it is represented as follows:

$$\tilde{H}(x) = \begin{cases} 0.8 & \text{if } x = 0, \\ 0.4 & \text{if } x = a, \\ 0.1 & \text{if } x = b, \\ 0.3 & \text{if } x = c. \end{cases}$$

Example 3.4. Consider a BCI-algebra $X = \{0, 1, 2, a, b\}$ with a Cayley table as follows:

$*$	0	1	2	a	b
0	0	0	0	a	a
1	1	0	1	b	a
2	2	2	0	a	a
a	a	a	a	0	0
b	b	a	b	1	0

Let $(\Omega, \mathbb{A}, P) = ([0, 1], \mathbb{A}, m)$ and define a random set $\xi : [0, 1] \rightarrow \mathbb{P}(X)$ as follows:

$$\xi(t) := \begin{cases} \emptyset & \text{if } t \in [0, 0.4), \\ \{0, 1\} & \text{if } t \in [0.4, 0.8), \\ \{0, 2, a\} & \text{if } t \in [0.8, 1]. \end{cases}$$

Then the falling shadow $\tilde{H}$ of ξ is a falling fuzzy BCI-commutative ideal of X and it is represented as follows:

$$\tilde{H}(x) = \begin{cases} 0.6 & \text{if } x = 0, \\ 0.4 & \text{if } x = 1, \\ 0.2 & \text{if } x \in \{2, a\}, \\ 0 & \text{if } x = b. \end{cases}$$

Theorem 3.5. *Every falling fuzzy BCI-commutative ideal is a falling fuzzy ideal.*

Proof. Let $\tilde{H}(u) = P(\omega \mid x \in \xi(\omega))$ be a falling fuzzy BCI-commutative ideal of X . Let $x, y \in X$ and $\omega \in \Omega$ be such that $x*y \in \xi(\omega)$ and $y \in \xi(\omega)$. Then $(x*0)*y = x*y \in \xi(\omega)$, and so

$$x = x*((0*(0*x))*(0*(0*(x*0)))) \in \xi(\omega)$$

because $\xi(\omega)$ is a BCI-commutative ideal of X . Thus $\tilde{H}$ is a falling fuzzy ideal of X . $\square$

The following examples shows that the converse of Theorem 3.5 may not be true.

Example 3.6. Consider a BCI-algebra $X = \{0, 1, 2, 3, 4\}$ with the following Cayley table:

$*$	0	1	2	3	4
0	0	0	0	0	0
1	1	0	1	0	0
2	2	2	0	0	0
3	3	3	3	0	0
4	4	4	4	3	0

Let $(\Omega, \mathbb{A}, P) = ([0, 1], \mathbb{A}, m)$ and define a random set $\xi : [0, 1] \rightarrow \mathbb{P}(X)$ as follows:

$$\xi(t) := \begin{cases} \{0, 1\} & \text{if } t \in [0, 0.2), \\ \{0, 2\} & \text{if } t \in [0.2, 0.5), \\ \{0, 1, 2\} & \text{if } t \in [0.5, 0.9), \\ X & \text{if } t \in [0.9, 1]. \end{cases}$$

Then the falling shadow $\tilde{H}$ of ξ is a falling fuzzy ideal of X , but it is not a falling fuzzy BCI-commutative ideal of X because if we take $t \in [0, 0.2)$ then $\xi(t) = \{0, 1\}$ is not a BCI-commutative ideal of X since $(2*3)*1 = 0 \in \xi(t)$ and $2*((3*(3*2))*(0*(0*(2*3)))) = 2 \notin \xi(t)$.

Let $(\Omega, \mathbb{A}, P)$ be a probability space and $\tilde{H}$ a falling shadow of a random set $\xi : \Omega \rightarrow \mathbb{P}(X)$. For any $x \in X$, let

$$(3.3) \quad \Omega(x; \xi) := \{\omega \in \Omega \mid x \in \xi(\omega)\}.$$

Then $\Omega(x; \xi) \in \mathbb{A}$.

Theorem 3.7. *Let $\tilde{H} = P(\omega \mid u \in \xi(\omega))$ be a falling shadow of a random set $\xi : \Omega \rightarrow \mathbb{P}(X)$. Then the following are equivalent:*

- (1) $\tilde{H}$ is a falling fuzzy BCI-commutative ideal of X .
- (2) $\tilde{H}$ is a falling fuzzy ideal of X that satisfies the following inclusion for all $x, y, z \in X$,

$$(3.4) \quad \begin{aligned} & \Omega((x * y) * z; \xi) \cap \Omega(z; \xi) \\ & \subseteq \Omega(x * ((y * (y * x)) * (0 * (0 * (x * y))))) ; \xi). \end{aligned}$$

Proof. Assume that $\tilde{H}$ is a falling fuzzy BCI-commutative ideal of X . Let $\omega \in \Omega$ and $x, y, z \in X$ be such that $\omega \in \Omega((x * y) * z; \xi) \cap \Omega(z; \xi)$. Then $(x * y) * z \in \xi(\omega)$ and $z \in \xi(\omega)$. Since $\xi(\omega)$ is a BCI-commutative ideal of X , it follows from (b3) that

$$x * ((y * (y * x)) * (0 * (0 * (x * y)))) \in \xi(\omega).$$

Hence $\omega \in \Omega(x * ((y * (y * x)) * (0 * (0 * (x * y))))) ; \xi$, and so

$$\Omega((x * y) * z; \xi) \cap \Omega(z; \xi) \subseteq \Omega(x * ((y * (y * x)) * (0 * (0 * (x * y))))) ; \xi.$$

Conversely, suppose that $\tilde{H}$ is a falling fuzzy ideal of X satisfying the condition (3.4). Let $\omega \in \Omega$ and $x, y, z \in X$ be such that $(x * y) * z \in \xi(\omega)$ and $z \in \xi(\omega)$. Then

$$\begin{aligned} \omega & \in \Omega((x * y) * z; \xi) \cap \Omega(z; \xi) \\ & \subseteq \Omega(x * ((y * (y * x)) * (0 * (0 * (x * y))))) ; \xi, \end{aligned}$$

and thus $x * ((y * (y * x)) * (0 * (0 * (x * y)))) \in \xi(\omega)$. Hence $\tilde{H}$ is a falling fuzzy BCI-commutative ideal of X . $\square$

Theorem 3.8. *Let $\tilde{H} = P(\omega \mid u \in \xi(\omega))$ be a falling shadow of a random set $\xi : \Omega \rightarrow \mathbb{P}(X)$. Then the following are equivalent:*

- (1) $\tilde{H}$ is a falling fuzzy BCI-commutative ideal of X .
- (2) $\tilde{H}$ is a falling fuzzy ideal of X that satisfies the following inclusion for all $x, y, z \in X$,

$$(3.5) \quad \Omega(x * y; \xi) \subseteq \Omega(x * ((y * (y * x)) * (0 * (0 * (x * y))))) ; \xi.$$

Proof. Suppose that $\tilde{H}$ is a falling fuzzy BCI-commutative ideal of X . Let $x, y \in X$ and $\omega \in \Omega$ be such that $\omega \in \Omega(x * y; \xi)$. Then $(x * y) * 0 = x * y \in \xi(\omega)$ and $0 \in \xi(\omega)$. Since $\xi(\omega)$ is a BCI-commutative ideal of X , it follows from (b3) that

$$x * ((y * (y * x)) * (0 * (0 * (x * y)))) \in \xi(\omega)$$

so that $\omega \in \Omega(x * ((y * (y * x)) * (0 * (0 * (x * y))))); \xi)$. Hence (3.5) is valid.

Conversely, suppose that $\tilde{H}$ is a falling fuzzy ideal of X satisfying the condition (3.5). Assume that $(x * y) * z \in \xi(\omega)$ and $z \in \xi(\omega)$ for all $\omega \in \Omega$ and $x, y, z \in X$. Then $x * y = (x * y) * 0 \in \xi(\omega)$, and so

$$\omega \in \Omega(x * y; \xi) \subseteq \Omega(x * ((y * (y * x)) * (0 * (0 * (x * y))))); \xi)$$

by (3.5). Thus $x * ((y * (y * x)) * (0 * (0 * (x * y)))) \in \xi(\omega)$. Therefore $\tilde{H}$ is a falling fuzzy BCI-commutative ideal of X . $\square$

Definition 3.9. If a falling fuzzy ideal $\tilde{H}$ of X satisfies the following condition:

$$(3.6) \quad (\forall x \in X)(\Omega(x; \xi) \subseteq \Omega(0 * x; \xi),$$

then $\tilde{H}$ is called a falling fuzzy closed ideal of X .

Note that, in Example 3.6, $\tilde{H}$ is a falling fuzzy closed ideal of X which is not a falling fuzzy BCI-commutative ideal of X . Hence any falling fuzzy closed ideal may not be a falling fuzzy BCI-commutative ideal. We now provide conditions for a falling fuzzy closed ideal to be a falling fuzzy BCI-commutative ideal.

Lemma 3.10. Let $\tilde{H} = P(\omega \mid u \in \xi(\omega))$ be a falling shadow of a random set $\xi : \Omega \rightarrow \mathbb{P}(X)$. Then $\tilde{H}$ is a falling fuzzy ideal of X if and only if $\Omega \subseteq \Omega(0; \xi)$ and

$$(3.7) \quad (\forall x, y, z \in X)(x * y \leq z \Rightarrow \Omega(y; \xi) \cap \Omega(z; \xi) \subseteq \Omega(x; \xi)).$$

Proof. Assume that $\tilde{H}$ is a falling fuzzy ideal of X . Clearly $\Omega \subseteq \Omega(0; \xi)$. Let $x, y, z \in X$ and $\omega \in \Omega$ be such that $x * y \leq z$ and $\omega \in \Omega(y; \xi) \cap \Omega(z; \xi)$. Then $y \in \xi(\omega)$ and $z \in \xi(\omega)$. Since $\xi(\omega)$ is an ideal of X , it follows from (2.1) that $x * y \in \xi(\omega)$, and so $x \in \xi(\omega)$ by (b2). Thus $\omega \in \Omega(x; \xi)$, and therefore (3.7) is valid.

Conversely suppose that $\Omega \subseteq \Omega(0; \xi)$ and the condition (3.7) is valid. Obviously, $0 \in \xi(\omega)$ for all $\omega \in \Omega$. Let $x, y \in X$ and $\omega \in \Omega$ be such that $x * y \in \xi(\omega)$ and $y \in \xi(\omega)$. Since $x * (x * y) \leq y$, it follows from (3.7) that

$$\omega \in \Omega(x * y; \xi) \cap \Omega(y; \xi) \subseteq \Omega(x; \xi)$$

so that $x \in \xi(\omega)$. Hence $\tilde{H}$ is a falling fuzzy ideal of X . $\square$

Theorem 3.11. *Let $\tilde{H}$ be a falling fuzzy closed ideal of X . Then $\tilde{H}$ is a falling fuzzy BCI-commutative ideal of X if and only if the following condition is valid:*

$$(3.8) \quad (\forall x, y \in X) (\Omega(x * y; \xi) \subseteq \Omega(x * (y * (y * x)); \xi)).$$

Proof. Suppose that $\tilde{H}$ is a falling fuzzy BCI-commutative ideal of X . Let $x, y \in X$ and $\omega \in \Omega$ be such that $\omega \in \Omega(x * y; \xi)$. Then $\omega \in \Omega(0 * (x * y); \xi)$ by (3.6), and $\omega \in \Omega(x * ((y * (y * x)) * (0 * (0 * (x * y)))); \xi)$ by (3.5). Note that

$$\begin{aligned} & (x * (y * (y * x))) * (x * ((y * (y * x)) * (0 * (0 * (x * y))))) \\ & \leq ((y * (y * x)) * (0 * (0 * (x * y)))) * (y * (y * x)) \\ & = ((y * (y * x)) * (y * (y * x))) * (0 * (0 * (x * y))) \\ & = 0 * (0 * (0 * (x * y))) = 0 * (x * y). \end{aligned}$$

Using (3.7), we have

$$\begin{aligned} \omega & \in \Omega(x * ((y * (y * x)) * (0 * (0 * (x * y)))); \xi) \cap \Omega(0 * (x * y); \xi) \\ & \subseteq \Omega(x * (y * (y * x)); \xi). \end{aligned}$$

Hence $\Omega(x * y; \xi) \subseteq \Omega(x * (y * (y * x)); \xi)$ for all $x, y \in X$.

Conversely, let $\tilde{H}$ be a falling fuzzy closed ideal of X that satisfies the condition (3.8). Let $\omega \in \Omega$ and $x, y \in X$ be such that $\omega \in \Omega(x * y; \xi)$. Then

$$(3.9) \quad \omega \in \Omega(x * (y * (y * x)); \xi)$$

by (3.8), and so $x * (y * (y * x)) \in \xi(\omega)$. Since $0 * (0 * (x * y)) \leq x * y \in \xi(\omega)$ and $\xi(\omega)$ is an ideal of X , it follows from (2.1) that $0 * (0 * (x * y)) \in \xi(\omega)$, that is,

$$(3.10) \quad \omega \in \Omega(0 * (0 * (x * y)); \xi).$$

Note that

$$\begin{aligned} & (x * ((y * (y * x)) * (0 * (0 * (x * y))))) * (x * (y * (y * x))) \\ & \leq (y * (y * x)) * ((y * (y * x)) * (0 * (0 * (x * y)))) \\ & \leq 0 * (0 * (x * y)). \end{aligned}$$

Using (3.9), (3.10) and (3.7), we have

$$\begin{aligned} \omega & \in \Omega(x * (y * (y * x)); \xi) \cap \Omega(0 * (0 * (x * y)); \xi) \\ & \subseteq \Omega(x * ((y * (y * x)) * (0 * (0 * (x * y)))); \xi). \end{aligned}$$

Therefore $\Omega(x * y; \xi) \subseteq \Omega(x * ((y * (y * x)) * (0 * (0 * (x * y)))); \xi)$ for all $x, y \in X$, and thus $\tilde{H}$ is a falling fuzzy BCI-commutative ideal of X by Theorem 3.8. $\square$

Theorem 3.12. *Let X be a BCI-algebra that satisfies the following conditions:*

$$(3.11) \quad (\forall x, y \in X) (x \leq y \Rightarrow x = y * (y * x)).$$

Then every falling fuzzy closed ideal of X is a falling fuzzy BCI-commutative ideal of X .

Proof. Assume that a BCI-algebra X satisfies the condition (3.11). Let $\tilde{H}$ be a falling fuzzy closed ideal of X . Let $x, y \in X$ and $\omega \in \Omega$ be such that $\omega \in \Omega(x * y; \xi)$. Then $\omega \in \Omega(0 * (x * y); \xi)$ since $\tilde{H}$ is a falling fuzzy closed ideal. Since $x * (x * y) \leq y$, it follows from (3.11) that

$$x * (x * y) = y * (y * (x * (x * y)))$$

so from (a2), (a6), (I) and (III) that

$$\begin{aligned} (x * (y * (y * x))) * (x * y) &= (x * (x * y)) * (y * (y * x)) \\ &= (y * (y * (x * (x * y)))) * (y * (y * x)) \\ &= (y * (y * (y * x))) * (y * (x * (x * y))) \\ &= (y * x) * (y * (x * (x * y))) \\ &\leq (x * (x * y)) * x \\ &= (x * x) * (x * y) = 0 * (x * y). \end{aligned}$$

Using Lemma 3.10, we have

$$\omega \in \Omega(x * y; \xi) \cap \Omega(0 * (x * y); \xi) \subseteq \Omega(x * (y * (y * x)); \xi),$$

which shows that $\Omega(x * y; \xi) \subseteq \Omega(x * (y * (y * x)); \xi)$ for all $x, y \in X$. Applying Theorem 3.11, we conclude that $\tilde{H}$ is a falling fuzzy BCI-commutative ideal of X . $\square$

Corollary 3.13. *Let $(X, *, 0)$ be an algebra of type $(2, 0)$ that satisfies the conditions (I), (a1) and*

$$(3.12) \quad (\forall x, y \in X) (x * (x * y) = y * (y * (x * (x * y)))).$$

Then every falling fuzzy closed ideal of X is a falling fuzzy BCI-commutative ideal of X .

Theorem 3.14. *Let $\tilde{H}$ be a falling shadow of a random set $\xi : \Omega \rightarrow \mathbb{P}(X)$. If $\tilde{H}$ is a falling fuzzy BCI-commutative ideal of X , then*

$$(3.13) \quad \tilde{H}(x * ((y * (y * x)) * (0 * (0 * (x * y))))) \geq T_m(\tilde{H}((x * y) * z), \tilde{H}(z))$$

for all $x, y, z \in X$ where $T_m(s, t) = \max\{s + t - 1, 0\}$ for any $s, t \in [0, 1]$.

Proof. By Definition 3.2, $\xi(\omega)$ is a BCI-commutative ideal of X for any $\omega \in \Omega$. Hence

$$\begin{aligned} & \{\omega \in \Omega \mid (x * y) * z \in \xi(\omega)\} \cap \{\omega \in \Omega \mid z \in \xi(\omega)\} \\ & \subseteq \{\omega \in \Omega \mid x * ((y * (y * x)) * (0 * (0 * (x * y)))) \in \xi(\omega)\}, \end{aligned}$$

and thus

$$\begin{aligned} & \tilde{H}(x * ((y * (y * x)) * (0 * (0 * (x * y))))) \\ & = P(\omega \mid x * ((y * (y * x)) * (0 * (0 * (x * y))))) \in \xi(\omega) \\ & \geq P(\{\omega \mid (x * y) * z \in \xi(\omega)\} \cap \{\omega \mid z \in \xi(\omega)\}) \\ & \geq P(\omega \mid (x * y) * z \in \xi(\omega)) + P(\omega \mid z \in \xi(\omega)) \\ & \quad - P(\omega \mid (x * y) * z \in \xi(\omega) \text{ or } z \in \xi(\omega)) \\ & \geq \tilde{H}((x * y) * z) + \tilde{H}(z) - 1. \end{aligned}$$

Hence

$$\begin{aligned} & \tilde{H}(x * ((y * (y * x)) * (0 * (0 * (x * y))))) \\ & \geq \max\{\tilde{H}((x * y) * z) + \tilde{H}(z) - 1, 0\} \\ & = T_m(\tilde{H}((x * y) * z), \tilde{H}(z)). \end{aligned}$$

This completes the proof. $\square$

Theorem 3.14 means that every falling fuzzy BCI-commutative ideal of X is a T_m -fuzzy BCI-commutative ideal of X .

Let $(\Omega, \mathbb{A}, P)$ be a probability space and let

$$F(X) := \{f \mid f : \Omega \rightarrow X \text{ is a mapping}\}.$$

Define an operation $\odot$ on $F(X)$ by

$$(\forall \omega \in \Omega) ((f \odot g)(\omega) = f(\omega) * g(\omega))$$

for all $f, g \in F(X)$. Let $\theta \in F(X)$ be defined by $\theta(\omega) = 0$ for all $\omega \in \Omega$.

Then $(F(X); \odot, \theta)$ is a BCI-algebra (see [6]).

For any subset A of X and $f \in F(X)$, let

$$A_f := \{\omega \in \Omega \mid f(\omega) \in A\}$$

and

$$\xi : \Omega \rightarrow \mathbb{P}(F(X)), \omega \mapsto \{f \in F(X) \mid f(\omega) \in A\}.$$

Then $A_f \in \mathbb{A}$.

Theorem 3.15. *If A is a BCI-commutative ideal of X , then*

$$\xi(\omega) = \{f \in F(X) \mid f(\omega) \in A\}$$

is a BCI-commutative ideal of $F(X)$.

Proof. Assume that A is a BCI-commutative ideal of X and let $\omega \in \Omega$. Since $\theta(\omega) = 0 \in A$, we see that $\theta \in \xi(\omega)$. Let $f, g, h \in F(X)$ be such that $(f \odot g) \odot h \in \xi(\omega)$ and $h \in \xi(\omega)$. Then

$$(f(\omega) * g(\omega)) * h(\omega) = ((f \odot g) \odot h)(\omega) \in A$$

and $h(\omega) \in A$. Since A is a BCI-commutative ideal of X , it follows from (b3) that

$$\begin{aligned} & f(\omega) * ((g(\omega) * (g(\omega) * f(\omega))) * (\theta(\omega) * (\theta(\omega) * (f(\omega) * g(\omega))))) \\ &= (f \odot ((g \odot (g \odot f)) \odot (\theta \odot (\theta \odot (f \odot g))))) (\omega) \in A \end{aligned}$$

and so $f \odot ((g \odot (g \odot f)) \odot (\theta \odot (\theta \odot (f \odot g)))) \in \xi(\omega)$. Hence $\xi(\omega)$ is a BCI-commutative ideal of $F(X)$. $\square$

Lemma 3.16. *A fuzzy set μ in X is a fuzzy BCI-commutative ideal of X if and only if*

$$(3.14) \quad (\forall t \in [0, 1]) (\mu_t \neq \emptyset \Rightarrow \mu_t \text{ is a BCI-commutative ideal of } X),$$

where $\mu_t := \{x \in X \mid \mu(x) \geq t\}$.

Proof. The proof is easy, and so we omit the proof. $\square$

Theorem 3.17. *Every fuzzy BCI-commutative ideal of X is a falling fuzzy BCI-commutative ideal of X .*

Proof. Consider the probability space $(\Omega, \mathbb{A}, P) = ([0, 1], \mathbb{A}, m)$, where $\mathbb{A}$ is a Borel field on $[0, 1]$ and m is the usual Lebesgue measure. Let μ be a fuzzy BCI-commutative ideal of X . Then μ_t is a BCI-commutative ideal of X for all $t \in [0, 1]$ by Lemma 3.16. Let

$$\xi : [0, 1] \rightarrow \mathbb{P}(X)$$

be a random set and $\xi(t) = \mu_t$ for every $t \in [0, 1]$. Then μ is a falling fuzzy BCI-commutative ideal of X . $\square$

The converse of Theorem 3.17 may not be true since the falling shadow $\tilde{H}$ of ξ in Example 3.4 is a falling fuzzy BCI-commutative ideal of X , but it is not a fuzzy BCI-commutative ideal of X .

4. Conclusion

We have established some connections between fuzzy mathematics and probability theory via the fuzzy BCI-commutative ideals of BCI-algebras. As an algebraic approach of the theory of falling shadows, we have introduced the notion of falling fuzzy BCI-commutative ideals

in BCI-algebras. We have discussed relations between falling fuzzy ideals and falling fuzzy BCI-commutative ideals. We have also provided relations between fuzzy BCI-commutative ideals and falling fuzzy BCI-commutative ideals. We have shown that every falling fuzzy BCI-commutative ideal is a Tm-fuzzy BCI-commutative ideal. We have considered characterizations of a falling fuzzy BCI-commutative ideal, and gave conditions for a falling fuzzy (closed) ideal to be a falling fuzzy BCI-commutative ideal. On the basis of these results, we will apply the theory of falling shadows to the another type of ideals, filters and deductive systems in BCK/BCI-algebras, Hilbert algebras, MV-algebras, MTL-algebras and BL-algebras, etc., in future study.

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