A NEW FORM OF FUZZY GENERALIZED BI-IDEALS IN ORDERED SEMIGROUPS

HIDAYAT ULLAH KHAN*, NOR HANIZA SARMIN, AND ASGHAR KHAN

Abstract. In several applied disciplines like control engineering, computer sciences, error-correcting codes and fuzzy automata theory, the use of fuzzified algebraic structures especially ordered semigroups and their fuzzy subsystems play a remarkable role. In this paper, we introduce the notion of $(\in, \in \vee \bar{q}_k)$ -fuzzy subsystems of ordered semigroups namely $(\in, \in \vee \bar{q}_k)$ -fuzzy generalized bi-ideals of ordered semigroups. The important milestone of the present paper is to link ordinary generalized bi-ideals and $(\in, \in \vee \bar{q}_k)$ -fuzzy generalized bi-ideals. Moreover, different classes of ordered semigroups such as regular and left weakly regular ordered semigroups are characterized by the properties of this new notion. Finally, the upper part of a $(\in, \in \vee \bar{q}_k)$ -fuzzy generalized bi-ideal is defined and some characterizations are discussed.

1. Introduction

Fuzzy set theory [1] is a useful tool to describe situations in which the data are imprecise or vague and handle such situations by attributing a degree to which a certain object belongs to a set. Further, utilizing this fundamental concept of fuzzy set Rosenfeld introduced the notion of fuzzy subgroup [2]. Bhakat and Das [3] generalized Rosenfeld's fuzzy group theory and defined $(\in, \in \vee q)$ -fuzzy subgroup by utilizing the combine notions of “belongingness” and “quasi-coincidence” of fuzzy point

Received June 16, 2014. Accepted August 26, 2014.

2010 Mathematics Subject Classification. 08A72, 20N25, 06F05, 20M12.

Key words and phrases. Ordered semigroups, regular, left (right) regular, completely regular and weakly regular ordered semigroups, generalized bi-ideals, fuzzy left (right) ideals, fuzzy generalized bi-ideals, $(\bar{\in}, \bar{\in} \vee \bar{q}_k)$ -fuzzy generalized bi-ideals, $(\bar{\in}, \bar{\in} \vee \bar{q}_k)$ -fuzzy left (right) ideals, $(\in, \in \vee \bar{q}_k)$ -fuzzy bi-ideals.

*Corresponding author

and fuzzy set [4]. In addition, Davvaz and Khan [5] discussed some characterization regular ordered semigroups in terms of (α, β) -fuzzy generalized bi-ideals, where $\alpha, \beta \in \{\in, q, \in \vee q, \in \wedge q\}$ and $\alpha \neq \in \wedge q$. Moreover, in the semigroup theory the notions of generalized fuzzy (interior, bi-, left, right, quasi) ideals was studied respectively in ([6-9]). Kazanchi and Yamak [7] gave $(\overline{\in}, \overline{\in} \vee \overline{q}_k)$ -fuzzy bi-ideals of a semigroup and in [10] Shabir *et. al* studied $(\overline{\in}, \overline{\in} \vee \overline{q}_k)$ -fuzzy ideals, generalized bi-ideals and quasi-ideals of a semigroup and characterized regular semigroups by the properties of these ideals. The reader is referred to [11-21] for further study regarding (α, β) -fuzzy subsets and its generalization.

The aim of this paper to investigate more general form of fuzzy generalized bi-ideals. In this connection the notion of $(\overline{\in}, \overline{\in} \vee \overline{q}_k)$ -fuzzy generalized bi-ideals of ordered semigroup is introduced. In addition, Characterizations of regular, left weakly regular ordered semigroups by means of $(\overline{\in}, \overline{\in} \vee \overline{q}_k)$ -fuzzy generalized bi-ideals and $(\overline{\in}, \overline{\in} \vee \overline{q}_k)$ -fuzzy bi-ideals are discussed. Further, the concepts of $(\overline{\in}, \overline{\in} \vee \overline{q}_k)$ -fuzzy left (right) ideals are also presented and some related properties are discussed.

2. Basic Definitions and Preliminaries

By an *ordered semigroup* (or *po-semigroup*) we mean a structure $(S, \cdot, \leq)$ in which the following conditions are satisfied:

(OS1) $(S, \cdot)$ is a semigroup,

(OS2) $(S, \leq)$ is a poset,

(OS3) $a \leq b \longrightarrow ax \leq bx$ and $a \leq b \longrightarrow xa \leq xb$ for all $a, b, x \in S$.

For subsets A, B of an ordered semigroup S , we denote by

$$AB = \{ab \in S \mid a \in A, b \in B\},$$

$$[A] = \{t \in S \mid t \leq h \text{ for some } h \in A\}.$$

If $A = \{a\}$, then we write $(a]$ instead of $(\{a\}]$. For any $A, B \subseteq S$ we have $A \subseteq [A]$, $[A](B) \subseteq [AB]$ and $([A]) = [A]$.

Let $(S, \cdot, \leq)$ be an ordered semigroup. A non-empty subset A of S is called a *subsemigroup* of S if $A^2 \subseteq A$. A non-empty subset A of S is called a *left (right) ideal* of S if

(i) $(\forall a \in S)(\forall b \in A) (a \leq b \longrightarrow a \in A)$,

(ii) $AS \subseteq A (SA \subseteq A)$.

A non-empty subset A of an ordered semigroup S is called a *generalized bi-ideal* [22] of S if

(i) $(\forall a \in S)(\forall b \in A) (a \leq b \longrightarrow a \in A)$,

(ii) $ASA \subseteq A$.

A non-empty subset A of an ordered semigroup S is called a *bi-ideal* [23] of S if

- (i) $(\forall a \in S)(\forall b \in A) (a \leq b \longrightarrow a \in A)$,
- (ii) $A^2 \subseteq A$,
- (iii) $ASA \subseteq A$.

Note that, every bi-ideal of S is a generalized bi-ideal of S , but the converse is not true, as shown in [22].

An ordered semigroup S is *regular* [23] if for every $a \in S$, there exists $x \in S$ such that $a \leq axa$, or equivalently, we have (i) $a \in (aSa]$ $\forall a \in S$ and (ii) $A \subseteq (ASA]$ $\forall A \subseteq S$. An ordered semigroup S is called *left (right) regular* [23] if for every $a \in S$ there exists $x \in S$ such that $a \leq xa^2(a \leq a^2x)$, or equivalently, (i) $a \in (Sa^2](a \in (a^2S])$ $\forall a \in S$ and (ii) $A \subseteq (SA^2](A \subseteq (A^2S])$ $\forall A \subseteq S$. An ordered semigroup S is called *left (right) simple* [23] if for every left (right) ideal A of S we have $A = S$ and S is called *simple* [23] if it is both left and right simple. An ordered semigroup S is *left (right) regular* [23] if for every $a \in S$, there exists $x \in S$, such that $a \leq xa^2$ ($a \leq a^2x$), or equivalently, (i) $a \in (Sa^2]$ ($a \in (a^2S])$ $\forall a \in S$ and (ii) $A \subseteq (SA^2]$ ($A \subseteq (A^2S])$ $\forall A \subseteq S$. An ordered semigroup S is called *completely regular* [23] if it is left regular, right regular and regular. An ordered semigroup S is called *left weakly regular* [22] if for every $a \in S$, there exist $x, y \in S$ such that $a \leq xay$, or equivalently, (i) $a \in ((Sa)^2]$ $\forall a \in S$ and (ii) $A \subseteq ((SA)^2]$ $\forall A \subseteq S$. *Right weakly regular* ordered semigroups are defined similarly. An ordered semigroup S is called *weakly regular* if it is both a left and right weakly regular.

Note that if S is commutative, then the concepts of regular and weakly regular ordered semigroups coincide.

By $B(a)$ ($L(a)$, $R(a)$ and $I(a)$) we mean the generalized bi-(left, right and two-sided) ideal of S generated by a ($a \in S$) denoted by

$$\begin{aligned} B(a) &= (a \cup aSa], L(a) = (a \cup Sa], R(a) = (a \cup aS], \\ I(a) &= (a \cup Sa \cup aS \cup SaS] \text{ (see [22, 23]).} \end{aligned}$$

Now, we give some fuzzy logic concepts.

A function $F : S \longrightarrow [0, 1]$ (unit closed interval) is called a *fuzzy subset* of S .

The study of fuzzification of algebraic structures has started in the pioneering paper of Rosenfeld [3] in 1971. Rosenfeld introduced the notion of fuzzy groups and successfully extended many results from groups in the theory of fuzzy groups. Kuroki [24] studied fuzzy ideals, fuzzy bi-ideals and semiprime fuzzy ideals in semigroups (also see [25-27]).

If F and G are fuzzy subsets of S then $F \preceq G$ means $F(x) \leq G(x)$ for all $x \in S$ and the symbols $\wedge$ and $\vee$ will mean the following fuzzy subsets are defined as follow for all $x \in S$:

$$\begin{aligned} F \wedge G &: S \longrightarrow [0, 1] | x \longmapsto (F \wedge G)(x) = F(x) \wedge G(x) = \min\{F(x), G(x)\}, \\ F \vee G &: S \longrightarrow [0, 1] | x \longmapsto (F \vee G)(x) = F(x) \vee G(x) = \max\{F(x), G(x)\}. \end{aligned}$$

A fuzzy subset F of S is called a *fuzzy subsemigroup* if

$$F(xy) \geq \min\{F(x), F(y)\} \text{ for all } x, y \in S.$$

A fuzzy subset F of S is called a *fuzzy generalized bi-ideal* [22] of S if:

- (i) $x \leq y \longrightarrow F(x) \geq F(y)$,
- (ii) $F(xyz) \geq \min\{F(x), F(z)\}$ for all $x, y, z \in S$.

A fuzzy subset F of S is called a *fuzzy left (right)-ideal* [23] of S if:

- (i) $x \leq y \longrightarrow F(x) \geq F(y)$,
- (ii) $F(xy) \geq F(y)$ ($F(xy) \geq F(x)$) for all $x, y \in S$.

A fuzzy subset of S is called a *fuzzy ideal* if it is both a fuzzy left and right ideal of S .

A fuzzy subsemigroup F is called a *fuzzy bi-ideal* [23] of S if:

- (i) $x \leq y \longrightarrow F(x) \geq F(y)$,
- (ii) $F(xyz) \geq \min\{F(x), F(z)\}$ for all $x, y, z \in S$.

Note that every fuzzy bi-ideal is a generalized fuzzy bi-ideal of S . But the converse is not true, as given in [22].

Let F be a fuzzy subset of an ordered semigroup S , then for all $t \in (0, 1]$, the set $U(F; t) = \{x \in S | F(x) \geq t\}$ is called a *level set* of F .

Theorem 2.1. ([28]) *A fuzzy subset F of an ordered semigroup S is a fuzzy left (right)-ideal of S if and only if $U(F; t) (\neq \emptyset)$ where $t \in (0, 1]$ is a left (right)-ideal of S .*

Theorem 2.2. ([5]) *A fuzzy subset F of an ordered semigroup S is a fuzzy generalized bi-ideal of S if and only if $U(F; t) (\neq \emptyset)$ where $t \in (0, 1]$ is a generalized bi-ideal of S .*

Theorem 2.3. ([28]) *A non-empty subset A of an ordered semigroup S is a left (right)-ideal of S if and only if*

$$\chi_A : S \longrightarrow [0, 1] | x \longmapsto \chi_A(x) = \begin{cases} 1, & \text{if } x \in A, \\ 0, & \text{if } x \notin A. \end{cases}$$

is a fuzzy left (right)-ideal of S .

Theorem 2.4. ([5]) *A non-empty subset A of an ordered semigroup S is a generalized bi-ideal of S if and only if*

$$\chi_A : S \longrightarrow [0, 1] | x \longmapsto \chi_A(x) = \begin{cases} 1, & \text{if } x \in A, \\ 0, & \text{if } x \notin A. \end{cases}$$

is a fuzzy generalized bi-ideal of S .

If $a \in S$ and A is a non-empty subset of S . Then,

$$A_a = \{(y, z) \in S \times S \mid a \leq yz\}.$$

For any two fuzzy subsets F and G of an ordered semigroup S , the product $F \circ G$ is defined by:

$$F \circ G : S \longrightarrow [0, 1] | a \longmapsto (F \circ G)(a) = \begin{cases} \bigvee_{(y,z) \in A_a} (F(y) \wedge G(z)), & \text{if } A_a \neq \emptyset, \\ 0, & \text{if } A_a = \emptyset. \end{cases}$$

Let F be a fuzzy subset of S , then the set of the form:

$$F(y) := \begin{cases} t \in (0, 1], & \text{if } y = x, \\ 0, & \text{otherwise,} \end{cases}$$

is called a *fuzzy point* with support x and value t and is denoted by $(x; t)$ [2]. A fuzzy point $(x; t)$ is said to *belong to* (*quasi-coincident with*) a fuzzy set F , written as $(x; t) \in F$ ($(x; t)qF$) if $F(x) \geq t$ ($F(x) + t > 1$) [2]. If $(x; t) \in F$ or $(x; t)qF$, then we write $(x; t) \in \vee qF$. The symbol $\overline{\in \vee q}$ means $\in \vee q$ does not hold.

Generalizing the concept of $(x; t)qF$, in semigroups, Khan *et. al.* [9] defined $(x; t)q_k F$, as $F(x) + t + k > 1$, where $k \in [0, 1]$.

Throughout in this paper S will denote an ordered semigroup unless otherwise specified.

3. $(\overline{\in}, \overline{\in} \vee \overline{q}_k)$ -fuzzy generalized bi-ideals

In this section, we define a more generalized form of (α, β) -fuzzy generalized bi-ideals of an ordered semigroup S and introduce $(\overline{\in}, \overline{\in} \vee \overline{q}_k)$ -fuzzy generalized bi-ideals of S , where $\alpha \in \{\overline{\in}, \overline{q}, \overline{\in} \wedge \overline{q}, \overline{\in} \vee \overline{q}\}$ and k is an arbitrary element of $[0, 1]$ unless otherwise specified.

Definition 3.1. A fuzzy subset F of S is called a $(\overline{\in}, \overline{\in} \vee \overline{q}_k)$ -fuzzy subsemigroup of S if for all $x, y \in S$ and $t \in (0, 1]$ the following holds:

$$(xy; t) \overline{\in} F \longrightarrow (x; t) \overline{\in} \vee \overline{q}_k F \text{ or } (y; t) \overline{\in} \vee \overline{q}_k F.$$

Definition 3.2. A fuzzy subset F of S is called a $(\bar{\in}, \bar{\in} \vee \bar{q}_k)$ -fuzzy generalized bi-ideal of S if for all $x, y, z \in S$ and $t \in (0, 1]$ the following conditions hold:

- (1) $(\forall x \leq y), (x; t) \bar{\in} F \longrightarrow (y; t) \bar{\in} \vee \bar{q}_k F$,
- (2) $(xyz; t) \bar{\in} F \longrightarrow (x; t) \bar{\in} \vee \bar{q}_k F$ or $(y; t) \bar{\in} \vee \bar{q}_k F$.

Lemma 3.3. For any fuzzy subset F of an ordered semigroup S and for all $x, y \in S$ and $t \in (0, 1]$ the following conditions are equivalent:

- (1a) $(\forall x \leq y), (x; t) \bar{\in} F \longrightarrow (y; t) \bar{\in} \vee \bar{q}_k F$,
- (1b) $(\forall x \leq y), \max \{F(x), \frac{1-k}{2}\} \geq F(y)$.

Proof. (1a) $\implies$ (1b): Suppose that there exist $x, y \in S$ with $x \leq y$ such that

$$\max \left\{ F(x), \frac{1-k}{2} \right\} < F(y),$$

then

$$\max \left\{ F(x), \frac{1-k}{2} \right\} < t \leq F(y) \text{ for some } t \in \left(\frac{1-k}{2}, 1 \right].$$

Shows that $(x; t) \bar{\in} F$ but $(y; t) \in \wedge q_k F$, a contradiction. Therefore we accept that

$$\max \left\{ F(x), \frac{1-k}{2} \right\} \geq F(y) \text{ for all } x, y \in S, \text{ with } x \leq y.$$

(1b) $\implies$ (1a): Let $(x; t) \bar{\in} F$ for all $x, y \in S$ such that $x \leq y$ and $t \in (0, 1]$, then $F(x) < t$. If

$$\max \left\{ F(x), \frac{1-k}{2} \right\} = F(x),$$

then $F(y) \leq F(x) < t$ follows that $(y; t) \bar{\in} F$. On the other hand if

$$\max \left\{ F(x), \frac{1-k}{2} \right\} = \frac{1-k}{2},$$

then $F(y) \leq \frac{1-k}{2}$. Suppose that $(y; t) \in F$, then $t \leq F(y) \leq \frac{1-k}{2}$ follows that $(y; t) \bar{q} F$ consequently $(y; t) \bar{\in} \vee \bar{q} F$. $\square$

Lemma 3.4. Let F be a fuzzy subset of an ordered semigroup S . Then the following conditions are equivalent for all $x, y \in S$ and $t \in (0, 1]$:

- (2a) $(xy; t) \bar{\in} F \implies (x; t) \bar{\in} \vee \bar{q}_k \text{ or } (y; t) \bar{\in} \vee \bar{q}_k$,
- (2b) $\max \{F(xy), \frac{1-k}{2}\} \geq \min \{F(x), F(y)\}$.

Proof. (2a) $\implies$ (2b): Let $\max\{F(xy), \frac{1-k}{2}\} < \min\{F(x), F(y)\}$ for some $x, y \in S$, then

$$\max\left\{F(xy), \frac{1-k}{2}\right\} < t \leq \min\{F(x), F(y)\} \text{ for some } t \in \left(\frac{1-k}{2}, 1\right].$$

shows that $(xy; t) \bar{\in} F$ but $(x; t) \in \wedge_{q_k} F$ and $(y; t) \in \wedge_{q_k} F$ a contradiction. Hence

$$\max\left\{F(xy), \frac{1-k}{2}\right\} \geq \min\{F(x), F(y)\} \text{ for all } x, y \in S.$$

(2b) $\implies$ (2a): If $(xy; t) \bar{\in} F$ for all $x, y \in S$ such that $x \leq y$ and $t \in (0, 1]$, then $F(xy) < t$. Consider

$$\max\left\{F(xy), \frac{1-k}{2}\right\} = F(xy),$$

then

$$\min\{F(x), F(y)\} \leq \max\left\{F(xy), \frac{1-k}{2}\right\} = F(xy) < t,$$

follows that $(x; t) \bar{\in} F$ or $(y; t) \bar{\in} F$. But if

$$\max\left\{F(xy), \frac{1-k}{2}\right\} = \frac{1-k}{2},$$

then

$$\min\{F(x), F(y)\} \leq \max\left\{F(xy), \frac{1-k}{2}\right\} = \frac{1-k}{2}.$$

Suppose that $(x; t) \in F$ or $(y; t) \in F$. Then $t \leq F(x) < \frac{1-k}{2}$ or $t \leq F(y) < \frac{1-k}{2}$ follows that $F(x) + t < \frac{1-k}{2} + \frac{1-k}{2} = 1 - k$ or $F(y) + t < \frac{1-k}{2} + \frac{1-k}{2} = 1 - k$. Hence $(x; t) \bar{q}_k F$ or $(y; t) \bar{q}_k F$. Thus $(x; t) \bar{\in} \vee \bar{q}_k F$ or $(y; t) \bar{\in} \vee \bar{q}_k F$. $\square$

Lemma 3.5. Let F be a fuzzy subset of an ordered semigroup S . Then for all $x, y, z \in S$ and $t \in (0, 1]$, the following conditions are equivalent:

$$(3a) \quad (xyz; t) \bar{\in} F \implies (x; t) \bar{\in} \vee \bar{q}_k F \text{ or } (z; t) \bar{\in} \vee \bar{q}_k F,$$

$$(3b) \quad \max\{F(xyz), \frac{1-k}{2}\} \geq \min\{F(x), F(z)\}.$$

Proof. Follows from the proofs of Lemma 3.3 and 3.4. $\square$

From Lemma 3.3 and 3.5, we have the following theorem:

Theorem 3.6. A fuzzy subset F of S is a $(\bar{\in}, \bar{\in} \vee \bar{q}_k)$ -fuzzy generalized bi-ideal of S if and only if it satisfies the condition (1b) and (3b).

Definition 3.7. A fuzzy subset F which is both a $(\bar{\epsilon}, \bar{\epsilon} \vee \bar{q}_k)$ -fuzzy generalized bi-ideal and a $(\bar{\epsilon}, \bar{\epsilon} \vee \bar{q}_k)$ -fuzzy subsemigroup of S is called a $(\bar{\epsilon}, \bar{\epsilon} \vee \bar{q}_k)$ -fuzzy bi-ideal of S .

From Lemma 3.3, 3.4 and 3.5, we have the following theorem:

Theorem 3.8. A fuzzy subset F of S is a $(\bar{\epsilon}, \bar{\epsilon} \vee \bar{q}_k)$ -fuzzy bi-ideal of S if and only if it satisfies the condition (1b), (2b) and (3b).

Theorem 3.9. Let F be a fuzzy subset of an ordered semigroup S . Then F is $(\bar{\epsilon}, \bar{\epsilon} \vee \bar{q}_k)$ -fuzzy generalized bi-ideal of S if and only if

$$U(F; t) = \{x \in S \mid F(x) \geq t\} \neq \emptyset$$

is generalized bi-ideal of S for all $t \in (\frac{1-k}{2}, 1]$.

Proof. Let F be $(\bar{\epsilon}, \bar{\epsilon} \vee \bar{q})$ -fuzzy generalized bi-ideal of S and $t \in (\frac{1-k}{2}, 1]$ be such that $U(F; t) \neq \emptyset$. Then by Lemma 3.3 (1b),

$$F(b) \leq \max \left\{ F(a), \frac{1-k}{2} \right\} \text{ for } a \leq b \in U(F; t),$$

follows that $t \leq F(b) \leq \max \{F(a), \frac{1-k}{2}\}$ that is $F(a) \geq t$ (as $t \in (\frac{1-k}{2}, 1]$) hence $a \in U(F; t)$.

Next, we let $a, c \in U(F; t)$, then by Lemma 3.5 (3b),

$$\max \left\{ F(abc), \frac{1-k}{2} \right\} \geq \min \{F(a), F(c)\} \geq \min \{t, t\} = t.$$

follows that $F(abc) \geq t$ (as $t \in (\frac{1-k}{2}, 1]$), hence $abc \in U(F; t)$.

Conversely, let $U(F; t) = \{x \in S \mid F(x) \geq t\} \neq \emptyset$ be a generalized bi-ideal of S for all $t \in (\frac{1-k}{2}, 1]$. Let $a, b \in S$ with $a \leq b$ such that $F(b) > \max \{F(a), \frac{1-k}{2}\}$, then $F(b) \geq t_0 > \max \{F(a), \frac{1-k}{2}\}$ for some $t_0 \in (\frac{1-k}{2}, 1]$. This shows that $b \in U(F; t_0)$ but $a \notin U(F; t_0)$, a contradiction and hence $F(x) \leq \max \{F(y), \frac{1-k}{2}\}$ for all $x \leq y$. $\square$

Example 3.10. Consider the ordered semigroup $S = \{a, b, c, d\}$ with the following multiplication table “.” and order relation “ $\leq$ ”:

.	a	b	c	d
a	a	a	a	a
b	a	a	a	a
c	a	a	b	a
d	a	a	b	b

$$\leq := \{(a, a), (b, b), (c, c), (d, d), (a, b)\}.$$

Then $\{a\}, \{a, b\}, \{a, c\}, \{a, d\}, \{a, b, c\}, \{a, c, d\}$ and $\{a, b, c, d\}$ are generalized bi-ideals of S . However, $\{a, c\}, \{a, d\}$ and $\{a, c, d\}$ are not bi-ideals of S . Define a fuzzy subset F of S as follows:

$$F : S \longrightarrow [0, 1] | x \longmapsto F(x) = \begin{cases} 0.50, & \text{if } x = a, \\ 0.10, & \text{if } x = b, \\ 0.30, & \text{if } x = c, \\ 0.40, & \text{if } x = d. \end{cases}$$

Then

$$U(F; t) = \begin{cases} S, & \text{if } 0.00 < t \leq 0.10, \\ \{a, c, d\}, & \text{if } 0.10 < t \leq 0.30, \\ \{a, d\}, & \text{if } 0.30 < t \leq 0.40, \\ \{a\}, & \text{if } 0.40 < t \leq 0.50, \\ \emptyset, & \text{if } 0.50 < t \leq 1.00. \end{cases}$$

Thus, by Theorem 3.9, F is a $(\overline{\epsilon}, \overline{\epsilon} \vee \overline{q}_k)$ -fuzzy generalized bi-ideal of S for all $t \in (\frac{1-k}{2}, 1]$ with $k = 0.3$.

Note that, every $(\overline{\epsilon}, \overline{\epsilon} \vee \overline{q}_k)$ -fuzzy bi-ideal of S is a $(\overline{\epsilon}, \overline{\epsilon} \vee \overline{q}_k)$ -fuzzy generalized bi-ideal of S . However, the converse is not true, in general, as shown in the following example.

Example 3.11. Consider the ordered semigroup as shown in Example 3.10. Then F is a $(\overline{\epsilon}, \overline{\epsilon} \vee \overline{q}_k)$ -fuzzy generalized bi-ideal of S but F is not a $(\overline{\epsilon}, \overline{\epsilon} \vee \overline{q}_k)$ -fuzzy bi-ideal of S because by condition (2b) of Lemma 3.4, we have

$$\max \left\{ F(xy), \frac{1-k}{2} \right\} \geq \min \{ F(x), F(y) \}.$$

If $x = y = d$ and $k = 0.3$ then

$$\max \left\{ F(dd) = 0.1, \frac{1-k}{2} = 0.35 \right\} \not\geq F(d) = 0.4.$$

Proposition 3.12. If F is $(\overline{\epsilon}, \overline{\epsilon} \vee \overline{q})$ -fuzzy generalized bi-ideal of S and

$$F_{\frac{1-k}{2}} = \left\{ a \in S | F(a) > \frac{1-k}{2} \right\},$$

then $F_{\frac{1-k}{2}}$ is a generalized bi-ideal of S .

Proof. Let $a, b \in S$ such that $a \leq b \in F_{\frac{1-k}{2}}$. Then by Lemma 3.3 (1b),

$$\max \left\{ F(a), \frac{1-k}{2} \right\} \geq F(b) > \frac{1-k}{2}.$$

This implies $F(a) > \frac{1-k}{2}$ (since $\frac{1-k}{2} \not\geq \frac{1-k}{2}$) i.e. $a \in F_{\frac{1-k}{2}}$.

Next, we let $a, b, c \in S$ such that $a, c \in F_{\frac{1-k}{2}}$. Then by Lemma 3.5 (3b),

$$\begin{aligned} \max \left\{ F(abc), \frac{1-k}{2} \right\} &\geq \min \{F(a), F(c)\} \\ &> \frac{1-k}{2}. \end{aligned}$$

From this we see that $F(abc) > \frac{1-k}{2}$ (since $\frac{1-k}{2} \not\geq \frac{1-k}{2}$) and we write $abc \in F_{\frac{1-k}{2}}$. Hence $F_{\frac{1-k}{2}}$ is a generalized bi-ideal of S . $\square$

Consider a fuzzy subset F of S and $t \in (0, 1]$. We define two sets as in the following:

$$Q^k(F; t) = \{x \in S \mid [x; t] q_k F\} \text{ and } [F]_t^k = \{x \in S \mid [x; t] \in \vee q_k F\}.$$

Theorem 3.13. If F is $(\overline{\in}, \overline{\in} \vee \overline{q}_k)$ -fuzzy generalized bi-ideal of S and

$$Q^k(F; t) = \{x \in S \mid [x; t] q_k F\} \neq \emptyset,$$

then $Q^k(F; t)$ is generalized bi-ideal of S for all $t \in (\frac{1-k}{2}, 1]$.

Proof. Let F be $(\overline{\in}, \overline{\in} \vee \overline{q})$ -fuzzy generalized bi-ideal of S . Let $a, b \in S$ such that $a \leq b \in Q^k(F; t)$ and $t \in (\frac{1-k}{2}, 1]$, then by Lemma 3.3 (1b)

$$\max \left\{ F(a), \frac{1-k}{2} \right\} \geq F(b) > 1-k-t > (1-k) - \left(\frac{1-k}{2} \right) = \frac{1-k}{2}.$$

This shows $F(a) > 1-k-t$ and we write $a \in Q^k(F; t)$.

Next, we let $a, b, c \in S$ such that $a, c \in Q^k(F; t)$, then by Lemma 3.5 (3b),

$$\begin{aligned} \max \left\{ F(abc), \frac{1-k}{2} \right\} &\geq \min \{F(a), F(c)\} \\ &> \min \{1-k-t, 1-k-t\} \\ &= 1-k-t. \end{aligned}$$

This shows that $F(abc) > 1-k-t$ (as $1-k-t > \frac{1-k}{2}$) and we write $abc \in Q^k(F; t)$. Hence $Q^k(F; t)$ is generalized bi-ideal of S . $\square$

From Theorem 3.9 and Theorem 3.13 we can prove the following Theorem.

Theorem 3.14. *Let F be a fuzzy subset of S . Then F is $(\bar{\epsilon}, \bar{\epsilon} \vee \bar{q}_k)$ -fuzzy generalized bi-ideal of S if and only if $[F]_t^k \neq \emptyset \Rightarrow [F]_t^k$ is a generalized bi-ideal of S for all $t \in (\frac{1-k}{2}, 1]$.*

Proposition 3.15. *Every $(\bar{\epsilon}, \bar{\epsilon} \vee \bar{q}_k)$ -fuzzy generalized bi-ideal of a regular ordered semigroup S is a $(\bar{\epsilon}, \bar{\epsilon} \vee \bar{q}_k)$ -fuzzy bi-ideal of S .*

Proof. Let $a, b \in S$ and F be a $(\bar{\epsilon}, \bar{\epsilon} \vee \bar{q}_k)$ -fuzzy generalized bi-ideal of S . Since S is regular, there exists $x \in S$ such that $b \leq bxb$. Then

$$\max \left\{ F(ab), \frac{1-k}{2} \right\} = F(ab) \geq F(a(bxb)) = F(a(bx)b) \geq \{F(a) \wedge F(b)\}.$$

This means that F is a $(\bar{\epsilon}, \bar{\epsilon} \vee \bar{q}_k)$ -fuzzy subsemigroup of S . Thus F is a $(\bar{\epsilon}, \bar{\epsilon} \vee \bar{q}_k)$ -fuzzy bi-ideal of S . $\square$

Proposition 3.16. *Every $(\bar{\epsilon}, \bar{\epsilon} \vee \bar{q}_k)$ -fuzzy generalized bi-ideal of a left weakly regular ordered semigroup S is a $(\bar{\epsilon}, \bar{\epsilon} \vee \bar{q}_k)$ -fuzzy bi-ideal of S .*

Proof. Let $a, b \in S$ and F be a $(\bar{\epsilon}, \bar{\epsilon} \vee \bar{q}_k)$ -fuzzy generalized bi-ideal of S . Since S is left weakly regular, there exist $x, y \in S$ such that $b \leq xbyb$. Then

$$\max \left\{ F(ab), \frac{1-k}{2} \right\} = F(ab) \geq F(a(xbyb)) = F(a(xby)b) \geq \{F(a) \wedge F(b)\}.$$

This means that F is a $(\bar{\epsilon}, \bar{\epsilon} \vee \bar{q}_k)$ -fuzzy subsemigroup of S . Thus F is a $(\bar{\epsilon}, \bar{\epsilon} \vee \bar{q}_k)$ -fuzzy bi-ideal of S . $\square$

Remark 3.17. From Proposition 3.15 and 3.16, it follows that in regular and left weakly regular ordered semigroups, the concepts of $(\bar{\epsilon}, \bar{\epsilon} \vee \bar{q}_k)$ -fuzzy generalized bi-ideals and $(\bar{\epsilon}, \bar{\epsilon} \vee \bar{q}_k)$ -fuzzy bi-ideals coincide.

Lemma 3.18. *A non empty subset A of S is generalized bi-ideal of S if and only if the characteristic χ_A of A is $(\bar{\epsilon}, \bar{\epsilon} \vee \bar{q}_k)$ -fuzzy generalized bi-ideal of S .*

Proof. The proof is obvious and is omitted. $\square$

Definition 3.19. A fuzzy subset F of S is called $(\bar{\epsilon}, \bar{\epsilon} \vee \bar{q}_k)$ -fuzzy left (right) ideal of S if

- (i) $(\forall x, y \in S, t \in (0, 1] \text{ and } x \leq y) (x; t) \bar{\epsilon} F \rightarrow (y; t) \bar{\epsilon} \vee \bar{q}_k F$,
- (ii) $(\forall x, y \in S, t \in (0, 1]) (xy; t) \bar{\epsilon} F \rightarrow (y; t) \bar{\epsilon} \vee \bar{q}_k F ((x; t) \bar{\epsilon} \vee \bar{q}_k F).$

Theorem 3.20. A fuzzy subset F of S is $(\overline{\in}, \overline{\in} \vee \overline{q}_k)$ -fuzzy left (right) ideal of S if and only if the following conditions hold for all $x, y \in S$, $t \in (0, 1]$:

(iii) $\max \left\{ F(x), \frac{1-k}{2} \right\} \geq F(y)$ for all $x \leq y$.

(iv) $\max \left\{ F(xy), \frac{1-k}{2} \right\} \geq F(y)$ ($\max \left\{ F(xy), \frac{1-k}{2} \right\} \geq F(x)$).

Proof. Let F be $(\overline{\in}, \overline{\in} \vee \overline{q}_k)$ -fuzzy left ideal of S and

$$\max \left\{ F(a), \frac{1-k}{2} \right\} < F(b) \text{ for some } a, b \in S \text{ with } a \leq b.$$

Then

$$\max \left\{ F(a), \frac{1-k}{2} \right\} < t \leq F(b) \text{ for some } t \in \left(\frac{1-k}{2}, 1 \right].$$

This show that $(a; t) \overline{\in} F$ but $(b; t) \in F$, a contradiction and hence

$$\max \left\{ F(x), \frac{1-k}{2} \right\} \geq F(y) \text{ for all } x \leq y.$$

Next, we consider

$$\max \left\{ F(ab), \frac{1-k}{2} \right\} < F(b) \text{ for some } a, b \in S.$$

Then there exist some $t \in \left(\frac{1-k}{2}, 1 \right]$ such that

$$\max \left\{ F(ab), \frac{1-k}{2} \right\} < t \leq F(b).$$

We see that $(ab; t) \overline{\in} F$ but $(b; t) \in F$, a contradiction and thus

$$\max \left\{ F(xy), \frac{1-k}{2} \right\} \geq F(y).$$

Conversely, let (iii) and (iv) are satisfied for all $x, y \in S$. Let $x, y \in S$ with $x \leq y$ such that $(x; t) \overline{\in} F$. Then by (iii)

$$\begin{aligned} F(y) &\leq \max \left\{ F(x), \frac{1-k}{2} \right\} \\ &= \begin{cases} F(x) < t, & \text{if } F(x) \geq \frac{1-k}{2}, \\ \frac{1-k}{2}, & \text{if } F(x) < \frac{1-k}{2}. \end{cases} \end{aligned}$$

From here we observe that $(y; t) \overline{\in} F$ or $F(y) + t + k < 1$ (if $t \leq \frac{1-k}{2}$) i.e. $(y; t) \overline{q}_k F$. On the other hand, if $t > \frac{1-k}{2}$ then $(y; t) \overline{\in} F$ and consequently $(y; t) \overline{\in} \vee \overline{q}_k F$.

Lastly we choose $x, y \in S$ such that $(xy; t) \bar{\in} F$, then by (iv),

$$\begin{aligned} F(y) &\leq \max \left\{ F(xy), \frac{1-k}{2} \right\} \\ &= \begin{cases} F(xy) < t, & \text{if } F(xy) \geq \frac{1-k}{2}, \\ \frac{1-k}{2}, & \text{if } F(xy) < \frac{1-k}{2}. \end{cases} \end{aligned}$$

It follows that $(y; t) \bar{\in} F$ or $F(y) + t + k < 1$ (if $t \leq \frac{1-k}{2}$) i.e. $(y; t) \bar{q}_k F$. On the other hand, if $t > \frac{1-k}{2}$ then $(y; t) \bar{\in} F$. Hence $(y; t) \bar{\in} \vee \bar{q}_k F$. Consequently, F is $(\bar{\in}, \bar{\in} \vee \bar{q}_k)$ -fuzzy left ideal of S . Similarly one can show this for right ideal. $\square$

Theorem 3.21. A fuzzy subset λ of S is a $(\bar{\in}, \bar{\in} \vee \bar{q}_k)$ -fuzzy left (right)-ideal of S if and only if $U(\lambda; t) (\neq \emptyset)$ is a left (right)-ideal of S for all $t \in (0, \frac{1-k}{2}]$.

Proof. The proof follows from Theorem 3.9. $\square$

4. Upper parts of $(\bar{\in}, \bar{\in} \vee \bar{q}_k)$ -fuzzy generalized bi-ideals

In this section, we define the upper/lower parts of an $(\bar{\in}, \bar{\in} \vee \bar{q}_k)$ -fuzzy generalized bi-ideal and characterize regular and left weakly regular ordered semigroups in terms of $(\bar{\in}, \bar{\in} \vee \bar{q}_k)$ -fuzzy generalized bi-ideals and $(\bar{\in}, \bar{\in} \vee \bar{q}_k)$ -fuzzy left (right)-ideals.

Definition 4.1. ([10]) For any fuzzy subsets F and G of S , then for all $x \in S$ the fuzzy subsets F^{+k} , $(F \wedge^k G)^+$, $(F \vee^k G)^+$ and $(F \circ^k G)^+$ of S are defined as follows:

$$\begin{aligned} F^{+k} &: S \longrightarrow [0, 1] | x \longmapsto F^k(x) = F(x) \vee \frac{1-k}{2}, \\ (F \wedge^k G)^+ &: S \longrightarrow [0, 1] | x \longmapsto (F \wedge^k G)(x) = (F \wedge G)(x) \vee \frac{1-k}{2}, \\ (F \vee^k G)^+ &: S \longrightarrow [0, 1] | x \longmapsto (F \vee^k G)(x) = (F \vee G)(x) \vee \frac{1-k}{2}, \\ (F \circ^k G)^+ &: S \longrightarrow [0, 1] | x \longmapsto (F \circ^k G)(x) = (F \circ G)(x) \vee \frac{1-k}{2}. \end{aligned}$$

Lemma 4.2. Let F and G be fuzzy subsets of S . Then the following hold:

- (i) $(F \wedge^k G)^+ = (F^{+k} \wedge G^{+k})$,
- (ii) $(F \vee^k G)^+ = (F^{+k} \vee G^{+k})$,

(iii) $(F \circ^k G)^+ \succeq (F^{+k} \circ G^{+k})$, if $A_x = \emptyset$ and $(F \circ^k G)^+ = (F^{+k} \circ G^{+k})$, if $A_x \neq \emptyset$.

Proof. (i) and (ii) follows from [10, Proposition 13].

(iii) Let $a \in S$. If $A_a = \emptyset$, then $(F \circ^k G)^+(a) = (F \circ G)(a) \vee \frac{1-k}{2} = 0 \vee \frac{1-k}{2} = \frac{1-k}{2}$. On the other hand, $(F^{+k} \circ G^{+k})(a) = 0$ and hence $(F^{+k} \circ G^{+k}) \preceq (F \circ^k G)^+$. Let $A_a \neq \emptyset$, then

$$\begin{aligned}
 (F \circ^k G)^+(a) &= (F \circ G)(a) \vee \frac{1-k}{2} \\
 &= \left(\bigvee_{(y,z) \in A_a} (F(y) \wedge G(z)) \right) \vee \frac{1-k}{2} \\
 &= \bigvee_{(y,z) \in A_a} ((F(y) \wedge G(z)) \vee \frac{1-k}{2}) \\
 &= \bigvee_{(y,z) \in A_a} \left(\left(F(y) \vee \frac{1-k}{2} \right) \wedge \left(G(z) \vee \frac{1-k}{2} \right) \right) \\
 &= \bigvee_{(y,z) \in A_a} (F^{+k}(y) \wedge G^{+k}(z)) \\
 &= (F^{+k} \circ G^{+k})(a). \quad \square
 \end{aligned}$$

Let A be a non-empty subset of S , then the upper part of the characteristic function χ_A^k is defined as follows:

$$\chi_A^{+k} : S \longrightarrow [0, 1] | x \longmapsto \chi_A^{+k}(x) = \begin{cases} 1, & \text{if } x \in A, \\ \frac{1-k}{2}, & \text{otherwise.} \end{cases}$$

Lemma 4.3. *Let A and B be non-empty subset of S . Then the following hold:*

- (i) $(\chi_A \wedge^k \chi_B)^+ = \chi_{A \cap B}^{+k}$,
- (ii) $(\chi_A \vee^k \chi_B)^+ = \chi_{A \cup B}^{+k}$,
- (iii) $(\chi_A \circ^k \chi_B)^+ = \chi_{(AB]}^{+k}$.

Proof. The proofs of (i) and (ii) are obvious.

(iii) Let $x \in (AB]$ then $\chi_{(AB]}(x) = 1$ and hence $\chi_{(AB]}^{+k}(x) = 1 \vee \frac{1-k}{2} = 1$. Since $x \in (AB]$, we have $x \leq ab$ for some $a \in A$ and $b \in B$. Then

$(a, b) \in A_x$ and $A_x \neq \emptyset$. Thus

$$\begin{aligned} (\chi_A \circ^k \chi_B)^+(x) &= (\chi_A \circ \chi_B)(x) \vee \frac{1-k}{2} \\ &= \left[\bigvee_{(y,z) \in A_x} (\chi_A(y) \wedge \chi_B(z)) \right] \vee \frac{1-k}{2} \\ &\geq (\chi_A(a) \wedge \chi_B(b)) \vee \frac{1-k}{2}. \end{aligned}$$

Since $a \in A$ and $b \in B$, we have $\chi_A(a) = 1$ and $\chi_B(b) = 1$ and so

$$\begin{aligned} (\chi_A \circ^k \chi_B)^+(x) &\geq (\chi_A(a) \wedge \chi_B(b)) \vee \frac{1-k}{2} \\ &= (1 \wedge 1) \vee \frac{1-k}{2} = 1. \end{aligned}$$

Thus $(\chi_A \circ^k \chi_B)^+(x) = 1 = \chi_{(AB]}^{+k}(x)$. Let $x \notin (AB]$, then $\chi_{(AB]}(x) = 0$ and hence, $\chi_{(AB]}^{+k}(x) = 0 \vee \frac{1-k}{2} = \frac{1-k}{2}$. Let $(y, z) \in A_x$. Then

$$\begin{aligned} (\chi_A \circ^k \chi_B)^+(x) &= (\chi_A \circ \chi_B)(x) \vee \frac{1-k}{2} \\ &= \left[\bigvee_{(y,z) \in A_x} (\chi_A(y) \wedge \chi_B(z)) \right] \vee \frac{1-k}{2}. \end{aligned}$$

Since $(y, z) \in A_x$, then $x \leq yz$. If $y \in A$ and $z \in B$, we have $yz \in AB$ and so $x \in (AB]$. This is a contradiction. If $y \notin A$ and $z \in B$, then

$$\left[\bigvee_{(y,z) \in A_x} (\chi_A(y) \wedge \chi_B(z)) \right] \vee \frac{1-k}{2} = \left[\bigvee_{(y,z) \in A_x} (0 \wedge 1) \right] \vee \frac{1-k}{2} = \frac{1-k}{2}.$$

Hence, $\chi_{(AB]}^{+k}(x) = \frac{1-k}{2} = (\chi_A \circ^k \chi_B)^+(x)$. Similarly, for $y \in A$ and $z \notin B$, we have $\chi_{(AB]}^{+k}(x) = 0 = (\chi_A \circ^k \chi_B)^+(x)$. $\square$

Lemma 4.4. *The upper part χ_A^{+k} of the characteristic function χ_A of A is a $(\overline{\in}, \overline{\in} \vee \overline{q}_k)$ -fuzzy generalized bi-ideal of S if and only if A is a generalized bi-ideal of S .*

Proof. Let A be a generalized bi-ideal of S . Then by Theorem 2.4 and 3.19, χ_A^{+k} is an $(\overline{\in}, \overline{\in} \vee \overline{q}_k)$ -fuzzy generalized bi-ideal of S . Conversely, assume that χ_A^{+k} is an $(\overline{\in}, \overline{\in} \vee \overline{q}_k)$ -fuzzy generalized bi-ideal of S . Let $x, y \in S$, $x \leq y$. If $y \in A$, then $\chi_A^{+k}(y) = 1$. Since χ_A^{+k} is an $(\overline{\in}, \overline{\in} \vee \overline{q}_k)$ -fuzzy generalized bi-ideal of S and $x \leq y$, we have, $\chi_A^{+k}(x) \geq \chi_A^{+k}(y) = 1$.

It follows that $\chi_A^{+k}(x) = 1$ and so $x \in A$. Let $x, z \in A$ and $y \in S$. Then, $\chi_A^{+k}(x) = 1$ and $\chi_A^{+k}(z) = 1$. Now,

$$\chi_A^{+k}(xyz) \geq \chi_A^{+k}(x) \wedge \chi_A^{+k}(z) = 1.$$

Hence $\chi_A^{+k}(xyz) = 1$ and so $xyz \in A$. Therefore A is a generalized bi-ideal of S . $\square$

Lemma 4.5. *The upper part χ_A^{+k} of the characteristic function χ_A of A is a $(\overline{\epsilon}, \overline{\epsilon} \vee \overline{q}_k)$ -fuzzy left (right)-ideal of S if and only if A is a left (right)-ideal of S .*

Proof. The proof follows from Lemma 4.4. $\square$

In the following proposition, we show that if F is a $(\overline{\epsilon}, \overline{\epsilon} \vee \overline{q}_k)$ -fuzzy generalized bi-ideal of S , then F^{+k} is a fuzzy generalized bi-ideal of S .

Proposition 4.6. *If F is a $(\overline{\epsilon}, \overline{\epsilon} \vee \overline{q}_k)$ -fuzzy generalized bi-ideal of S , then F^{+k} is a fuzzy generalized bi-ideal of S .*

Proof. Let $x, y \in S$, $x \leq y$. Since F is an $(\overline{\epsilon}, \overline{\epsilon} \vee \overline{q}_k)$ -fuzzy generalized bi-ideal of S and $x \leq y$, then we have $F^{+k}(x) = F(x) \vee \frac{1-k}{2} \geq F(y)$. It follows that $F^{+k}(x) = F(x) \vee \frac{1-k}{2} = (F(x) \vee \frac{1-k}{2}) \vee \frac{1-k}{2} \geq F(y) \vee \frac{1-k}{2} = F^{+k}(y)$.

For $x, y, z \in S$, we have $F^{+k}(xyz) = F(xyz) \vee \frac{1-k}{2} \geq F(x) \wedge F(z)$. Then

$$\begin{aligned} F^{+k}(xyz) &= F(xyz) \vee \frac{1-k}{2} = \left(F(xyz) \vee \frac{1-k}{2} \right) \vee \frac{1-k}{2} \\ &\geq (F(x) \wedge F(z)) \vee \frac{1-k}{2} \\ &= \left(F(x) \vee \frac{1-k}{2} \right) \wedge \left(F(z) \vee \frac{1-k}{2} \right) \\ &= F^{+k}(x) \wedge F^{+k}(z). \end{aligned}$$

Consequently, F^{+k} is a fuzzy generalized bi-ideal of S . $\square$

In [33], regular and left weakly regular ordered semigroups are characterized by the properties of their fuzzy left (right) and fuzzy generalized bi-ideals. In the following we characterize regular, left weakly regular, left and right simple and completely regular ordered semigroups in terms of $(\overline{\epsilon}, \overline{\epsilon} \vee \overline{q}_k)$ -fuzzy left (right) and $(\overline{\epsilon}, \overline{\epsilon} \vee \overline{q}_k)$ -fuzzy generalized bi-ideals. We first need the following result.

Proposition 4.7. ([23]) *An ordered semigroup S is left (right) simple if and only if $(Sa] = S$ ($(aS] = S$) for every $a \in S$.*

Proposition 4.8. *If S is regular, left and right simple ordered semi-group then for every $(\overline{\epsilon}, \overline{\epsilon} \vee \overline{q}_k)$ -fuzzy generalized bi-ideal F of S we have $F^{+k}(a) = F^{+k}(b)$, for every $a, b \in S$.*

Proof. Assume that S is regular, left and right simple and F a $(\overline{\epsilon}, \overline{\epsilon} \vee \overline{q}_k)$ -fuzzy generalized bi-ideal of S . We consider, $E_S = \{e \in S | e \leq e^2\}$, then E_S is obviously non-empty. Let $s, t \in E_S$. Since S is left and right simple, by Proposition 4.7, it follows that $S = (Ss]$ and $S = (tS]$. Since $t \in S$, we have $t \in (Ss]$ and $t \in (tS]$, then $t \leq xs$ and $t \leq sy$ for some $x, y \in S$, and we have

$$t^2 \leq (sy)(xs) = s(yx)s.$$

Since F is a $(\overline{\epsilon}, \overline{\epsilon} \vee \overline{q}_k)$ -fuzzy generalized bi-ideal of S , we have

$$\begin{aligned} F(t^2) \vee \frac{1-k}{2} &\geq F(s(yx)s) \vee \frac{1-k}{2} \\ &\geq (F(s) \wedge F(s)) \\ &= F(s). \end{aligned}$$

Thus $F^{+k}(t^2) = F(t^2) \vee \frac{1-k}{2} = (F(t^2) \vee \frac{1-k}{2}) \vee \frac{1-k}{2} \geq F(s) \vee \frac{1-k}{2} = F^{+k}(s)$ and we have

$$(I) \quad F^{+k}(t^2) \geq F^{+k}(s).$$

Since $t \in E_S$, we have $t \leq t^2$ and so $F(t) \vee \frac{1-k}{2} \geq F(t^2)$. It follows that

$$\begin{aligned} F^{+k}(t) &= \left(F(t) \vee \frac{1-k}{2} \right) \vee \frac{1-k}{2} \\ &\geq F(t^2) \vee \frac{1-k}{2}, \end{aligned}$$

and so $F^{+k}(t) \geq F^{+k}(t^2)$. Thus, by (I), we have $F^{+k}(t) \geq F^{+k}(s)$. On the other hand, since $t \in S$, by Proposition 4.7, we have $(St] = S = (tS]$. Since $s \in S$, we have $s \in (St]$ and $s \in (tS]$, then $s \leq at$ and $s \leq tb$ for some $a, b \in S$. Thus, by the same arguments as above, we get $F^{+k}(s) \geq F^{+k}(t)$. It follows that $F^{+k}(t) = F^{+k}(s)$ and hence F^{+k} is constant on E_S .

Now, let $a \in S$. Then there exists $x \in S$ such that $a \leq axa$. It follows that

$$\begin{aligned} ax &\leq (axa)x \\ &= (ax)(ax) \\ &= (ax)^2, \end{aligned}$$

and

$$\begin{aligned} xa &\leq x(axa) \\ &= xa)(xa) \\ &= (xa)^2. \end{aligned}$$

Thus, $ax, xa \in E_S$. By previous arguments, we get, $F^{+k}(ax) = F^{+k}(b) = F^{+k}(xa)$. Since $(ax)a(xa) = (axa)xa \geq axa \geq a$, we have

$$\begin{aligned} F^{+k}(a) &= F(a) \vee \frac{1-k}{2} \geq F((ax)a(xa)) \vee \frac{1-k}{2} \\ &\geq (F(ax) \wedge F(xa)). \end{aligned}$$

Thus

$$\begin{aligned} F^{+k}(a) &= F(a) \vee \frac{1-k}{2} = \left(F(a) \vee \frac{1-k}{2} \right) \vee \frac{1-k}{2} \\ &\geq (F(ax) \wedge F(xa)) \vee \frac{1-k}{2} \\ &= \left(F(ax) \vee \frac{1-k}{2} \right) \wedge \left(F(xa) \vee \frac{1-k}{2} \right), \end{aligned}$$

and thus $F^{+k}(a) \geq F^{+k}(ax) \wedge F^{+k}(xa) = F^{+k}(b)$. Since $b \in (Sa]$ and $b \in (aS]$, we have $b \leq pa$ and $b \leq aq$ for some $p, q \in S$. Then $b^2 \leq (aq)(pa) = a(qp)a$ and thus

$$\begin{aligned} F(b^2) \vee \frac{1-k}{2} &\geq F(a(qp)a) \\ &\geq (F(a) \wedge F(a)) \\ &= F(a). \end{aligned}$$

Hence, $F^{+k}(b^2) = F(b^2) \vee \frac{1-k}{2} = (F(b^2) \vee \frac{1-k}{2}) \vee \frac{1-k}{2} \geq F(a) \vee \frac{1-k}{2}$ and we have, $F^{+k}(b^2) \geq F^{+k}(a)$. Since $b \in E_S$, $b^2 \geq b$, then $F(b) \vee \frac{1-k}{2} \geq F(b^2)$ and hence $F^{+k}(b) = F(b) \vee \frac{1-k}{2} = (F(b) \vee \frac{1-k}{2}) \vee \frac{1-k}{2} \geq F(b^2) \vee \frac{1-k}{2}$, it follows that $F^{+k}(b) \geq F^{+k}(b^2)$ and so $F^{+k}(b) \geq F^{+k}(a)$. Thus, $F^{+k}(b) = F^{+k}(a)$ and so, F^{+k} is a constant function on S . $\square$

Proposition 4.9. ([23]) *An ordered semigroup S is completely regular if and only if for every $A \subseteq S$, we have $A \subseteq (A^2SA^2]$.*

Proposition 4.10. *Let S be a completely regular ordered semigroup. Then for every $(\overline{\epsilon}, \overline{\epsilon} \vee \overline{q}_k)$ -fuzzy generalized bi-ideal F of S , we have*

$$F^{+k}(a) = F^{+k}(a^2) \text{ for every } a \in S.$$

Proof. Let $a \in S$. Since S is completely regular, by Proposition 4.9, $a \in (a^2Sa^2]$. Then there exists $x \in S$ such that $a \leq a^2xa^2$. Since F is a $(\overline{\epsilon}, \overline{\epsilon} \vee \overline{q}_k)$ -fuzzy generalized bi-ideal of S , we have

$$\begin{aligned} F(a) \vee \frac{1-k}{2} &\geq F(a^2xa^2) \vee \frac{1-k}{2} \\ &\geq (F(a^2) \wedge F(a^2)) \\ &= F(a^2) \\ &\geq (F(a) \wedge F(a)) \\ &= F(a). \end{aligned}$$

Thus, $F^{+k}(a) = F(a) \vee \frac{1-k}{2} = (F(a) \vee \frac{1-k}{2}) \vee \frac{1-k}{2} \geq F(a^2) \vee \frac{1-k}{2} \geq F(a) \vee \frac{1-k}{2}$, and it follows that $F^{+k}(a) \geq F^{+k}(a^2) \geq F^{+k}(a)$. Thus $F^{+k}(a) = F^{+k}(a^2)$ for every $a \in S$. $\square$

Theorem 4.11. *If every $(\overline{\epsilon}, \overline{\epsilon} \vee \overline{q}_k)$ -fuzzy generalized bi-ideal F of S satisfies the condition, $F^{+k}(t) = F^{+k}(t^2)$ for every $t \in S$. Then S is completely regular.*

Proof. Let $t \in S$. We consider the generalized bi-ideal $B(t^2) = (t^2 \cup t^2St^2]$ of S , generated by $t^2(t \in S)$. Then by Lemma 4.4,

$$\chi_{B(t^2)}^{+k}(t) = \begin{cases} 1, & \text{if } t \in B(t^2), \\ \frac{1-k}{2}, & \text{otherwise,} \end{cases}$$

is a $(\overline{\epsilon}, \overline{\epsilon} \vee \overline{q}_k)$ -fuzzy generalized bi-ideal of S . By hypothesis, we have

$$\overline{\chi}_{B(t^2)}^k(t^2) = \overline{\chi}_{B(t^2)}^k(t).$$

Since $t^2 \in B(t^2)$, we have $\chi_{B(t^2)}^{+k}(t^2) = 1$ and hence, $\chi_{B(t^2)}^{+k}(t) = 1$, thus $t \in B(t^2)$ and hence, $t \leq t^2$ or $t \leq t^2xt^2$. If $t \leq t^2$, then $t \leq t^2 = tt \leq t^2t^2 = ttt^2 \leq t^2tt^2 \in t^2St^2$ and $t \in (t^2St^2]$. If $t \leq t^2xt^2$, then $t \in t^2St^2$ and $t \in (t^2St^2]$. Thus, S is completely regular. $\square$

Proposition 4.12. ([22]) *Let S be an ordered semigroup, then the following conditions are equivalent:*

- (i) S is regular,
- (ii) $B \cap L \subseteq (BL]$ for every generalized bi-ideal B and left ideal L of S ,
- (iii) $B(a) \cap L(a) \subseteq (B(a)L(a)]$ for every $a \in S$.

Theorem 4.13. *An ordered semigroup S is regular if and only if for every $(\overline{\epsilon}, \overline{\epsilon} \vee \overline{q}_k)$ -fuzzy generalized bi-ideal F and $(\overline{\epsilon}, \overline{\epsilon} \vee \overline{q}_k)$ -fuzzy left*

ideal G of S , we have

$$(F \wedge^k G)^+ \preceq (F \circ^k G)^+.$$

Proof. Suppose that F is a $(\overline{\epsilon}, \overline{\epsilon} \vee \overline{q}_k)$ -fuzzy generalized bi-ideal and G a $(\overline{\epsilon}, \overline{\epsilon} \vee \overline{q}_k)$ -fuzzy left ideal of a regular ordered semigroup S . Let $a \in S$. Since S is regular, there exists $x \in S$ such that $a \leq axa \leq (axa)(xa)$. Then $(axa, xa) \in A_a$ and $A_a \neq \emptyset$. Thus,

$$\begin{aligned} (F \circ^k G)^+(a) &= (F \circ G)(a) \vee \frac{1-k}{2} \\ &= \left[\bigvee_{(y,z) \in A_a} (F(y) \wedge G(z)) \right] \vee \frac{1-k}{2} \\ &\geq (F(axa) \wedge G(xa)) \vee \frac{1-k}{2} \\ &= \left(F(axa) \vee \frac{1-k}{2} \right) \wedge \left(G(xa) \vee \frac{1-k}{2} \right) \\ &= F^{+k}(axa) \wedge G^{+k}(xa). \end{aligned}$$

Since F is a $(\overline{\epsilon}, \overline{\epsilon} \vee \overline{q}_k)$ -fuzzy generalized bi-ideal and G an $(\overline{\epsilon}, \overline{\epsilon} \vee \overline{q}_k)$ -fuzzy left ideal of S , we have $F^{+k}(axa) \geq F^{+k}(a) \wedge F^{+k}(a) = F^{+k}(a)$ and $G^{+k}(xa) \geq G^{+k}(a)$. Therefore,

$$\left[F^{+k}(axa) \wedge G^{+k}(xa) \right] \geq F^{+k}(a) \wedge G^{+k}(a) = (F \wedge^k G)^+(a).$$

Thus $(F \circ^k G)^+(a) \geq (F \wedge^k G)^+(a)$.

Conversely, assume that $(F \wedge^k G)^+ \preceq (F \circ^k G)^+$ for every $(\overline{\epsilon}, \overline{\epsilon} \vee \overline{q}_k)$ -fuzzy generalized bi-ideal F and every $(\overline{\epsilon}, \overline{\epsilon} \vee \overline{q}_k)$ -fuzzy left ideal G of S . To prove that S is regular, by Proposition 4.12 it is enough to prove that,

$$B \cap L \subseteq (BL] \text{ for generalized bi-ideal } B \text{ and left ideal } L \text{ of } S.$$

Let $x \in B \cap L$. Then $x \in B$ and $x \in L$. Since B is a generalized bi-ideal and L a left ideal of S , by Lemma 4.4 and 4.5, χ_B^{+k} is a $(\overline{\epsilon}, \overline{\epsilon} \vee \overline{q}_k)$ -fuzzy generalized bi-ideal and χ_L^{+k} a $(\overline{\epsilon}, \overline{\epsilon} \vee \overline{q}_k)$ -fuzzy left ideal of S . By hypothesis, $(\chi_B \circ^k \chi_L)^+(x) \geq (\chi_B \wedge^k \chi_L)^+(x) = (\chi_B^k \wedge \chi_L^k)(x) \vee \frac{1-k}{2}$. Since $x \in B$ and $x \in L$, we have $\chi_B^{+k}(x) = 1$ and $\chi_L^{+k}(x) = 1$. Thus $(\chi_B^{+k} \wedge \chi_L^{+k})(x) = 1$. It follows that $(\chi_B \circ^k \chi_L)^+(x) = 1$. By using Lemma 4.3 (iii), we have $(\chi_B \circ^k \chi_L)^+ = \chi_{(BL]}^{+k}$. Therefore, $\chi_{(BL]}^{+k}(x) = 1$ and so $x \in (BL]$. Consequently, S is regular. $\square$

Proposition 4.14. ([22]) *Let S be an ordered semigroup, then the following conditions are equivalent:*

- (i) S is regular,
- (ii) $B \cap I = (BIB]$ for every generalized bi-ideal B and ideal I of S ,
- (iii) $B(a) \cap I(a) = (B(a)I(a)B(a)]$ for every $a \in S$.

Theorem 4.15. *An ordered semigroup S is regular if and only if for every $(\in, \in \vee q_k)$ -fuzzy generalized bi-ideal F and every $(\overline{\in}, \overline{\in} \vee \overline{q}_k)$ -fuzzy ideal G of S , we have*

$$(F \wedge^k G)^+ \preceq (F \circ^k G \circ^k F)^+.$$

Proof. Suppose that F is a $(\overline{\in}, \overline{\in} \vee \overline{q}_k)$ -fuzzy generalized bi-ideal and G a $(\overline{\in}, \overline{\in} \vee \overline{q}_k)$ -fuzzy ideal of a regular ordered semigroup S . Let $a \in S$. Since S is regular, there exists $x \in S$ such that $a \leq axa \leq (axa)(xa) = a(xaxa)$. Then $(a, xaxa) \in A_a$ and $A_a \neq \emptyset$. Thus,

$$\begin{aligned} & (F \circ^k G \circ^k F)^+(a) \\ &= (F \circ^k G \circ F)(a) \vee \frac{1-k}{2} \\ &= \left[\bigvee_{(y,z) \in A_a} (F \circ^k G)(y) \wedge F(z) \right] \vee \frac{1-k}{2} \\ &= \bigvee_{(y,z) \in A_a} \left[\bigvee_{(p,q) \in A_a} \left((F(p) \wedge G(q)) \vee \frac{1-k}{2} \right) \wedge F(z) \right] \vee \frac{1-k}{2} \\ &= \bigvee_{(y,z) \in A_a} \bigvee_{(p,q) \in A_a} ((F(p) \wedge G(q)) \wedge F(z)) \vee \frac{1-k}{2} \\ &= \bigvee_{(y,z) \in A_a} \bigvee_{(p,q) \in A_a} (F(p) \wedge G(q) \wedge F(z)) \vee \frac{1-k}{2} \\ &\geq (F(a) \wedge G(xax) \wedge F(a)) \vee \frac{1-k}{2}. \end{aligned}$$

Since G a $(\overline{\in}, \overline{\in} \vee \overline{q}_k)$ -fuzzy ideal of S , we have $G(xax) \vee \frac{1-k}{2} \geq G(ax) \wedge \frac{1-k}{2} \geq G(a)$. Therefore

$$\begin{aligned} & \left[F(a) \wedge G(xax) \wedge F(a) \wedge \frac{1-k}{2} \right] \\ &= \left[F(a) \wedge \left(G(xax) \vee \frac{1-k}{2} \right) \wedge F(a) \vee \frac{1-k}{2} \right] \\ &\geq \left[F(a) \wedge G(a) \wedge F(a) \vee \frac{1-k}{2} \right] \end{aligned}$$

$$\begin{aligned}
&\geq \left(F(a) \vee \frac{1-k}{2} \right) \wedge \left(G(a) \vee \frac{1-k}{2} \right) \\
&= F^+(a) \wedge^k G^+(a).
\end{aligned}$$

Thus $(F \circ^k G \circ^k F)^+(a) \geq (F \wedge^k G)^+(a)$.

Conversely, assume that $(F \wedge^k G)^+ \preceq (F \circ^k G \circ^k F)^+$ for every $(\overline{\epsilon}, \overline{\epsilon} \vee \overline{q}_k)$ -fuzzy generalized bi-ideal F and every $(\overline{\epsilon}, \overline{\epsilon} \vee \overline{q}_k)$ -fuzzy ideal F of S . To prove that S is regular, by Proposition 4.14, it is enough to prove that $B \cap I \subseteq (BIB]$ for generalized bi-ideal B and ideal I of S .

Let $x \in B \cap I$. Then $x \in B$ and $x \in I$. Since B is a generalized bi-ideal and I an ideal of S , by Lemma 4.4 and 4.5, χ_B^{+k} is an $(\overline{\epsilon}, \overline{\epsilon} \vee \overline{q}_k)$ -fuzzy generalized bi-ideal and χ_I^{+k} an $(\overline{\epsilon}, \overline{\epsilon} \vee \overline{q}_k)$ -fuzzy ideal of S . By hypothesis, $(\chi_B \circ^k \chi_I \circ^k \chi_B)^+(x) \geq (\chi_B \wedge^k \chi_I)^+(x) = (\chi_B \wedge \chi_I)(x) \vee \frac{1-k}{2}$. Since $x \in B$ and $x \in I$, we have $\chi_B^{+k}(x) = 1$ and $\chi_I^{+k}(x) = 1$. Thus $(\chi_B \wedge \chi_I)(x) \vee \frac{1-k}{2} = 1$. It follows that $(\chi_B \circ^k \chi_I \circ^k \chi_B)^+(x) = 1$. By using Lemma 4.3 (iii), we have $(\chi_B \circ^k \chi_I \circ^k \chi_B)^+(x) = \chi_{(BIB]}^{+k}(x)$. Therefore, $\chi_{(BIB]}^{+k}(x) = 1$ and so $x \in (BIB]$. Consequently, S is regular. $\square$

Proposition 4.16. ([22]) *Let S be an ordered semigroup, then the following conditions are equivalent:*

- (i) S is regular,
- (ii) $R \cap B \cap L \subseteq (RBL]$ for every right ideal R , generalized bi-ideal B and left ideal L of S ,
- (iii) $R(a) \cap B(a) \cap L(a) \subseteq (R(a)B(a)L(a))$ for every $a \in S$.

Theorem 4.17. *An ordered semigroup S is regular if and only if for every $(\overline{\epsilon}, \overline{\epsilon} \vee \overline{q}_k)$ -fuzzy generalized bi-ideal F , every $(\overline{\epsilon}, \overline{\epsilon} \vee \overline{q}_k)$ -fuzzy right ideal G and every $(\overline{\epsilon}, \overline{\epsilon} \vee \overline{q}_k)$ -fuzzy left ideal H of S , we have,*

$$(F \wedge G \wedge H)^+ \preceq (G \circ G \circ H)^+.$$

Proof. Let S be a regular ordered semigroup, F a $(\overline{\epsilon}, \overline{\epsilon} \vee \overline{q}_k)$ -fuzzy generalized bi-ideal, G a $(\overline{\epsilon}, \overline{\epsilon} \vee \overline{q}_k)$ -fuzzy right ideal and H a $(\overline{\epsilon}, \overline{\epsilon} \vee \overline{q}_k)$ -fuzzy left ideal of S . Let $a \in S$. Since S is regular, there exists $x \in S$ such that $a \leq axa = axa \leq (axa)(xa) \leq (axa)x(axa) = (ax)(axa)(xa)$. Then $((ax)(axa), xa) \in A_a$ and $A_a \neq \emptyset$. Thus,

$$\begin{aligned}
&\left(F \circ^k G \circ^k H \right)^+(a) \\
&= (F \circ^k G \circ H)(a) \vee \frac{1-k}{2}
\end{aligned}$$

$$\begin{aligned}
&= \left[\bigvee_{(y,z) \in A_a} (F \circ^k G)(y) \wedge H(z) \right] \vee \frac{1-k}{2} \\
&= \bigvee_{(y,z) \in A_a} \left[\bigvee_{(p,q) \in A_a} \left((F(p) \wedge G(q)) \vee \frac{1-k}{2} \right) \wedge H(z) \right] \vee \frac{1-k}{2} \\
&= \bigvee_{(y,z) \in A_a} \bigvee_{(p,q) \in A_a} ((F(p) \wedge G(q)) \wedge H(z)) \vee \frac{1-k}{2} \\
&= \bigvee_{(y,z) \in A_a} \bigvee_{(p,q) \in A_a} (F(p) \wedge G(q) \wedge H(z)) \vee \frac{1-k}{2} \\
&\geq (F(ax) \wedge G(axa) \wedge H(xa)) \vee \frac{1-k}{2}.
\end{aligned}$$

Since F is a $(\overline{\epsilon}, \overline{\epsilon} \vee \overline{q}_k)$ -fuzzy right ideal G , a $(\overline{\epsilon}, \overline{\epsilon} \vee \overline{q}_k)$ -fuzzy generalized ideal and H a $(\overline{\epsilon}, \overline{\epsilon} \vee \overline{q}_k)$ -fuzzy left ideal of S , we have $F(ax) \vee \frac{1-k}{2} \geq F(a) \vee \frac{1-k}{2}$, $G(axa) \vee \frac{1-k}{2} \geq G(a) \wedge G(a) \vee \frac{1-k}{2}$ and $H(xa) \vee \frac{1-k}{2} \geq H(a) \vee \frac{1-k}{2}$. Therefore,

$$\begin{aligned}
&[F(ax) \wedge G(axa) \wedge H(xa)] \vee \frac{1-k}{2} \\
&= \left[\left(F(ax) \vee \frac{1-k}{2} \right) \wedge \left(G(axa) \vee \frac{1-k}{2} \right) \wedge \left(H(xa) \vee \frac{1-k}{2} \right) \right] \\
&\geq \left[\left(F(a) \vee \frac{1-k}{2} \right) \wedge \left(G(a) \vee \frac{1-k}{2} \right) \wedge \left(H(a) \vee \frac{1-k}{2} \right) \right] \\
&= F^+(a) \wedge^k G^+(a) \wedge^k H^+(a).
\end{aligned}$$

Thus $(F \circ^k G \circ^k H)^+(a) \geq (F \wedge^k G \wedge^k H)^+(a)$.

Conversely, assume that $(\lambda \wedge^k \mu \wedge^k \rho)^+ \preceq (\lambda \circ^k \mu \circ^k \rho)^+$ for every $(\overline{\epsilon}, \overline{\epsilon} \vee \overline{q}_k)$ -fuzzy right ideal F , every $(\overline{\epsilon}, \overline{\epsilon} \vee \overline{q}_k)$ -fuzzy generalized ideal G and every $(\overline{\epsilon}, \overline{\epsilon} \vee \overline{q}_k)$ -fuzzy left ideal H of S . To prove that S is regular, by Proposition 4.16, it is enough to prove that $R \cap B \cap L \subseteq (RBL)$ for right ideal R , generalized bi-ideal B and left ideal L of S . Let $x \in R \cap B \cap L$. Then $x \in R$, $x \in B$ and $x \in L$. Since R is a right ideal, B a generalized bi-ideal and L a left ideal of S , by Lemma 4.4 and 4.5, χ_R^{+k} is an $(\overline{\epsilon}, \overline{\epsilon} \vee \overline{q}_k)$ -fuzzy right ideal, χ_B^{+k} an $(\overline{\epsilon}, \overline{\epsilon} \vee \overline{q}_k)$ -fuzzy generalized bi-ideal and χ_L^{+k} an $(\overline{\epsilon}, \overline{\epsilon} \vee \overline{q}_k)$ -fuzzy left ideal of S . By hypothesis, $(\chi_R \circ^k \chi_B \circ^k \chi_L)^+(x) \geq (\chi_R \wedge^k \chi_B \wedge^k \chi_L)^+(x) = (\chi_R \wedge \chi_B \wedge \chi_L)(x) \vee \frac{1-k}{2}$. Since $x \in R$, $x \in B$ and $x \in L$, we have $\chi_R^{+k} = 1$, $\chi_B^{+k} = 1$ and $\chi_L^{+k} = 1$. Thus $(\chi_R \wedge^k \chi_B \wedge^k \chi_L)^+(x) = (\chi_R \wedge \chi_B \wedge \chi_L)(x) \vee \frac{1-k}{2} = 1$. Follows that $(\chi_R \circ^k \chi_B \circ^k \chi_L)^+(x) = 1$. By using Lemma 4.4 (iii), we

have $(\chi_R \circ^k \chi_B \circ^k \chi_L)^+ = \chi_{(RBL)}^{+k}$. Therefore, $\chi_{(RBL)}^{+k}(x) = 1$ and so $x \in (RBL]$. Consequently, S is regular. $\square$

Proposition 4.18. ([22]) *Let $(S, \cdot, \leq)$ be an ordered semigroup, then the following conditions are equivalent:*

- (1) S is left weakly regular,
- (2) $I \cap L \subseteq (IL]$ for every ideal I and left ideal L of S ,
- (3) $I(a) \cap L(a) \subseteq (I(a)L(a)]$ for every $a \in S$.

Theorem 4.19. *An ordered semigroup $(S, \cdot, \leq)$ is left weakly regular if and only if for every $(\overline{\epsilon}, \overline{\epsilon} \vee \overline{q}_k)$ -fuzzy ideal F and every $(\overline{\epsilon}, \overline{\epsilon} \vee \overline{q}_k)$ -fuzzy left ideal G of S ,*

$$(F \wedge^k G)^+ \preceq (F \circ^k G)^+.$$

Proof. Suppose that F is a $(\overline{\epsilon}, \overline{\epsilon} \vee \overline{q}_k)$ -fuzzy ideal and G a $(\overline{\epsilon}, \overline{\epsilon} \vee \overline{q}_k)$ -fuzzy left ideal of a left weakly regular ordered semigroup S . Let $a \in S$. Since S is left weakly regular, there exists $x, y \in S$ such that $a \leq xaya = (xa)(ya)$. Then $(xa, ya) \in A_a$ and $A_a \neq \emptyset$. Thus,

$$\begin{aligned} (F \circ^k G)^+(a) &= (F \circ G)(a) \vee \frac{1-k}{2} \\ &= \left[\bigvee_{(y,z) \in A_a} (F(y) \wedge G(z)) \right] \vee \frac{1-k}{2} \\ &\geq (F(xa) \wedge G(ya)) \vee \frac{1-k}{2}. \end{aligned}$$

Since F is an $(\overline{\epsilon}, \overline{\epsilon} \vee \overline{q}_k)$ -fuzzy ideal and G an $(\overline{\epsilon}, \overline{\epsilon} \vee \overline{q}_k)$ -fuzzy left ideal of S , we have $F(xa) \vee \frac{1-k}{2} \geq F(a) \vee \frac{1-k}{2}$ and $G(ya) \vee \frac{1-k}{2} \geq G(a) \vee \frac{1-k}{2}$. Therefore,

$$\begin{aligned} (F(xa) \wedge G(ya)) \vee \frac{1-k}{2} &= \left[\left(F(xa) \vee \frac{1-k}{2} \right) \wedge \left(G(ya) \vee \frac{1-k}{2} \right) \right] \\ &\geq \left(F(a) \vee \frac{1-k}{2} \right) \wedge \left(G(a) \vee \frac{1-k}{2} \right) \\ &= F^+(a) \wedge^k G^+(a). \end{aligned}$$

Thus $(F \circ^k G)^+(a) \geq (F \wedge^k G)^+(a)$.

Conversely, assume that $(\lambda \wedge^k \mu)^- \preceq (\lambda \circ^k \mu)^-$ for every $(\overline{\epsilon}, \overline{\epsilon} \vee \overline{q}_k)$ -fuzzy ideal λ and every $(\overline{\epsilon}, \overline{\epsilon} \vee \overline{q}_k)$ -fuzzy left ideal μ of S . To prove that S is left weakly regular, by Proposition 4.18, it is enough to prove that

$$I \cap L \subseteq (IL] \text{ for ideal } I \text{ and left ideal } L \text{ of } S.$$

Let $x \in I \cap L$. Then $x \in I$ and $x \in L$. Since I is an ideal and L a left ideal of S , by Lemma 4.4 and 4.5, χ_I^{+k} is an $(\bar{\epsilon}, \bar{\epsilon} \vee \bar{q}_k)$ -fuzzy ideal and $\bar{\chi}_L^k$ an $(\bar{\epsilon}, \bar{\epsilon} \vee \bar{q}_k)$ -fuzzy left ideal of S . By hypothesis, $(\chi_I \circ^k \chi_L)^+(x) \geq (\chi_I \wedge^k \chi_L)^+(x) = (\chi_I^k \wedge \chi_L^k)(x) \vee \frac{1-k}{2}$. Since $x \in I$ and $x \in L$, we have $\chi_I^{+k} = 1$ and $\chi_L^{+k} = 1$. Thus $(\chi_I^{+k} \wedge \chi_L^{+k})(x) \vee \frac{1-k}{2} = 1$. It follows that $(\chi_I \circ^k \chi_L)^+(x) = 1$. By using Lemma 4.3 (iii), we have $(\chi_I \circ^k \chi_L)^+ = \chi_{(IL)}^{+k}(x)$. Therefore, $\chi_{(IL)}^{+k}(x) = 1$ and so $x \in (IL)$. Consequently, S is left weakly regular. $\square$

Proposition 4.20. ([22]) *Let S be an ordered semigroup, then the following conditions are equivalent:*

- (i) S is left weakly regular,
- (ii) $I \cap B \subseteq (IB)$ for every generalized bi-ideal B and every ideal I of S ,
- (iii) $I(a) \cap B(a) \subseteq (I(a)B(a))$ for every $a \in S$.

Theorem 4.21. *An ordered semigroup $(S, \cdot, \leq)$ is left weakly regular if and only if for every $(\bar{\epsilon}, \bar{\epsilon} \vee \bar{q}_k)$ -fuzzy ideal F and every $(\bar{\epsilon}, \bar{\epsilon} \vee \bar{q}_k)$ -fuzzy generalized bi-ideal G of S ,*

$$(F \wedge^k G)^+ \preceq (F \circ^k G)^+.$$

Proof. Suppose that F is a $(\bar{\epsilon}, \bar{\epsilon} \vee \bar{q}_k)$ -fuzzy ideal and G a $(\bar{\epsilon}, \bar{\epsilon} \vee \bar{q}_k)$ -fuzzy generalized bi-ideal of a left weakly regular ordered semigroup S . Let $a \in S$. Since S is left weakly regular, there exists $x, y \in S$ such that $a \leq xaya \leq x(xaya)ya = (x^2ay)(aya)$. Then $(x^2ay, aya) \in A_a$ and $A_a \neq \emptyset$. Thus,

$$\begin{aligned} (F \circ^k G)^+(a) &= (F \circ G)(a) \vee \frac{1-k}{2} \\ &= \left[\bigvee_{(y,z) \in A_a} (F(y) \wedge G(z)) \right] \vee \frac{1-k}{2} \\ &\geq (F(x^2ay) \wedge G(aya)) \vee \frac{1-k}{2}. \end{aligned}$$

Since F is a $(\bar{\epsilon}, \bar{\epsilon} \vee \bar{q}_k)$ -fuzzy ideal and G a $(\bar{\epsilon}, \bar{\epsilon} \vee \bar{q}_k)$ -fuzzy generalized bi-ideal of S , we have $F(x^2ay) \vee \frac{1-k}{2} \geq F(ay) \vee \frac{1-k}{2} \geq F(a) \vee \frac{1-k}{2}$ and $G(aya) \vee \frac{1-k}{2} \geq (G(a) \wedge G(a)) \vee \frac{1-k}{2} = G(a) \vee \frac{1-k}{2}$. Therefore

$$\begin{aligned} &F(x^2ay) \wedge G(aya) \vee \frac{1-k}{2} \\ &= \left[\left(F(x^2ay) \vee \frac{1-k}{2} \right) \wedge \left(G(aya) \vee \frac{1-k}{2} \right) \right] \end{aligned}$$

$$\begin{aligned}
&\geq \left(F(a) \wedge \frac{1-k}{2} \right) \wedge \left(G(a) \wedge \frac{1-k}{2} \right) \\
&= F^+(a) \wedge^k G^+(a)
\end{aligned}$$

Thus $(F \circ^k G)^+(a) \geq (F \wedge^k G)^+(a)$.

Conversely, assume that $(\lambda \wedge^k \mu)^+ \preceq (\lambda \circ^k \mu)^+$ for every $(\overline{\epsilon}, \overline{\epsilon} \vee \overline{q}_k)$ -fuzzy ideal F and every $(\overline{\epsilon}, \overline{\epsilon} \vee \overline{q}_k)$ -fuzzy generalized bi-ideal G of S . To prove that S is left weakly regular, by Proposition 4.20, it is enough to prove that

$$I \cap B \subseteq (IL] \text{ for ideal } I \text{ and generalized bi-ideal } B \text{ of } S.$$

Let $x \in I \cap B$. Then $x \in I$ and $x \in B$. Since I is an ideal and B a generalized bi-ideal of S , by Lemma 4.4 and 4.5, χ_I^{+k} is an $(\overline{\epsilon}, \overline{\epsilon} \vee \overline{q}_k)$ -fuzzy ideal and χ_B^{+k} an $(\overline{\epsilon}, \overline{\epsilon} \vee \overline{q}_k)$ -fuzzy generalized bi-ideal of S . By hypothesis, $(\chi_I \circ^k \chi_B)^+(x) \geq (\chi_I \wedge^k \chi_B)^+(x) = (\chi_I \wedge \chi_B)(x) \vee \frac{1-k}{2}$. Since $x \in I$ and $x \in B$, we have $\chi_I^{+k}(x) = 1$ and $\chi_B^{+k}(x) = 1$. Thus $(\chi_I \wedge \chi_B)(x) \vee \frac{1-k}{2} = 1$. It follows that $(\chi_I \circ^k \chi_B)^+(x) = 1$. By using Lemma 4.3 (iii), we have $(\chi_I \circ^k \chi_B)^+ = \chi_{[IB]}^{+k}$. Therefore, $\chi_{[IB]}^{+k}(x) = 1$ and so $x \in [IB]$. Consequently, S is left weakly regular. $\square$

References

- [1] L. A. Zadeh, *Fuzzy sets*, Information and control, **8** (1965), 338-3353.
- [2] A. Rosenfeld, *Fuzzy groups*, J. Math. Anal. Appl., **35** (1971), 512-517.
- [3] S. K. Bhakat and P. Das, $(\in, \in \vee q)$ -fuzzy subgroups, Fuzzy Sets and Systems **80** (1996), 359-368.
- [4] P. M. Pu and Y. M. Liu, *Fuzzy topology I, neighborhood structure of a fuzzy point and Moore-Smith convergence*, J. Math. Anal. Appl., **76** (1980), 571-599.
- [5] B. Davvaz and A. Khan, *Characterizations of regular ordered semigroups in terms of (α, β) -fuzzy generalized bi-ideals*, Inform. Sci., **181** (2011), 1759-1770.
- [6] Y. B. Jun and S. Z. Song, *Generalized fuzzy interior ideals in semigroups*, Inform. Sci. **176** (2006), 3079-3093.
- [7] O. Kazanci and S. Yamak, *Generalized fuzzy bi-ideals of semigroup*, Soft Computing, **12** (2008), 1119-1124.
- [8] M. Shabir, Y. B. Jun and Y. Nawaz, *Semigroups characterized by (α, β) -fuzzy ideals*, Computer and Mathematics with Applications **59** (2010), 161-175.
- [9] M. Shabir, Y. B. Jun and Y. Nawaz, *Characterization of regular semigroups by $(\in, \in \vee q_k)$ -fuzzy ideals*, Computer and Mathematics with Applications **60** (2010), 1473-1493.
- [10] M. Shabir, Y. Nawaz and M. Ali, *Characterizations of Semigroups by $(\overline{\epsilon}, \overline{\epsilon} \vee \overline{q}_k)$ -fuzzy Ideals*, World Applied Sciences, **14(12)** (2011), 1866-1878.

- [11] B. Davvaz, O. Kazanci and S. Yamak, *Generalized fuzzy n -ary subpolygroups endowed with interval valued membership functions*, Journal Intelligent and Fuzzy Systems, **20(4-5)** (2009), 159-168.
- [12] B. Davvaz and Z. Mozafar, *$(\in, \in \vee q)$ -fuzzy Lie subalgebra and ideals*, International Journal of Fuzzy Systems, **11(2)** (2009), 123-129.
- [13] B. Davvaz and P. Corsini, *On (α, β) -fuzzy H_v -ideals of H_v -rings*, Iranian Journal of Fuzzy System **5** (2008), 35-47.
- [14] B. Davvaz, J. Zhan, K.P. Shum, *Generalized fuzzy polygroups endowed with interval valued membership functions*, Journal of Intelligent and Fuzzy Systems, **19(3)** (2008), 181-188.
- [15] B. Davvaz, *Fuzzy R -subgroups with thresholds of near-rings and implication operators*, Soft Computing, **12** (2008), 875-879.
- [16] O. Kazanci and B. Davvaz, *Fuzzy n -ary polygroups related to fuzzy points*, Computers & Mathematics with Applications, **58(7)** (2009), 1466-1474.
- [17] X. Ma, J. Zhan and Y. B. Jun, *Interval valued $(\in, \in \vee q)$ -fuzzy ideals of pseudo-MV algebras*, International Journal of Fuzzy Systems, **10(2)** (2008), 84-91.
- [18] J. N. Mordeson, D. S. Malik and N. Kuroki, *Fuzzy Semigroups, Studies in Fuzziness and Soft Computing*, Vol. **131**, Springer-Verlag Berlin (2003).
- [19] Y. Yin and J. Zhan, *New types of fuzzy filters of BL-algebras*, Computer Mathematics with Applications, **60** (2010), 2115-2125.
- [20] X. Yuan, C. Zhang and Y. Ren, *Generalized fuzzy groups and many-valued implications*, Fuzzy Sets and Systems, **138** (2003), 205-211.
- [21] J. Zhan, B. Davvaz, K. P. Shum, *A new view of fuzzy hyperquasigroups*, Journal of Intelligent and Fuzzy Systems, **20** (2009), 147-157.
- [22] M. Shabir, A. Khan, *Characterizations of ordered semigroups by the properties of their fuzzy generalized bi-ideals*, New Mathematics and Natural Computation, **4(2)** (2008), 237-250.
- [23] N. Kehayopulu and M. Tsingelis, *Fuzzy bi-ideals in ordered semigroups*, Information Sciences, **171** (2005), 13-28.
- [24] N. Kuroki, *On fuzzy ideals and fuzzy bi-ideals in semigroups*, Fuzzy Sets and Systems, **5** (1981), 203-215.
- [25] N. Kuroki, *Fuzzy congruences on T^* -pure semigroups*, Information Sciences, **84** (1995), 239-246.
- [26] N. Kuroki, *Fuzzy semiprime quasi-ideals in semigroups*, Information Sciences, **75** (1993), 201-211.
- [27] N. Kuroki, *On fuzzy semigroups*, Information Sciences, **53** (1991), 203-236.
- [28] A. Khan, Y. B. Jun, Z. Abbasi, *Characterizations of ordered semigroups in terms of $(\in, \in \vee q)$ -fuzzy interior ideals*, Neural Comp., Appli., **21(3)** (2012), 433-440.

Hidayat Ullah Khan

Department of Mathematical Sciences, Faculty of Science, Universiti Teknologi Malaysia,
81310 UTM Johor Bahru, Johor, Malaysia.

E-mail: hidayatullak@yahoo.com

Nor Haniza Sarmin

Department of Mathematical Sciences, Faculty of Science, Universiti
Teknologi Malaysia,

81310 UTM Johor Bahru, Johor, Malaysia.

E-mail: nhs@utm.my

Asghar Khan

Department of Mathematics, Abdul Wali Khan University Mardan,
Mardan, Khyber Pakhtunkhwa, Pakistan.

E-mail: azhar4set@yahoo.com