COMBINATORIC CONVOLUTION SUMS CONTAINING σ AND $\tilde{\sigma}$ OF THE FORM $2^m p$

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Abstract. In this paper, we study combinatoric convolution sums of divisor functions and get values of this sum when $n = 2^m p$. We find that the value of this convolution sum is represented by a sum of powers of 2 and Bernoulli or Euler number.

1. Definitions and Facts

Bernoulli polynomials $B_n(x)$ and Bernoulli numbers B_n are usually defined by the following generating functions :

$$(1.1) \quad \frac{ue^{ux}}{e^u - 1} = \sum_{k=0}^{\infty} B_k(x) \frac{u^k}{k!},$$

$$(1.2) \quad B_k = B_k(0).$$

Also, Euler polynomials $E_n(x)$ and Euler numbers E_n are defined by followings :

$$(1.3) \quad \frac{2ue^{ux}}{e^u + 1} = \sum_{k=0}^{\infty} E_k(x) \frac{u^k}{k!},$$

$$(1.4) \quad E_k = 2^k E_k\left(\frac{1}{2}\right).$$

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For $k, m, n \in \mathbb{N}$ and $l \in \mathbb{Z}$, we define divisor sigma functions

$$\begin{aligned}\sigma_k(n) &= \sum_{d|n} d^k, \\ \sigma_{k,l}(n; m) &= \sum_{d|n, d \equiv l \pmod{m}} d^k, \\ \tilde{\sigma}_k(n) &= \sum_{d|n} (-1)^{d-1} d^k.\end{aligned}$$

These functions appear in the Besge [2] and Ramanujan [8] by the identity

$$(1.5) \quad \sum_{k=1}^{n-1} \sigma_1(k) \sigma_1(n-k) = \frac{5}{12} \sigma_3(n) + \left(\frac{1}{12} - \frac{1}{2} n \right) \sigma_1(n).$$

We are motivated by this function and also by the recent development on the convolution sums [3], [4], [5], [6], [7], [9]. In this paper, we compute the values of four combinatoric convolution sums containing divisor functions σ and $\tilde{\sigma}$. Here $\binom{k}{r} = \frac{k!}{r!(k-r)!}$. Now, we state the main results.

Theorem 1.1. *When p is an odd prime and $k \geq 1$, the following equality holds for all positive integer $n \geq 2$:*

$$\begin{aligned}& \sum_{s=0}^{k-1} \binom{2k}{2s+1} \sum_{m=1, m \text{ even}}^{2^np-1} \sigma_{2k-2s-1,1}(m; 2) \sigma_{2s+1,1}(2^np - m; 2) \\ &= \frac{1}{4} \times \left\{ 2^{2k+1} + (2p)^{2k+1} \right\} \times \left\{ 1 + 2^{2k+1} + (2^{2k+1})^2 + \cdots + (2^{2k+1})^{n-2} \right\} \\ &+ \frac{2^{2k}}{2k+1} B_{2k+1} \left(\frac{p+1}{2} \right).\end{aligned}$$

In particular when $n = 1$,

$$\begin{aligned}& \sum_{s=0}^{k-1} \binom{2k}{2s+1} \sum_{m=1, m \text{ even}}^{2p-1} \sigma_{2k-2s-1,1}(m; 2) \sigma_{2s+1,1}(2p - m; 2) \\ &= \frac{2^{2k}}{2k+1} B_{2k+1} \left(\frac{p+1}{2} \right).\end{aligned}$$

Corollary 1.2. *When p is an odd prime, the following equality holds for all positive integer n and given $k \geq 1$:*

$$\begin{aligned} & \sum_{s=0}^{k-1} \binom{2k}{2s+1} \sum_{m=1, m \text{ odd}}^{2^n p-1} \sigma_{2k-2s-1,1}(m; 2) \sigma_{2s+1,1}(2^n p - m; 2) \\ &= \sum_{s=0}^{k-1} \binom{2k}{2s+1} \sum_{m=1, m \text{ odd}}^{2^n p-1} \tilde{\sigma}_{2k-2s-1}(m) \tilde{\sigma}_{2s+1}(2^n p - m) \\ &= \frac{1}{4} \times (2^{2k+1})^{n-1} \times \left\{ 2^{2k+1} + (2p)^{2k+1} \right\}. \end{aligned}$$

Corollary 1.3. *When p is an odd prime, the following equality holds for all positive integer n and given $k \geq 1$:*

$$\begin{aligned} & \sum_{s=0}^{k-1} \binom{2k}{2s+1} \sum_{m=1, m \text{ even}}^{2^n p-1} \tilde{\sigma}_{2k-2s-1}(m) \tilde{\sigma}_{2s+1}(2^n p - m) \\ &= -\frac{1}{2} \left\{ 1 + p^{2k+1} - \sum_{u=1}^n (2^u)^{2k+1} - \sum_{v=1}^n (2^v p)^{2k+1} \right\} \\ &+ \frac{1}{2} \left\{ 1 + p^{2k} - \sum_{u=1}^n (2^u)^{2k} - \sum_{v=1}^n (2^v p)^{2k} \right\} \\ &+ (2^n p) \left\{ 1 + p^{2k-1} - \sum_{u=1}^n (2^u)^{2k-1} - \sum_{v=1}^n (2^v p)^{2k-1} \right\} \\ &- \frac{1}{2} \left\{ E_{2k}(2) + E_{2k}(p+1) - \sum_{u=1}^n E_{2k}(2^u + 1) - \sum_{v=1}^n E_{2k}(2^v p + 1) \right\} \\ &- \frac{1}{4} \times (2^{2k+1})^{n-1} \times \left\{ 2^{2k+1} + (2p)^{2k+1} \right\}. \end{aligned}$$

2. Proofs of Main Results

To get the results, we use next two propositions [1] inductively and recursively. We cite here without proof.

Proposition 2.1. *Let $k \geq 1$ and $n \geq 2$. Then*

$$\begin{aligned} & \sum_{s=0}^{k-1} \binom{2k}{2s+1} \sum_{m=1}^{n-1} \sigma_{2k-2s-1,1}(m; 2) \sigma_{2s+1,1}(n-m; 2) \\ &= \frac{1}{4} \sigma_{2k+1,0}(n; 2) + \frac{2^{2k}}{2k+1} \sum_{d|n, d \text{ odd}} B_{2k+1} \left(\frac{d+1}{2} \right). \end{aligned}$$

Proposition 2.2. *Let $k \geq 1$ and $n \geq 2$. Then*

$$\begin{aligned} & \sum_{s=0}^{k-1} \binom{2k}{2s+1} \sum_{m=1}^{n-1} \tilde{\sigma}_{2k-2s-1}(m) \tilde{\sigma}_{2s+1}(n-m) \\ &= -\frac{1}{2} \tilde{\sigma}_{2k+1}(n) + \frac{1}{2} \tilde{\sigma}_{2k}(n) + n \tilde{\sigma}_{2k-1}(n) - \frac{1}{2} \sum_{d|n} (-1)^{d-1} E_{2k}(d+1). \end{aligned}$$

We state Lemmas with or without proof.

Lemma 2.3. *For all $k \geq 1$*

$$B_{2k+1}(1) = 0.$$

Lemma 2.4. *When p is an odd prime and $k \geq 1$,*

$$\begin{aligned} & \sum_{s=0}^{k-1} \binom{2k}{2s+1} \sum_{m=1}^{p-1} \sigma_{2k-2s-1,1}(m; 2) \sigma_{2s+1,1}(p-m; 2) \\ &= \frac{1}{4} \sigma_{2k+1,0}(p; 2) + \frac{2^{2k}}{2k+1} \sum_{d|p, d \text{ odd}} B_{2k+1} \left(\frac{d+1}{2} \right) \\ &= \frac{2^{2k}}{2k+1} B_{2k+1} \left(\frac{p+1}{2} \right). \end{aligned}$$

Lemma 2.5. *If m is odd, then*

$$\begin{aligned} \sigma_{2k-2s-1,1}(2m; 2) &= \sum_{d|2m, d \text{ odd}} d^{2k-2s-1} = \sum_{d'|m, d \text{ odd}} d'^{2k-2s-1} \\ &= \sigma_{2k-2s-1,1}(m; 2). \end{aligned}$$

Note that when p is an odd prime and $k \geq 1$, we have

$$\begin{aligned} (2.1) \quad & \sum_{s=0}^{k-1} \binom{2k}{2s+1} \sum_{m=1}^{2^n p-1} \sigma_{2k-2s-1,1}(m; 2) \sigma_{2s+1,1}(2^n p-m; 2) \\ &= \frac{1}{4} \sigma_{2k+1,0}(2^n p; 2) + \frac{2^{2k}}{2k+1} \sum_{d|2^n p, d \text{ odd}} B_{2k+1} \left(\frac{d+1}{2} \right). \end{aligned}$$

But since $d \mid 2^n p$ and d is odd, then $d = 1$ or p , the second term of the right-hand side of (2.1) is always computed as

$$\begin{aligned} & \frac{2^{2k}}{2k+1} \sum_{d \mid 2^n p, d \text{ odd}} B_{2k+1} \left(\frac{d+1}{2} \right) \\ &= \frac{2^{2k}}{2k+1} \left\{ B_{2k+1} \left(\frac{1+1}{2} \right) + B_{2k+1} \left(\frac{p+1}{2} \right) \right\} = \frac{2^{2k}}{2k+1} B_{2k+1} \left(\frac{p+1}{2} \right). \end{aligned}$$

Let us compute the first term $\frac{1}{4}\sigma_{2k+1,0}(2^n p; 2)$ of right-hand side of (2.1) for all $n \geq 1$. But this term can be computed inductively and recursively using Proposition [1]. When $n = 1$, then the even divisors d of $2p$ are $d = 2$ or $2p$. Thus

$$(2.2) \quad \frac{1}{4}\sigma_{2k+1,0}(2p; 2) = \frac{1}{4} \left\{ 2^{2k+1} + (2p)^{2k+1} \right\}.$$

Also if $d \mid 2^n p$ and d is even, then $d = 2^e$ or $2^e p$ where $e = 1, 2, \dots, n$. Thus

$$\begin{aligned} \frac{1}{4}\sigma_{2k+1,0}(n; 2) &= \frac{1}{4} \left\{ 2^{2k+1} + (2p)^{2k+1} \right\} + \frac{1}{4} \times 2^{2k+1} \times \left\{ 2^{2k+1} + (2p)^{2k+1} \right\} \\ &\quad + \frac{1}{4} \times 2^{2k+1} \times 2^{2k+1} \left\{ 2^{2k+1} + (2p)^{2k+1} \right\} + \dots \\ &\quad + \frac{1}{4} \times (2^{2k+1})^{n-1} \times \left\{ 2^{2k+1} + (2p)^{2k+1} \right\}. \end{aligned}$$

All these together, we get the value of the following convolution sum :

$$\begin{aligned} & \sum_{s=0}^{k-1} \binom{2k}{2s+1} \sum_{m=1}^{2^n p-1} \sigma_{2k-2s-1,1}(m; 2) \sigma_{2s+1,1}(2^n p - m; 2) \\ (2.3) \quad &= \frac{1}{4} \left\{ 2^{2k+1} + (2p)^{2k+1} \right\} + \frac{1}{4} \times 2^{2k+1} \times \left\{ 2^{2k+1} + (2p)^{2k+1} \right\} \\ &\quad + \frac{1}{4} \times 2^{2k+1} \times 2^{2k+1} \times \left\{ 2^{2k+1} + (2p)^{2k+1} \right\} + \dots \\ &\quad + \frac{1}{4} \times (2^{2k+1})^{n-1} \times \left\{ 2^{2k+1} + (2p)^{2k+1} \right\} + \frac{2^{2k}}{2k+1} B_{2k+1} \left(\frac{p+1}{2} \right). \end{aligned}$$

Now we are going to divide into two cases, one is m is even and the other is m is odd. Then prove Theorem 1.1.

Proof of Theorem 1.1

Proof. Let us check when $n = 1$. By Lemma 2.4 and Lemma 2.5

$$\begin{aligned}
& \sum_{s=0}^{k-1} \binom{2k}{2s+1} \sum_{m=1, m \text{ even}}^{2p-1} \sigma_{2k-2s-1,1}(m; 2) \sigma_{2s+1,1}(2p-m; 2) \\
&= \sum_{s=0}^{k-1} \binom{2k}{2s+1} \{ \sigma_{2k-2s-1,1}(2; 2) \sigma_{2s+1,1}(2p-2; 2) \\
&\quad + \sigma_{2k-2s-1,1}(4; 2) \sigma_{2s+1,1}(2p-4; 2) + \cdots \\
&\quad + \sigma_{2k-2s-1,1}(2p-2; 2) \sigma_{2s+1,1}(2; 2) \} \\
&= \sum_{s=0}^{k-1} \binom{2k}{2s+1} \{ \sigma_{2k-2s-1,1}(1; 2) \sigma_{2s+1,1}(p-1; 2) \\
&\quad + \sigma_{2k-2s-1,1}(2; 2) \sigma_{2s+1,1}(p-2; 2) + \cdots \\
&\quad + \sigma_{2k-2s-1,1}(p-1; 2) \sigma_{2s+1,1}(1; 2) \} \\
&= \sum_{s=0}^{k-1} \binom{2k}{2s+1} \sum_{m=1}^{p-1} \sigma_{2k-2s-1,1}(p; 2) \sigma_{2s+1,1}(p-m; 2) \\
&= \frac{2^{2k}}{2k+1} B_{2k+1} \left(\frac{p+1}{2} \right).
\end{aligned}$$

When $n = 2$. We compute the equality by the same manner :

$$\begin{aligned}
& \sum_{s=0}^{k-1} \binom{2k}{2s+1} \sum_{m=1, m \text{ even}}^{4p-1} \sigma_{2k-2s-1,1}(m; 2) \sigma_{2s+1,1}(4p-m; 2) \\
&= \sum_{s=0}^{k-1} \binom{2k}{2s+1} \{ \sigma_{2k-2s-1,1}(2; 2) \sigma_{2s+1,1}(4p-2; 2) \\
&\quad + \sigma_{2k-2s-1,1}(6; 2) \sigma_{2s+1,1}(4p-6; 2) + \cdots \\
&\quad + \sigma_{2k-2s-1,1}(4p-2; 2) \sigma_{2s+1,1}(2; 2) \} \\
&\quad + \sum_{s=0}^{k-1} \binom{2k}{2s+1} \{ \sigma_{2k-2s-1,1}(4; 2) \sigma_{2s+1,1}(4p-4; 2) \\
&\quad + \sigma_{2k-2s-1,1}(8; 2) \sigma_{2s+1,1}(4p-8; 2) + \cdots \\
&\quad + \sigma_{2k-2s-1,1}(4p-4; 2) \sigma_{2s+1,1}(4; 2) \} \\
&= \sum_{s=0}^{k-1} \binom{2k}{2s+1} \left\{ \sum_{m=1, m \text{ odd}}^{2p-1} \sigma_{2k-2s-1,1}(m; 2) \sigma_{2s+1,1}(2p-m; 2) \right\}
\end{aligned}$$

$$\begin{aligned}
& + \sum_{s=0}^{k-1} \binom{2k}{2s+1} \left\{ \sum_{m=1, m \text{ odd}}^{p-1} \sigma_{2k-2s-1,1}(m;2) \sigma_{2s+1,1}(p-m;2) \right\} \\
& = \frac{1}{4} \left\{ 2^{2k+1} + (2p)^{2k+1} \right\} + \frac{2^{2k}}{2k+1} B_{2k+1} \left(\frac{p+1}{2} \right).
\end{aligned}$$

Let us proof Theorem by mathematical induction on n . Note that $m = 1, 2, \dots, 2^n p - 1$ and m is even, so $m = 2, 4, 6, 8, \dots, 2^n p, 2^n p + 2, 2^n p + 4, 2^n p + 6, 2^n p + 8, \dots, 2^n p - 2$. We divide into cases and then apply induction hypothesis on each case :

Case. When $m = 2^t, t = 1, 2, \dots, p-1$. Then by Lemma 2.5, we get

$$\begin{aligned}
& \sum_{s=0}^{k-1} \binom{2k}{2s+1} \sum_{m=2^t, 1 \leq t \leq p-1} \sigma_{2k-2s-1,1}(m;2) \sigma_{2s+1,1}(2^n p - m;2) \\
& = \sum_{s=0}^{k-1} \binom{2k}{2s+1} \left\{ \sigma_{2k-2s-1,1}(2^n;2) \sigma_{2s+1,1}(2^n p - 2^n;2) \right. \\
& \quad + \sigma_{2k-2s-1,1}(2^n \times 2;2) \sigma_{2s+1,1}(2^n p - 2^n \times 2;2) + \dots \\
& \quad \left. + \sigma_{2k-2s-1,1}(2^n p - 2^n;2) \sigma_{2s+1,1}(2^n;2) \right\} \\
& = \sum_{s=0}^{k-1} \binom{2k}{2s+1} \left\{ \sum_{m=1}^{p-1} \sigma_{2k-2s-1,n}(m;2) \sigma_{2s+1,1}(p-m;2) \right\} \\
& = \frac{2^{2k}}{2k+1} B_{2k+1} \left(\frac{p+1}{2} \right).
\end{aligned}$$

Case. When $m = 2^t + 2^{n-1}, t = 1, 2, \dots, p-1$. Then by Lemma 2.5, we get

$$\begin{aligned}
& \sum_{s=0}^{k-1} \binom{2k}{2s+1} \sum_{m=2^t+2^{n-1}, 1 \leq t \leq p-1} \sigma_{2k-2s-1,1}(m;2) \sigma_{2s+1,1}(2^n p - m;2) \\
& = \sum_{s=0}^{k-1} \binom{2k}{2s+1} \sum_{m=1, m \text{ odd}}^{2p-1} \sigma_{2k-2s-1,1}(m;2) \sigma_{2s+1,1}(2p - m;2) \\
& = \frac{1}{4} \times \left\{ 2^{2k+1} + (2p)^{2k+1} \right\}.
\end{aligned}$$

Case. When $m = 2^t \pm 2^{n-2}, t = 1, 2, \dots, p-1$. Then by Lemma 2.5, we get

$$\begin{aligned}
& \sum_{s=0}^{k-1} \binom{2k}{2s+1} \sum_{m=2^t, 1 \leq t \leq p-1} \sigma_{2k-2s-1,1}(m; 2) \sigma_{2s+1,1}(2^np - m; 2) \\
&= \sum_{s=0}^{k-1} \binom{2k}{2s+1} \sum_{m=1, m \text{ odd}}^{4p-1} \sigma_{2k-2s-1,1}(m; 2) \sigma_{2s+1,1}(4p - m; 2) \\
&= \frac{1}{4} \times 2^{2k+1} \times \left\{ 2^{2k+1} + (2p)^{2k+1} \right\}.
\end{aligned}$$

Case. When $m = 2^nt \pm 2^{n-3}$, $\pm(2^{n-3} + 2^{n-2})$, $t = 1, 2, \dots, p-1$. Then by Lemma 2.5, we get

$$\begin{aligned}
& \sum_{s=0}^{k-1} \binom{2k}{2s+1} \sum_{m=2^t, 1 \leq t \leq p-1} \sigma_{2k-2s-1,1}(m; 2) \sigma_{2s+1,1}(2^np - m; 2) \\
&= \sum_{s=0}^{k-1} \binom{2k}{2s+1} \sum_{m=1, m \text{ odd}}^{8p-1} \sigma_{2k-2s-1,1}(m; 2) \sigma_{2s+1,1}(8p - m; 2) \\
&= \frac{1}{4} \times 2^{2k+1} \times 2^{2k+1} \times \left\{ 2^{2k+1} + (2p)^{2k+1} \right\}.
\end{aligned}$$

Case. When $m = 2^nt \pm 2^{n-4}$, $\pm(2^{n-4} + 2^{n-3})$, $\pm(2^{n-4} + 2^{n-3} + 2^{n-2})$, $\pm(2^{n-4} + 2^{n-3} + 2^{n-2} + 2^{n-1})$, $t = 1, 2, \dots, p-1$. Then by Lemma 2.5, we get

$$\begin{aligned}
& \sum_{s=0}^{k-1} \binom{2k}{2s+1} \sum_{m=2^t, 1 \leq t \leq p-1} \sigma_{2k-2s-1,1}(m; 2) \sigma_{2s+1,1}(2^np - m; 2) \\
&= \sum_{s=0}^{k-1} \binom{2k}{2s+1} \sum_{m=1, m \text{ odd}}^{16p-1} \sigma_{2k-2s-1,1}(m; 2) \sigma_{2s+1,1}(16p - m; 2) \\
&= \frac{1}{4} \times 2^{2k+1} \times 2^{2k+1} \times 2^{2k+1} \times \left\{ 2^{2k+1} + (2p)^{2k+1} \right\}.
\end{aligned}$$

$\vdots$

Case. When $m = 2^nt \pm 2, \pm 6, \pm 10, \pm 14, \pm 18, \dots, t = 1, 2, \dots, p-1$. Then by induction hypothesis, we get

$$\begin{aligned} & \sum_{s=0}^{k-1} \binom{2k}{2s+1} \sum_{m=2^t, 1 \leq t \leq p-1} \sigma_{2k-2s-1,1}(m; 2) \sigma_{2s+1,1}(2^np - m; 2) \\ &= \sum_{s=0}^{k-1} \binom{2k}{2s+1} \sum_{m=1, m \text{ odd}} \sigma_{2k-2s-1,1}(m; 2) \sigma_{2s+1,1}(2^{n-1}p - m; 2) \\ &= \frac{1}{4} \times (2^{2k+1})^{n-2} \times \left\{ 2^{2k+1} + (2p)^{2k+1} \right\}. \end{aligned}$$

All theses cases together, we get the theorem :

$$\begin{aligned} & \sum_{s=0}^{k-1} \binom{2k}{2s+1} \sum_{m=1, m \text{ even}}^{2^np-1} \sigma_{2k-2s-1,1}(m; 2) \sigma_{2s+1,1}(2^np - m; 2) \\ &= \frac{2^{2k}}{2k+1} B_{2k+1} \left(\frac{p+1}{2} \right) + \frac{1}{4} \{ 2^{2k+1} + (2p)^{2k+1} \} \\ & \quad + \frac{1}{4} \times 2^{2k+1} \times \left\{ 2^{2k+1} + (2p)^{2k+1} \right\} + \dots \\ & \quad + \frac{1}{4} \times (2^{2k+1})^{n-2} \times \left\{ 2^{2k+1} + (2p)^{2k+1} \right\}. \quad \square \end{aligned}$$

Lemma 2.6. When $m \geq 1$ and m is odd, then

$$\tilde{\sigma}_{2k-2s-1}(m) = \sigma_{2k-2s-1,1}(m; 2).$$

Proof of Corollary 1.2

Proof. By (2.3) and Theorem 1.1, we get the first equality. By Lemma 2.6 and Corollary 1.2,

$$\begin{aligned} & \sum_{s=0}^{k-1} \binom{2k}{2s+1} \sum_{m=1, m \text{ odd}}^{2^np-1} \tilde{\sigma}_{2k-2s-1}(m) \tilde{\sigma}_{2s+1}(2^np - m) \\ &= \sum_{s=0}^{k-1} \binom{2k}{2s+1} \sum_{m=1, m \text{ odd}}^{2^np-1} \sigma_{2k-2s-1,1}(m; 2) \sigma_{2s+1}(2^np - m; 2) \\ &= \frac{1}{4} \times (2^{2k+1})^{n-1} \times \left\{ 2^{2k} + (2p)^{2k+1} \right\}. \quad \square \end{aligned}$$

Proof of Corollary 1.3

Proof. By Proposition 2.2,

$$\begin{aligned}
& \sum_{s=0}^{k-1} \binom{2k}{2s+1} \sum_{m=1, m \text{ even}}^{2^n p-1} \tilde{\sigma}_{2k-2s-1}(m) \tilde{\sigma}_{2s+1}(2^n p - m) \\
&= \sum_{s=0}^{k-1} \binom{2k}{2s+1} \sum_{m=1}^{2^n p-1} \tilde{\sigma}_{2k-2s-1}(m) \tilde{\sigma}_{2s+1}(2^n p - m) \\
&\quad - \frac{1}{4} \times (2^{2k+1})^{n-1} \times \left\{ 2^{2k+1} + (2p)^{2k+1} \right\} \\
&= -\frac{1}{2} \tilde{\sigma}_{2k+1}(2^n p) + \frac{1}{2} \tilde{\sigma}_{2k}(2^n p) + 2^n p \tilde{\sigma}_{2k}(2^n p) \\
&\quad - \frac{1}{4} \times (2^{2k+1})^{n-1} \times \left\{ 2^{2k+1} + (2p)^{2k+1} \right\} \\
&= -\frac{1}{2} \left\{ 1 + p^{2k+1} - \sum_{u=1}^n (2^u)^{2k+1} - \sum_{v=1}^n (2^v p)^{2k+1} \right\} \\
&\quad + \frac{1}{2} \left\{ 1 + p^{2k} - \sum_{u=1}^n (2^u)^{2k} - \sum_{v=1}^n (2^v p)^{2k} \right\} \\
&\quad + (2^n p) \left\{ 1 + p^{2k-1} - \sum_{u=1}^n (2^u)^{2k-1} - \sum_{v=1}^n (2^v p)^{2k-1} \right\} \\
&\quad - \frac{1}{2} \left\{ E_{2k}(2) + E_{2k}(p+1) - \sum_{u=1}^n E_{2k}(2^u + 1) - \sum_{v=1}^n E_{2k}(2^v p + 1) \right\} \\
&\quad - \frac{1}{4} \times (2^{2k+1})^{n-1} \times \left\{ 2^{2k+1} + (2p)^{2k+1} \right\}.
\end{aligned}$$

We are done. $\square$

3. Computations

We use Mathematica 8.0 and get tables for various k , p , and n .

k	p	n	left hand side	right hand side
2	3	2	1,968	same with the left hand side
2	3	3	64,432	same with the left hand side
3	7	2	26,404,224	same with the left hand side
3	7	3	3,399,640,448	same with the left hand side
5	11	2	146,090,710,610,944	same with the left hand side
5	11	3	299,317,061,034,259,456	same with the left hand side
5	11	4	613,001,464,283,866,412,032	same with the left hand side

TABLE 1. Examples of Theorem 1.1

k	p	n	left hand side	right hand side
2	3	2	44, 368	same with the left hand side
2	3	3	1, 752, 080	same with the left hand side
3	7	2	2, 955, 560, 608	same with the left hand side
3	7	3	407, 589, 417, 312	same with the left hand side
5	11	2	273, 756, 111, 052, 080, 624	same with the left hand side
5	11	3	586, 188, 309, 588, 278, 573, 712	same with the left hand side
5	11	4	1, 227, 871, 497, 289, 179, 774, 458, 832	same with the left hand side

TABLE 2. Examples of Corollary 1.3

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