THREE CONVEX HULL THEOREMS ON TRIANGLES AND CIRCLES

BAHMAN KALANTARI AND JONG YOULL PARK*

Abstract. We prove three convex hull theorems on triangles and circles. Given a triangle $\triangle$ and a point p , let $\triangle'$ be the triangle each of whose vertices is the intersection of the orthogonal line from p to an extended edge of $\triangle$. Let $\triangle''$ be the triangle whose vertices are the centers of three circles, each passing through p and two other vertices of $\triangle$. The first theorem characterizes when $p \in \triangle$ via a *distance duality*. The *triangle algorithm* in [1] utilizes a general version of this theorem to solve the convex hull membership problem in any dimension. The second theorem proves $p \in \triangle$ if and only if $p \in \triangle'$. These are used to prove the third: Suppose p be does not lie on any extended edge of $\triangle$. Then $p \in \triangle$ if and only if $p \in \triangle''$.

1. Introduction

We begin with three figures in order to describe three convex hull properties of a general triangle to be proved here.

In Figure 1 we have a point p inside a triangle $\triangle ABC$. We have drawn three circles each centered at a vertex passing through p . Note that the three corresponding discs intersect only at p . Will this property always be valid? What happens if p is outside of $\triangle ABC$?

We mention that a well-known theorem, “Johnson’s three circle theorem” is a result on three circles of equal radius r passing through a common point: The circle that passes through the three centers also has radius r . Johnson’s Theorem is visually evident. However, a natural question is to characterize the cases when the circles do pass through a common point. This characterization will follow here as a special case.

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*Corresponding author

FIGURE 1. Three circles centered at the vertices of $\triangle ABC$ passing through p .

In Figure 2 we have a point p inside a triangle $\triangle ABC$. We have drawn three circles, each passing through p and two other vertices of $\triangle ABC$. As in the previous case the three discs intersect only at p . Note that p lies in the triangle whose vertices are the centers of these circles. Will this property always be valid? What happens if p is outside of $\triangle ABC$?

FIGURE 2. Three circles, each passing through p and two other vertices of $\triangle ABC$.

In Figure 3 also we have a point p inside a triangle $\triangle ABC$. We have drawn three perpendicular lines from p to each of the edges. Denoting the intersection points by b_1, b_2, b_3 , note that p is inside $\triangle b_1b_2b_3$. Will this property always be valid? What happens if p is outside of $\triangle ABC$?

FIGURE 3. The triangle of the three intersection of orthogonal lines from p to the edges of $\triangle ABC$.

These three figures should make it clear what we intend to prove. In the next section we will state precise statements and prove three convex hull theorems.

2. Three Convex Hull Theorems on a Triangle

The first theorem to be proved, Theorem 1, is proved in much more generality in [1] and is explored in an algorithm called *triangle algorithm*, solving a range of problems, including the convex hull membership problem in arbitrary dimensions, linear programming, and a linear system of equations. For some related articles that also make comparisons with some algorithms for solving the convex hull problem, linear programming and iterative methods for solving a linear system, see [1]-[6]. For the sake of completeness here we prove the theorem for the special case of a triangle considered in this article.

Theorem 1. (distance duality) *Given a triangle $\triangle ABC$, a point p in the plane of the triangle, draw three circles centered at the vertices, passing through p . Let D_1, D_2, D_3 be the corresponding discs. Then $p \in \triangle ABC$ if and only if $D_1 \cap D_2 \cap D_3 = \{p\}$, i.e. the three discs intersect only at p .*

Proof. Suppose $p \in \triangle ABC$ but $D_1 \cap D_2 \cap D_3$ contains a point $p' \neq p$. Consider the orthogonal bisecting line of the line segment pp' . Since both p and p' lie inside the triangle this line when extended must intersect the edges of the triangle, see left figure in Figure 4. This implies there is a vertex that is closer to p than to p' (B or C in the figure). But this implies that the disc centered at this vertex will exclude p' , a contradiction.

FIGURE 4. The orthogonal bisecting line of pp' does not separate p from $\triangle ABC$ (left). The orthogonal bisecting line of pp' (not drawn) separates p from $\triangle ABC$ (right).

Conversely, suppose $p \notin \triangle ABC$. Let $p' \in \triangle ABC$ be the closest point of the triangle to p . Such a point must exist. The orthogonal bisector of pp' defines a half-space that contains p' , and $\triangle ABC$. Thus each vertex is closer to p' than to p (see right figure in Figure 4). Hence $p' \in D_1 \cap D_2 \cap D_3$. $\square$

Theorem 2. Given a triangle $\triangle ABC$, a point p in the plane of the triangle, draw a perpendicular line from p to each of the extended edges, resulting in three intersection points b_1, b_2, b_3 . Then $p \in \triangle ABC$ if and only if $p \in \triangle b_1b_2b_3$ (see Figure 3 for an example).

Proof. Assume $p \in \triangle ABC$ but not a boundary point since otherwise the theorem is trivially true. Suppose each b_i lies in $\triangle ABC$, see Figure 3. Without loss of generality we assume $A = (0, 0)$ in the Euclidean plane, $B = (1, 0)$ and C a point with positive y -coordinate. Let $p = (p_1, p_2)$. Then $0 < p_1 < 1$. Note that $b_1 = (p_1, 0)$ is one of the intersection points. Since p is inside $\triangle ABC$, it is easy to argue the x -coordinate of b_2 and b_3 must lie to the left and right of p_1 . Since b_2, b_3 are in $\triangle ABC$, a convex set, the line segment b_2b_3 must lie in $\triangle ABC$. Hence there is

point of the form $(p_1, y) \in \triangle ABC$ where $y > p_1$. But this point can be written as a convex combination of b_2 and b_3 . We can write p as a convex combination of b_1 and (p_1, y) . Hence p can be written as a convex combination of b_1, b_2, b_3 . But a point lies in a triangle if and only if it can be written as a convex combination of its vertices. Hence $p \in \triangle b_1 b_2 b_3$.

Next suppose that $p \in \triangle ABC$ but some b_i is not in $\triangle ABC$. Without loss of generality we may assume the edge AB is the line segment $[0, 1]$ and $p = (p_1, p_2)$ with $p_1 < 0$, see Figure 5. One of the intersections, say b_1 , is $(p_1, 0)$ and it is not in $\triangle ABC$. Since $p \in \triangle ABC$ we can easily argue that both b_2, b_3 must lie in $\triangle ABC$. Thus the line segment $b_2 b_3$ lies in $\triangle ABC$ and contains a point of the form (p_1, y) with $y_1 > p_1$. Then p can be written as a convex combination of b_1 and (p_1, y) , hence a convex combination of b'_i s. Thus $p \in \triangle b_1 b_2 b_3$.

FIGURE 5. A case where one b_i is not in $\triangle ABC$, however $p \in \triangle b_1 b_2 b_3$.

Conversely, suppose $p \in \triangle b_1 b_2 b_3$. We claim $p \in \triangle ABC$. Since p is contained in $\triangle b_1 b_2 b_3$, it is also contained in the three inequalities (half-spaces) defined by the lines orthogonal to pb_i at b_i , $i = 1, 2, 3$. Let P be the intersection of the three half-spaces. The extreme points of P are precisely A, B, C , and lie in P . By convexity of P , the convex hull of A, B, C , namely $\triangle ABC$ coincides with P and hence contains $\triangle b_1 b_2 b_3$. $\square$

Theorem 3. Given a triangle $\triangle ABC$, let p be a point in the plane of the triangle, not lying on any extended edge. For each pair of vertices of $\triangle ABC$, draw a circle that passes through them and also p . Let D_i , $i = 1, 2, 3$ denote the corresponding discs, with centers c_i . The following conditions are equivalent

- (i) $p \in \triangle ABC$;

- (ii) $p \in \triangle c_1 c_2 c_3$;
- (iii) $D_1 \cap D_2 \cap D_3 = \{p\}$.

Proof. Assume (i) holds. By the assumption of the theorem p does not lie on an edge, p is an interior point of $\triangle ABC$ (see Figure 2 or Figure 6 for examples). Consider the triangle whose vertices are the midpoints of the line segments Ap , Bp , Cp . Let these midpoints be A' , B' , C' , respectively. Clearly, p is also in $\triangle A'B'C'$, see Figure 6. Note that A' , B' , C' are precisely the intersections of orthogonal lines from p to the extended edges of $\triangle c_1 c_2 c_3$. This follows by the simple fact that the orthogonal bisecting line of each of the line segments Ap , Bp , Cp must contain two of the centers of the circles. Thus by Theorem 2, $p \in \triangle c_1 c_2 c_3$. Thus (i) implies (ii). By Theorem 1, $D_1 \cap D_2 \cap D_3 = \{p\}$. Thus (ii) implies (iii).

Next we prove (iii) implies (i). Suppose $D_1 \cap D_2 \cap D_3 = \{p\}$. By Theorem 1, $p \in \triangle c_1 c_2 c_3$. By Theorem 2, p is in the triangle whose vertices are the intersection points with respect to orthogonal lines to extended edges of $\triangle c_1 c_2 c_3$. Let these be b_1, b_2, b_3 . Extending the line segments pb_1, pb_2, pb_3 , from p toward b_i , while doubling the corresponding length, we obtain the vertices A, B and C . Hence $p \in \triangle b_1 b_2 b_3$ implies $p \in \triangle ABC$. $\square$

FIGURE 6. Three circles, each passing through p and two other vertices of $\triangle ABC$.

To see what happens when p is not in $\triangle ABC$, we give an example in Figure 7.

FIGURE 7. Three circles, each passing through p and two other vertices of $\triangle$.

3. Conclusions

In this article we have proved three convex hull theorems on triangles and circles, revealing interesting properties on these fundamental geometric figures. The results could give rise to generalizations to convex polygons and more generally convex polytopes in arbitrary dimension. Specifically, a generalization of Theorem 1 is given in [1]. We will give generalizations of Theorem 2 and 3 elsewhere. For instance Theorem 2 is generalizable to a simplex of $n + 1$ points in the n dimensional Euclidean space.

References

- [1] B. Kalantari, *A characterization theorem and an algorithm for a convex hull problem*, to appear in Annals of Operations Research, available online August, 2014. arxiv.org/pdf/1204.1873v2.pdf, and <http://arxiv-web3.library.cornell.edu/pdf/1204.1873v4.pdf>, 2012. To appear in Annals of Operations Research, 2014.
- [2] B. Kalantari, *Finding a lost treasure in convex hull of points from known distances*, In the Proceedings of the 24th Canadian Conference on Computational Geometry (2012), 271-276.
- [3] B. Kalantari, *Solving linear system of equations via a convex hull algorithm*, arxiv.org/pdf/1210.7858v1.pdf, 2012.

- [4] B. Kalantari and M. Saks, *On the Triangle Algorithm for the Convex Hull Membership*, 2-page Extended Abstract, 23rd Annual Fall Workshop on Computational Geometry, City College of New York, 2013.
- [5] M. Li and B. Kalantari, *Experimental Study of the Convex Hull Decision Problem via a New Geometric Algorithm*, 2-page Extended Abstract, 23rd Annual Fall Workshop on Computational Geometry, City College of New York, 2013.
- [6] T. Gibson and B. Kalantari, *Experiments with the Triangle Algorithm for Linear Systems*, 2-page Extended Abstract, 23rd Annual Fall Workshop on Computational Geometry, City College of New York, 2013.
- [7] R. Johnson, *A circle theorem*, Amer. Math. Monthly **23** (1916), 161-162.

Bahman Kalantari

Department of Computer Science, Rutgers University,
New Brunswick, New Jersey, USA.

E-mail: kalantari@cs.rutgers.edu

Jong Youll Park

Department of Mathematics, Chonnam National University,
Gwangju, South Korea.

E-mail: parkjy@chonnam.ac.kr