A Uniform Asymptotic Solution for Transmitted Waves through a Plane Dielectric Interface from a Denser to a Rarer Mediums by Using Parabolic Cylinder Functions

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Abstract

When the cylindrical wave is incident on a plane dielectric interface from a denser medium to a rarer one, the asymptotic solution for the transmitted wave in the near region is different from the one in the far region. In this paper, we have derived a novel uniform asymptotic solution represented by using the parabolic cylinder function for the transmitted and scattered waves observed in the rarer medium when the cylindrical wave is incident on the plane dielectric interface from the denser medium. The validity of the uniform asymptotic solution has been confirmed by comparing with the reference solution calculated numerically. It has been clarified that the transition wave plays an important role to connect smoothly the asymptotic solution in the near region to the one in the far region through the transition region. We have shown the very interesting phenomenon that the lateral wave type transmitted wave is observed in the far and shallow region.

Key words: Parabolic Cylinder Function, Plane Dielectric Interface, Transmitted Wave, Uniform Asymptotic Solution.

1. Introduction

The problems of the high-frequency transmitted and scattered waves excited by a high-frequency incident wave on a plane dielectric interface consisting of two different mediums have been the important research subjects for many years in the areas of the electromagnetic theory [1]~[6], the antennas and the propagation [7], [8], the optics [9]~[12], the acoustics [13], [14], the metamaterials [15]~[18], and so forth. The phenomena of the lateral wave [1]~[6], [7], [10], [11], [13], the totally reflected ray, the evanescent wave [6], [7], [19]~[23] produced by an incident wave with the incident angle greater than the critical angle of the total reflection, and the transmitted geometrical ray [19]~[23] are related to the above mentioned problems.

When the cylindrical wave or the spherical wave is incident on a plane dielectric interface from a denser 1st medium (ε_1, μ_0) to a rarer 2nd one (ε_2, μ_0) where $\varepsilon_1 > \varepsilon_2$, the observation point placed in the rarer (2nd) medium may receive only the transmitted geometrical ray in the short distance from the source and both the transmitted geometrical ray and the evanescent wave in the long distance [1], [2], [6], [9]~[11], [13], [19]~

[23]. Thus, the asymptotic solution in the short distance is different from the solution in the long distance. Therefore, it is necessary to derive a novel uniform asymptotic solution [6], [19]~[23] which can connect smoothly two different solutions in the near region and in the far region through a transition region.

In the present study, we shall derive the novel uniform asymptotic solution for the transmitted and scattered waves observed in the rarer medium when the cylindrical wave radiated from a line source is incident on a plane dielectric interface from the side of the denser dielectric medium [6], [7], [19]~[23]. We will derive the uniform asymptotic solution for the transition wave which plays an important role to connect the one solution in the near region to the other solution in the far region smoothly through the transition region [6], [7], [19]~[23]. The uniform asymptotic solution may be represented by using the parabolic cylinder function [24], [25]. The validity and the applicable range of the uniform asymptotic solution are confirmed by comparing with the reference solution calculated numerically from the integral representation of the transmitted wave. We will show the very interesting phenomenon that the lateral wave type transmitted wave appears in the far and

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Fig. 1. Cartesian coordinate system (x, y, z) , plane dielectric interface at $z=0$ consisting of denser medium and rarer medium, and schematic figures for transmitted waves.

shallow region. The time convention $e^{-i\omega t}$ is adopted and suppressed here.

II. Formulation and Integral Representations

Fig. 1 shows (x, y, z) Cartesian coordinate system, a plane dielectric interface at $z = 0$ consisting of a denser medium (ϵ_1, μ_0) and a rarer medium (ϵ_2, μ_0) , i.e., $\epsilon_1 > \epsilon_2$, a electric line source Q located at $(0, h)$ in the two dimensional (x, z) coordinate system, totally reflected rays, transmitted rays, and an evanescent wave. The area in the rarer medium may be divided into the near region, the transition region, and the far region in the x -direction, and into the shallow region and the deep region in the $(-z)$ -direction.

When the electromagnetic wave is radiated from the electric line source $Q(x, z) = Q(0, h)$, the electric field $E_y(x, z)$ observed in the rarer medium (ϵ_2, μ_0) may be expressed by using the Green's function G as follows [6], [7], [19]~[23]

$$E_y = i\omega\mu_0 I G, \quad G = \frac{i}{4\pi} \int_{P_\theta} T(\theta) e^{ik_1 q(\theta)} d\theta \quad (1)$$

$$T(\theta) = \frac{2 \cos \theta}{\cos \theta + \sqrt{n^2 - \sin^2 \theta}}, \quad n = \sqrt{\epsilon_2/\epsilon_1} \quad (2)$$

$$q(\theta) = R \cos(\theta - \theta_0) + |z| \sqrt{n^2 - \sin^2 \theta} \quad (3)$$

where $T(\theta)$ corresponds to the transmission coefficient while I and n denote, respectively, the current and the refractive index $n = \sqrt{\epsilon_2/\epsilon_1}$. R and θ_0 are defined geometrically in Fig. 1.

In Fig. 2, we have shown the original integration path P_θ and branch points at $\theta = \pm\delta$ ($\delta = \sin^{-1}n$) associated with the branch cuts of the integrand of the Green's function G in the complex θ -plane. Note that the branch

point at $\theta = \delta$ corresponds to the critical angle of the total reflection (see Fig. 1).

As shown in Fig. 1, only the transmitted geometrical ray is observed at P_1 in the near region. While the transmitted geometrical ray and the evanescent wave are observed at P_2 located in the far region. Therefore, it is necessary to find out the uniform asymptotic solution which can connect smoothly two different asymptotic solutions in the near region and in the far region. In the following, we shall derive from the integral (1) a novel uniform asymptotic solution applicable in the transition region [6], [19]~[23] between two regions.

So far, we have considered the transverse electric (TE) case, i.e., the y -component of the electric field E_y excited by the line current I directing y -direction. When we consider the transverse magnetic (TM) case, the y -component of the magnetic field H_y excited by the magnetic current I_m , directing y -direction and located at $Q(x, z) = Q(0, h)$ (see Fig. 1), may be obtained directly from (1)~(3) via the duality replacements as follows;

$$H_y = i\omega\epsilon_2 I_m G_h, \quad G_h = \frac{i}{4\pi} \int_{P_\theta} T_h(\theta) e^{ik_1 q(\theta)} d\theta \quad (4a)$$

where $T_h(\theta)$ corresponds to the transmission coefficient for the TM case and may be given by

$$T_h(\theta) = \frac{2n \cos \theta}{n^2 \cos \theta + \sqrt{n^2 - \sin^2 \theta}}, \quad n = \sqrt{\epsilon_2/\epsilon_1} \quad (4b)$$

Note that the integrand of the Green's function G_h in (4a) associated with (4a) and (3) possesses the branch points at the same places $\theta = \pm\delta$ ($\delta = \sin^{-1}n$) with those of the integrand of the Green's function G in (1) associated with (2) and (3) (see Fig. 2).

Thus, the asymptotic analysis method for the TM case represented by (4a) associated with (4b) and (3) is very

Fig. 2. Original integration path P_θ and branch points $\theta = \pm\delta$ associated with branch cuts in the complex θ -plane.

similar to the one for the TE case represented by (1) associated with (2) and (3). Therefore, in the following, we shall consider only the TE case represented by (1) ~ (3). Note, however, that the Brewster angle θ_B ($= \tan^{-1}n$) at which the reflection coefficient of the geometrically reflected ray vanishes exists in the TM case. This phenomenon does not appear in the TE case.

III. Asymptotic Solutions for Transmitted Waves

3-1 Asymptotic Solution in Transition Region and in Near Region

In this section, we will derive from (1) the asymptotic solution for the transmitted wave observed in the transition region between the dotted curves AG and AH shown by the shaded region in Fig. 1.

In Fig. 3, we show two saddle points θ_s and θ_e obtained from the saddle point equation $(d/d\theta)q(\theta) = 0$ or

$$R \sin(\theta - \theta_0) + |z| \frac{\sin \theta \cos \theta}{\sqrt{n^2 - \sin^2 \theta}} = 0 \quad (5)$$

The saddle point equation in (5) can be solved analytically by obtaining first the quartic equation from (5) and then solving the quartic equation by applying the Ferrari's method [26]. Four solutions may be obtained from the quartic equation. However, one may show that only two solutions θ_s and θ_e satisfy the original saddle point equation in (5). One may also show that two solutions θ_s and θ_e obtained analytically agree excellently with the numerical solutions $\hat{\theta}_s$ and $\hat{\theta}_e$, respectively, obtained by applying the Muller's method [27].

In this case, although (5) produces two saddle points θ_s and θ_e (see Fig. 3), it can be shown that only the saddle point θ_s contributes to the integral (1) [6], [19] ~ [23]. Thus, by deforming the original integration path P_θ into the steepest descent path SDP_{θ_s} , one may obtain

$$G = \frac{i}{4\pi} \int_{SDP_{\theta_s}} T(\theta) e^{ik_1 q(\theta)} d\theta \quad (6)$$

Since the saddle point θ_s is located near the branch point at $\theta = \delta$, the asymptotic solution obtained from the isolated saddle point technique [28] produces the errors. Therefore, one may define the transmitted wave G as follows.

$$G = G_{go} + G_{tran} \quad (7)$$

where G_{go} denotes the transmitted geometrical ray solution obtained from the isolated saddle point technique [28] and is given by

Fig. 3. Saddle points θ_s and θ_e and steepest descent path SDP_{θ_s} through saddle point θ_s in the complex θ -plane. The steepest descent path SDP_{θ_s} and the path P_{θ_s} are shown by the solid lines (—) when those paths run on the upper sheet and by the dashed lines (---) when the paths run on the lower sheet.

$$G_{go} = \frac{i}{4\pi} T(\theta_s) \sqrt{\frac{2\pi}{k_1 |q''(\theta_s)|}} e^{ik_1 l_1 + ik_2 l_2 - i\pi/4} \quad (8)$$

$$q''(\theta_s) = -\cos \theta_s \left\{ R \cos \theta_0 \sec^2 \theta_s + |z| \frac{n^2 \cos \theta_s}{(n^2 - \sin^2 \theta_s)^{3/2}} \right\} \quad (9)$$

The asymptotic solution G_{go} represents clearly the transmitted geometrical ray $Q \rightarrow C \rightarrow P_{go}$ observed at P_{go} shown in Fig. 4. The geometrical parameters l_1 , l_2 , R , $|z|$, and θ_0 are defined in Fig. 4.

While G_{tran} in (7) denotes the transition wave which plays an important role only in the transition region between the curves AG and AH or in the transition region $P_3 < x < P_4$ for the given depth $|z|$ (see Fig. 4) and is defined as follows.

$$G_{tran} = G - G_{go} \quad (10)$$

Fig. 4. Transmitted geometrical ray solution G_{go} observed at P_{go} located in the transition region between the dotted curves AG and AH .

$$G = \frac{i}{4\pi} \int_{P_{\theta_s}} T(\theta) e^{ik_1 q(\theta)} d\theta \quad (11)$$

Note that G in (11) is the integral along the straight integration path P_{θ_s} (see Fig. 3). Since the steepest descent path SDP_{θ_s} in (6) is the curved integration path as shown in Fig. 3, it is difficult to derive from (6) the uniform asymptotic solution expressed by using the parabolic cylinder functions [24], [25]. If we deform the curved steepest descent path SDP_{θ_s} into the straight integration path P_{θ_s} as shown in Fig. 3, it becomes possible to derive the uniform asymptotic solution represented by using the parabolic cylinder functions as shown in the following. One may show that the integral in (6) is equal to the integral in (11) [19]~[21], [23].

To proceed the asymptotic analysis of the integral G in (11), the function $T(\theta)$ may be expressed as follows

$$T(\theta) = T(\theta_s) + T^d(\theta) + T^{(2)}(\theta) \sqrt{\sin(\delta + \theta) \sin(\delta - \theta)} \quad (12)$$

$$T^d(\theta) = T^{(1)}(\theta) - T^{(1)}(\theta_s) - T^{(2)}(\theta_s) \sqrt{\sin(\delta + \theta_s) \sin(\delta - \theta_s)} \quad (13)$$

$$T^{(1)}(\theta) = \frac{2 \cos^2 \theta}{1 - n^2}, T^{(2)}(\theta) = \frac{-2 \cos \theta}{1 - n^2} \quad (14)$$

Then the integral (11) may be represented by

$$G = G_1 + G_2 + G_3 \quad (15)$$

where G_1 , G_2 , and G_3 can be represented by (11) with the replacement of $T(\theta)$ in (11) by $T(\theta_s)$, $T^d(\theta)$, and $T^{(2)}(\theta) \sqrt{\sin(\delta + \theta) \sin(\delta - \theta)}$, respectively [20], [21], [23].

The integral G_1 for example may be represented by

$$G_1 = \frac{i}{4\pi} T(\theta_s) e^{ik_1 q(\theta_s)} \int_{P_{\theta_s}} e^{U(\theta) + V(\theta)} d\theta \quad (16)$$

$$U(\theta) = ik_1 |z| \left(\sqrt{n^2 - \sin^2 \theta} - \sqrt{n^2 - \sin^2 \theta_s} \right) \quad (17)$$

$$V(\theta) \doteq ik_1 R \left\{ (\theta - \theta_s)^2 \frac{\cos(\theta_s - \theta_0)}{2} + (\theta - \theta_s) \sin(\theta_s - \theta_0) \right\} \quad (18)$$

When $U(\theta)$ in $e^{U(\theta)}$ in (16) satisfies the condition $|U(\theta)| \ll 1$, then one may substitute the expansion

$$e^{U(\theta)} = \sum_{j=0}^{\infty} \frac{\{U(\theta)\}^j}{j!} \quad (19)$$

into (16) and perform the lengthy but straightforward

calculation to derive

$$G_1 = \frac{i}{4\pi} T(\theta_s) e^{ik_1 l_1 + ik_2 l_2 - i\pi/4} I_1 \quad (20)$$

$$I_1 = \sum_{j=0}^{\infty} \frac{\alpha^j}{\sqrt{k_1 R \cos(\theta_s - \theta_0)} j!} I(j) \quad (21)$$

$$I(j) = \int_{-\infty}^{\infty} (\sqrt{\beta + w} - \sqrt{\beta})^j e^{-\frac{w^2}{2} + \gamma w} dw \quad (22)$$

where α , β , and γ are defined as follows.

$$\alpha = ik_1 |z| e^{-i\pi/8} \frac{\sqrt{\sin(\delta + \theta_s) \cos(\delta - \theta_s)}}{\{k_1 R \cos(\theta_s - \theta_0)\}^{1/4}} \quad (23)$$

$$\beta = e^{i\pi/4} \tan(\delta - \theta_s) \sqrt{k_1 R \cos(\theta_s - \theta_0)} \quad (24)$$

$$\gamma = e^{i\pi/4} \frac{k_1 R \sin(\theta_0 - \theta_s)}{\sqrt{k_1 R \cos(\theta_s - \theta_0)}} \quad (25)$$

The integral $I(j)$ in (22) is proceeded furthermore by applying the binominal theorem [29]

$$(\sqrt{\beta + w} - \sqrt{\beta})^j = \sum_{m=0}^j {}_j C_m (\sqrt{\beta + w})^m (-\sqrt{\beta})^{j-m} \quad (26)$$

where ${}_j C_m$ denotes the binominal coefficient [29], and then transforming from the complex w -plane to the complex s -plane via $s = \beta + w$. The integral (22) may be represented by

$$I(j) = e^{-\frac{\beta^2}{2} - \gamma\beta} \sum_{m=0}^j {}_j C_m (-\sqrt{\beta})^{j-m} \cdot \{I_{m,1} + I_{m,2}\} \quad (27)$$

$$I_{m,1} = \int_0^{\infty} s^{m/2} e^{-\frac{s^2}{2} - (\gamma - \beta)s} ds \quad (28)$$

$$I_{m,2} = i^m \int_0^{\infty} s^{m/2} e^{-\frac{s^2}{2} + (\gamma - \beta)s} ds \quad (29)$$

Above integrals $I_{m,1}$ and $I_{m,2}$ are expressed by using the parabolic cylinder functions [24], [25]. One may obtain the following result for $I(j)$.

$$I(j) = \sum_{m=0}^j i^{m/2} \sqrt{2\pi} {}_j C_m (-\sqrt{\beta})^{j-m} \cdot e^{\gamma^2/2 - (\gamma - \beta)^2/4} D_{m/2}(i\{\gamma - \beta\}) \quad (30)$$

In deriving (30), we have applied the relation for the parabolic cylinder function as follows [24], [25];

$$D_n(z) = \frac{\Gamma(n+1)}{\sqrt{2\pi}} \left\{ e^{i\frac{n\pi}{2}} D_{-n-1}(iz) + e^{-i\frac{n\pi}{2}} D_{-n-1}(-iz) \right\} \quad (31)$$

where $\Gamma(z)$ and $D_v(z)$ denote, respectively, the Gamma function and the parabolic cylinder function [24], [25].

Similarly, one may obtain the asymptotic solution for $(G_2 + G_3)$ in (15) as follows;

$$G_2 + G_3 = \frac{i}{4\pi} T^{(2)}(\theta_s) \sqrt{\sin(\delta + \theta_s)} \cdot e^{ik_1 l_1 + ik_2 l_2 - \frac{i\pi}{4}} (I_2 + I_3) \quad (32)$$

$$I_2 + I_3 = \sum_{j=0}^{\infty} \frac{e^{-i\pi/8} \sqrt{\cos(\delta - \theta_s)} \alpha^j}{\{k_1 R \cos(\theta_s - \theta_0)\}^{3/4} j!} I(j+1) \quad (33)$$

Substituting G_1 in (20) associated with (30) and $G_2 + G_3$ in (32) associated with (33) into (15) produces the asymptotic solution for G and then substituting G thus obtained into (10) one may obtain the transition wave G_{tran} which plays an important role in the transition region between the dotted curves AG and AH (see Figs. 1 and 4).

When the observation point moves along the observation line $P_1 P_2$ to the P_1 -direction (see Fig. 4), the transition wave G_{tran} approaches zero, i.e., $G_{tran} \rightarrow 0$, outside the transition region where $x < P_3$ (see Fig. 4). Thus the transmitted wave G in (7) in the near region is represented only by the transmitted geometrical ray G_{go} . Since the asymptotic solution for (7) is applicable in the transition region and in the near region, we define the asymptotic solution for (7) as the uniform asymptotic solution. The uniform asymptotic solution for G in (7) derived in this section is new and plays an important role in the transition region between the dotted curves AG and AH and approaches the transmitted geometrical ray G_{go} in the near region.

3-2 Asymptotic Solution in Transition Region and in Far Region

When the observation point is located in the transition region between the dotted curves AH and AH' (see Fig. 1), two saddle points θ_s and θ_e are obtained from the saddle point equation (5). In Fig. 5, we have shown the steepest descent paths SDP_{θ_s} and SDP_{θ_e} passing through the saddle points θ_s and θ_e , respectively.

The transmitted wave G in (1) may be represented as [6], [20] ~ [23]

$$G = G_{eva} + \hat{G} \quad (34)$$

$$G_{eva} = \frac{i}{4\pi} \int_{SDP_{\theta_e}} T(\theta) e^{ik_1 q(\theta)} d\theta \quad (35)$$

$$\hat{G} = \frac{i}{4\pi} \int_{SDP_{\theta_s}} T(\theta) e^{ik_1 q(\theta)} d\theta \quad (36)$$

Fig. 5. Steepest descent paths SDP_{θ_s} and SDP_{θ_e} passing through saddle points θ_s and θ_e , respectively, for observation point placed in the transition region and in the far region.

Since the saddle point θ_e is located away from θ_s and the branch point at $\theta = \delta$ (see Fig. 5), the integral G_{eva} representing the evanescent wave is evaluated by applying the isolated saddle point technique [28]. The asymptotic solution for G_{eva} may be represented as

$$G_{eva} \sim \frac{i}{4\pi} \sqrt{\frac{2}{\pi k_1 R}} e^{ik_1 R - i\frac{\pi}{4} T(\theta_0)} \cdot e^{-k_1 |z| \sqrt{\sin^2 \theta_0 - n^2}} \quad (37)$$

where $\theta_0 \doteq \text{Re}[\theta_e]$. The evanescent wave G_{eva} , excited by the incident wave $Q \rightarrow E$ (see Fig. 1) with the incident angle θ_0 satisfying $\theta_0 > \delta$ (critical angle of the total reflection), decays exponentially rapidly in the $-z$ -direction as shown in Fig. 1.

3-2-1 Asymptotic Solution in Deep Region

When the observation point is located in the transition region between AH and AH' or in the far region in the deep region (see Fig. 1), by applying the isolated point technique [28] one may obtain from $\hat{G}$ in (36) the transmitted geometrical ray solution G_{go} , $Q(0, h) \rightarrow D \rightarrow P_2$ shown in Fig. 1 (Note that the observation point P_2 is placed in the deep region in this case). The transmitted geometrical ray G_{go} derived from (36) agrees with the solution G_{go} derived in (8) and (9) for the transmitted geometrical ray $Q(0, h) \rightarrow C \rightarrow P_{go}$.

Thus the asymptotic solution in the transition region and in the far region in the deep region may be obtained from the summation of the evanescent wave G_{eva} ($Q(0, h) \rightarrow E \rightarrow P_2$) and the transmitted geometrical ray G_{go} ($Q(0, h) \rightarrow D \rightarrow P_2$) (see Fig. 1). However, in the deep region, the evanescent G_{eva} in (37) is sufficiently

small so that the transmitted wave G in (34) may be obtained only from the transmitted geometrical ray solution as follows.

$$G \sim G_{go} \quad (38)$$

3-2-2 Asymptotic Solution in Shallow Region

In this section, we shall derive from (34)~(36) the uniform asymptotic solution for the observation point placed in the transition region between the curves AH and AH' and in the far region in the shallow region (see Fig. 1). In this case, the saddle point θ_s is located near the branch point at $\theta = \delta$ (see Fig. 5). The asymptotic solution for the evanescent wave G_{eva} in (37) is unchanged.

To derive the asymptotic solution for $\hat{G}$ in (36), one may express the function $T(\theta)$ in (36) as expressed in (12) associated with (13) and (14). Then the integral $\hat{G}$ may be represented by

$$\hat{G} = \hat{G}_1 + \hat{G}_2 + \hat{G}_3 \quad (39)$$

where $\hat{G}_1$, $\hat{G}_2$, and $\hat{G}_3$ are given by (36) with the replacement of $T(\theta)$ in (36) by $T(\theta_s)$, $T^d(\theta)$, and $T^{(2)}(\theta)\sqrt{\sin(\delta + \theta)\sin(\delta - \theta)}$, respectively (see (12)) [6], [20]~[23].

The integral $\hat{G}_3$ for example may be approximated by

$$\hat{G}_3 \doteq \frac{i}{4\pi} T^{(2)}(\theta_s) \sqrt{\sin(\delta + \theta_s)} e^{ik_1 q(\theta_s)} I_3 \quad (40)$$

$$I_3 = \int_{SDP_{\theta_s}} \sqrt{\sin(\delta - \theta)} e^{U(\theta) + V(\theta)} d\theta \quad (41)$$

where the functions $U(\theta)$ and $V(\theta)$ have been defined in (17) and (18), respectively.

Note that the integration path P_{θ_s} for G in (11) and G_1 , G_2 , and G_3 in (15) (see Fig. 3) is different from the steepest descent path SDP_{θ_s} for $\hat{G}$ in (36) and $\hat{G}_1$, $\hat{G}_2$, and $\hat{G}_3$ in (39). After passing through the saddle point θ_s , the SDP_{θ_s} turns 360° around the branch point at $\theta = \delta$ in the counterclockwise direction and runs along the straight line to the backward direction toward the valley located in the upper left region in the 1st quadrant of the complex θ -plane as shown in Fig. 5. Thus, the analysis method applied in the followings is different from the one applied in the Section 3-1.

Assuming $|U(\theta)| \ll 1$, the function $e^{U(\theta)}$ in (41) is expanded as in (19). Then the integral I_3 in (41) is evaluated along the steepest descent path SDP_{θ_s} shown in

Fig. 5. In the papers [6] and [23], we have derived the asymptotic solution for the reduced integral $I_3 (= I_3^- + I_3^+)$. The integrals $I_3^\mp$ derived in [20]~[22] have been solved by using the parabolic cylinder functions [6], [23] between the dotted curves AH and AH' for the saddle point θ_s located near the branch point δ , i.e., $\theta_s \doteq \delta$. By performing the straightforward calculations similar to those applied for $I_3^\mp$, we have also derived in [6] and [23] the asymptotic solution for the integrals $\hat{G}_1$ and $\hat{G}_2$ by using the parabolic cylinder functions [6], [23]~[25].

Thus we will obtain the novel uniform asymptotic solution for G in (34) as follows

$$G = G_{eva} + G_{tran} + G_{lat.wave} \quad (42)$$

where $G_{tran} (= \hat{G}_1 + \hat{G}_2)$ denotes the transition wave solution which plays the important roles only in the transition region [6], [20]~[23] (see Fig. 1) and $G_{lat.wave} (= \hat{G}_3)$ denotes the lateral wave type transmitted wave. $\hat{G}_1$, $\hat{G}_2$, and $\hat{G}_3$ may be represented by [6], [23]

$$\hat{G}_1 \sim \frac{i}{4\pi} T(\theta_s) e^{ik_1 L_1 + ik_2 L_2 - i\pi/4} (I_1^- + I_1^+) \quad (43)$$

$$I_1^\mp \sim \pm \sum_{j=0}^{\infty} \frac{1}{\sqrt{k_1 R \cos(\theta_s - \theta_0)}} \frac{(\alpha^\mp)^j}{j!} \cdot \Gamma\left(1 + \frac{j}{2}\right) e^{\frac{\gamma^2}{4}} D_{-1-\frac{j}{2}}(\gamma) \left(\frac{1}{i}\right)^j \quad (44)$$

$$\hat{G}_2 \sim -\frac{i}{4\pi} T^{(2)}(\theta_s) \sqrt{\sin(\delta + \theta_s) \sin(\delta - \theta_s)} \cdot e^{ik_1 L_1 + ik_2 L_2 - i\pi/4} (I_2^- + I_2^+), \quad I_2^\mp = I_1^\mp \quad (45)$$

$$\hat{G}_3 \sim -\frac{i}{4\pi} T^{(2)}(\theta_s) \sqrt{\sin(\delta + \theta_s)} \cdot e^{ik_1 L_1 + ik_2 L_2 - i\pi/4} (I_3^- + I_3^+) \quad (46)$$

$$I_3^\mp \sim \pm \sum_{j=0}^{\infty} \frac{\sqrt{\cos(\delta - \theta_s)}}{\{k_1 R \cos(\theta_s - \theta_0)\}^{3/4}} \frac{(\alpha^\mp)^j}{j!} \cdot \Gamma\left(\frac{j+3}{2}\right) e^{\frac{\gamma^2}{4}} D_{\frac{j+3}{2}}(\gamma) \left(e^{i3\pi/8}\right) \quad (47)$$

where $\alpha^\mp$ and γ are defined as follows.

$$\alpha^\mp = ik_1 |z| \frac{\sqrt{\sin(\delta + \theta_s) \cos(\delta - \theta_s)}}{\{k_1 R \cos(\theta_s - \theta_0)\}^{1/4}} \left(e^{-i\pi/8}\right) \quad (48)$$

$$\gamma = e^{i\pi/4} \frac{k_1 R \sin(\theta_0 - \theta_s)}{\sqrt{k_1 R \cos(\theta_s - \theta_0)}} \quad (49)$$

Note that I_1^+ in (44), for example, chooses the lower signs. That is - sign from $\pm$, α^+ from $\alpha^\mp$, and i^j from

(1 and i^j) in the open brackets ().

As far as authors know, (42) is the new solution applicable in the transition region between the dotted curves AH and AH' and in the far region.

When the observation point P_2 is located in the shallow and far region, the novel uniform asymptotic solution in (42) may be approximated by

$$G \sim G_{eva} + G_{lat.wave}^0 \quad (50)$$

where $G_{lat.wave}^0$ obtained from (46) and (47) by substituting $j = 0$ and the asymptotic approximation for the parabolic cylinder function $D_{-3/2}(\gamma)$ [24], [25], is given by [6], [20], [22], [23]

$$G_{lat.wave}^0 = \frac{i}{k_1} \sqrt{\frac{2n}{\pi k_1}} \frac{e^{i3\pi/4}}{1 - n^2} \frac{1}{L_2^{3/2}} e^{ik_1 L_1 + ik_2 L_2} \quad (51)$$

The geometrical distances L_1 and L_2 in (51) are defined in Fig. 1. It is very interesting to observe that in the shallow and far region the amplitude $1/(L_2^{3/2})$ of $G_{lat.wave}^0$ in (51) behaves like a lateral wave which is usually observed in the denser 1st medium (see Fig. 1). Thus we define the transmitted wave $G_{lat.wave}^0$ in (51) as the lateral wave type transmitted wave. While G_{eva} in (50) is the evanescent wave derived in (37).

IV. Numerical Results and Discussions

In this section, we will confirm the validity and applicability of the novel uniform asymptotic solutions in (7) and (42) and the asymptotic solution in (50) numerically. We have performed extensive numerical calculations to show the validity of the novel uniform asymptotic solution presented in this paper. However, we will present here three typical examples since the results calculated by using other numerical parameters are similar to those shown here.

In Fig. 6, we have shown the electric field magnitude vs. distance $x(\lambda)$ curve. We have calculated the transmitted wave observed in the rarer medium when the observation point is located in deep region $z = -2.5\lambda$. The bold solid curve (—) calculated by using the transmitted geometrical rays G_{go} in (8) and (38) agrees excellently with the reference solution (•••: closed circles) calculated numerically from (1). It is clarified that when the observation point is located in the deep region, only the transmitted geometrical ray provides the excellent solution.

For the observation point located in the shallow region, the typical numerical result is shown in Fig. 7 for the case when $z = -0.7\lambda$. The numerical parameters

Fig. 6. Comparison of transmitted geometrical ray solution with the reference solution at the depth $z = -2.5\lambda$. Numerical parameters: $h = 400\lambda$, $\varepsilon_1 = 2.3\varepsilon_0$, $\varepsilon_2 = 1.0\varepsilon_0$, $f = 3$ GHz, $n = 0.659$.

used in the calculation are the same as those used in the calculation of Fig. 6. The thin solid curve (—) is calculated by using the uniform asymptotic solution in (7) in the near region and in the transition region between the curves AG and AH and the uniform asymptotic solution in (42) in the transition region between the curves AH and AH' and in the far region (see Figs. 1 and 4). It is clarified that the uniform asymptotic solution (—) derived in this study agrees very well with the reference solution (•••). It is also shown that the transition waves (---) calculated from the solution G_{tran} in (10) and $G_{tran} = \hat{G}_1 + \hat{G}_2$ (see (42)) play an important role in the transition region $P_3 < x < P'_4$ shown in Figs. 1 and 4.

Fig. 7. Comparison of uniform asymptotic solution with reference solution at the depth $z = -0.7\lambda$.

Fig. 8. Comparison of asymptotic solution in (50) with the reference solution. Numerical parameters: $h = 400\lambda$, $\epsilon_1 = 2.3\epsilon_0$, $\epsilon_2 = 1.0\epsilon_0$, $f = 3\text{GHz}$, and $z = -0.05\lambda$ (shallow region).

In Fig. 8, we have calculated the transmitted wave G in the shallow ($z = -0.05\lambda$) and far region by using the asymptotic solution (50) consisting of the evanescent wave in (37) and the lateral wave type transmitted wave in (51). It is clarified that, the asymptotic solution (—) agrees excellently with the reference solution (•••: closed circles) calculated numerically from (1) in the far region shown in Fig. 8. However, the asymptotic solution deviates from the reference solution in the region where $x < 365\lambda$ (see Fig. 8) since the approximations we have applied become inaccurate in this region. Thus we have confirmed that the lateral wave type transmitted wave in (51) is valid in the far and shallow region. In Fig. 8, we have also plotted the lateral wave type transmitted wave in (51) by the dotted curve (.....). By the interference of the evanescent wave and the lateral wave type transmitted wave, the asymptotic solution oscillates as the function of the horizontal distance $x(\lambda)$.

It is very interesting to observe that the lateral wave type transmitted wave with the amplitude proportional to $1/(L_2^{3/2})$ (see Fig. 1) is excited near the plane dielectric interface where $|z|$ is small.

V. Conclusions

We have derived the novel uniform asymptotic solution represented by using the parabolic cylinder function for the transmitted wave observed in the rarer medium

when the cylindrical wave is incident on a plane dielectric interface from the denser medium. We have confirmed the validity and utility of the uniform asymptotic solution by comparing with the reference solution calculated numerically. It is clarified that the transition wave plays an important role to connect smoothly the two different solutions in the near region and the far region through the transition region. We have shown the very interesting phenomenon that the lateral wave type transmitted wave is produced in the shallow and far region.

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