Double Fuzzy Preproximity Spaces

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Abstract

In this paper, we introduce the concept of double fuzzy preproximity spaces as a generalization of a fuzzy preproximity spaces and investigate some of their properties. Also we study the relationships between double fuzzy preproximity spaces, double fuzzy topological spaces and double fuzzy closure spaces. In addition to this was the introduction of the concept of double fuzzy neighborhood system and has been studying the connection with double fuzzy preproximity, which resulted in the definition of the concept double fuzzy preproximal neighborhood.

Key words : Double fuzzy topology, double fuzzy preproximity, double fuzzy preproximal neighborhood.

1. Introduction

Šostak [16] introduced the fundamental concept of a fuzzy topological structure as an extension of both crisp topology and fuzzy topology [3]. Many authors [7, 13, 14, 15, 17, 12] gave various definitions of fuzzy proximities in a Šostak's sense.

As a generalization of fuzzy sets, the notion of intuitionistic fuzzy sets was introduced by Atanassove [1, 2, 18]. Recently, Çoker and his colleagues [4, 6, 5] introduced the notion of intuitionistic fuzzy topological space using intuitionistic fuzzy sets. Samanta and Mondal [10, 11] introduced the notion of intuitionistic gradation of openness as a generalization of intuitionistic fuzzy topological spaces [5] and smooth topological spaces.

Working under the name "intuitionistic" didn't continue because doubts were thrown about the suitability of this term, especially when working in the case of complete lattice L . These doubts were quickly ended in 2005 by Garcia and Rodabaugh [19]. They proved that this term is unsuitable in mathematics and applications. They concluded that they work under the name "double".

In this paper, we introduce the notion of double fuzzy preproximity spaces as a generalization of fuzzy preproximity spaces and investigate some of their properties. We define double fuzzy preproximal neighborhood systems. We study double fuzzy topologies induced by double fuzzy preproximity. Moreover, we investigate the relationships among double fuzzy preproximities, double fuzzy topolo-

gies and double fuzzy closure spaces.

2. Preliminaries

Throughout this paper, let X be nonempty set, $I = [0, 1]$ the unit interval, $I_0 = (0, 1]$ and $I_1 = [0, 1)$. For each $\alpha \in I$, let $\bar{\alpha}$ denote the constant fuzzy subset of X with value α , respectively. For $\lambda \in I^X$, $\lambda' = \bar{1} - \lambda$. Let $I \otimes I = \{(r, s) \mid r \in I_0, s \in I_1 \text{ and } r + s \leq 1\}$.

Definition 2.1 ([10]). Let X be a non-empty set. A double fuzzy topology $(\hat{T}, \check{T})$ on X is the maps $\hat{T}, \check{T}: I^X \rightarrow I \otimes I$ which satisfy the following properties:

- (1) $\hat{T}(\bar{0}) = \hat{T}(\bar{1}) = 1$, and $\check{T}(\bar{0}) = \check{T}(\bar{1}) = 0$.
- (2) $\hat{T}(\lambda_1 \wedge \lambda_2) \geq \hat{T}(\lambda_1) \wedge \hat{T}(\lambda_2)$, and $\check{T}(\lambda_1 \wedge \lambda_2) \leq \check{T}(\lambda_1) \vee \check{T}(\lambda_2)$.
- (3) $\hat{T}(\bigvee \lambda_i) \geq \bigwedge \hat{T}(\lambda_i)$, and $\check{T}(\bigvee \lambda_i) \leq \bigvee \check{T}(\lambda_i)$.

The triplet $(X, \hat{T}, \check{T})$ is said to be a double fuzzy topological space. Also, we call $\hat{T}(\lambda)$ a gradation of openness of λ and $\check{T}(\lambda)$ a gradation of non openness of λ . Let $(X, \hat{T}_1, \check{T}_1)$ and $(Y, \hat{T}_2, \check{T}_2)$ be double fuzzy topological spaces. A map $f^-: (X, \hat{T}_1, \check{T}_1) \rightarrow (Y, \hat{T}_2, \check{T}_2)$ is said to be a double fuzzy continuous if

$$\hat{T}_1(f^-(\nu)) \geq \hat{T}_2(\nu),$$

and

$$\check{T}_1(f^-(\nu)) \leq \check{T}_2(\nu)$$

for each $\nu \in I^Y$.

Notice that we didn't mention the condition $\hat{T}(\lambda) + \check{T}(\lambda) \leq 1$ for each $\lambda \in I^X$ as it concluded in the definition of $I \oplus I$.

Definition 2.2 ([9]). A double fuzzy closure space is an ordered pair $(X, \mathcal{C})$, where $\mathcal{C}: I^X \times I \otimes I \rightarrow I^X$ is a function satisfying the following axioms:

- (1) $\mathcal{C}(\underline{0}, r, s) = \underline{0}$.
- (2) $\lambda \leq \mathcal{C}(\lambda, r, s)$.
- (3) If $\lambda_1 \leq \lambda_2$, then $\mathcal{C}(\lambda_1, r, s) \leq \mathcal{C}(\lambda_2, r, s)$.
- (4) $\mathcal{C}(\lambda, r_1, s_1) \leq \mathcal{C}(\lambda, r_2, s_2)$ if $r_1 \leq r_2$ and $s_1 \geq s_2$.
- (5) $\mathcal{C}(\lambda \vee \mu, r, s) = \mathcal{C}(\lambda, r, s) \vee \mathcal{C}(\mu, r, s)$.

A double fuzzy closure space is called *topological* provided that

$$\mathcal{C}(\mathcal{C}(\lambda, r, s), r, s) = \mathcal{C}(\lambda, r, s)$$

for each $\lambda \in I^X$, $(r, s) \in I \otimes I$.

A continuous function between two double fuzzy closure spaces $(X, \mathcal{C}_1)$ and $(Y, \mathcal{C}_2)$ ($\mathcal{C}$ -map, for short) is a function $f^\rightarrow: (X, \mathcal{C}_1) \rightarrow (Y, \mathcal{C}_2)$ such that for each $\lambda \in I^X$, $(r, s) \in I \otimes I$,

$$f^\rightarrow(\mathcal{C}_1(\lambda, r, s)) \leq \mathcal{C}_2(f^\rightarrow(\lambda), r, s).$$

Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be two double fuzzy closure operators on X . We say $\mathcal{C}_1$ is finer than $\mathcal{C}_2$ ($\mathcal{C}_2$ is coarser than $\mathcal{C}_1$) iff $\mathcal{C}_2(\lambda, r, s) \leq \mathcal{C}_1(\lambda, r, s)$, for each $\lambda \in I^X$ and $(r, s) \in I \otimes I$.

Definition 2.3. A preproximity on X is a binary relation δ on I^X satisfying the following properties:

- (P1) $(\lambda, \mu) \in \delta$ implies $\lambda \neq \underline{0}, \mu \neq \underline{0}$.
- (P2) If $(\lambda, \mu) \in \delta$, $\lambda \leq \lambda_1$ and $\mu \leq \mu_1$, then $(\lambda_1, \mu_1) \in \delta$.
- (P3) $(\lambda, \mu) \in \delta$, implies $\lambda \leq \mu'$.
- (P4) $(\lambda \vee \mu, \nu) \in \delta$ if and only if $(\lambda, \nu) \in \delta$ or $(\mu, \nu) \in \delta$.

The pair (X, δ) is called a preproximity space. Let (X, δ_1) and (Y, δ_2) be two preproximity spaces and $f^\rightarrow: (X, \delta_1) \rightarrow (Y, \delta_2)$ be a map. Then $f^\rightarrow$ is called a preproximal map if $(\lambda, \mu) \in \delta_1$ implies $(f^\rightarrow(\lambda), f^\rightarrow(\mu)) \in \delta_2$, for each $\lambda, \mu \in I^X$.

3. Double Gradation Of Fuzzy Preproximity

Definition 3.1. Let X be a non-empty set. A double gradation of fuzzy preproximity $(\hat{\delta}, \check{\delta})$ on X is the maps $\hat{\delta}, \check{\delta}: I^X \times I^X \rightarrow I \otimes I$ which satisfy the following properties:

- (DFP1) $\hat{\delta}(\underline{0}, \underline{0}) = \hat{\delta}(\underline{1}, \underline{0}) = 0$ and $\check{\delta}(\underline{0}, \underline{1}) = \check{\delta}(\underline{1}, \underline{0}) = 1$.
- (DFP2) If $\hat{\delta}(\lambda, \mu) = 1$ and $\check{\delta}(\lambda, \mu) = 0$ implies to $\lambda \leq \mu'$.
- (DFP3) If $\lambda_1 \leq \lambda_2$, then $\hat{\delta}(\lambda_1, \mu) \leq \hat{\delta}(\lambda_2, \mu)$ and $\check{\delta}(\lambda_1, \mu) \geq \check{\delta}(\lambda_2, \mu)$.
- (DFP4) $\hat{\delta}(\lambda, \mu_1 \vee \mu_2) = \hat{\delta}(\lambda, \mu_1) \vee \hat{\delta}(\lambda, \mu_2)$, $\check{\delta}(\lambda_1 \vee \lambda_2, \mu) = \check{\delta}(\lambda_1, \mu) \vee \check{\delta}(\lambda_2, \mu)$

The triplet $(X, \hat{\delta}, \check{\delta})$ is said to be a double fuzzy preproximity space. Also, we call $\hat{\delta}(\lambda, \mu)$ a gradation of nearness and $\check{\delta}(\lambda, \mu)$ a gradation of non nearness between λ and μ .

Proposition 3.2. Let $(X, \hat{\delta}, \check{\delta})$ be a double fuzzy preproximity space. Then, for each $(\alpha, \beta) \in I \otimes I$, the family $\delta_{(\alpha, \beta)}$ defined by

$$\delta_{(\alpha, \beta)} = \{(\lambda, \mu) \in I^X \times I^X \mid \hat{\delta}(\lambda, \mu) \geq \alpha, \check{\delta}(\lambda, \mu) \leq \beta\}.$$

is preproximity space on X . In this case $\delta_{(\alpha, \beta)}$ is called the (α, β) -level double fuzzy preproximity space on X .

Definition 3.3. Let $(X, \hat{\delta}_1, \check{\delta}_1)$ and $(Y, \hat{\delta}_2, \check{\delta}_2)$ be two double fuzzy preproximity spaces and let $f^\rightarrow: (X, \hat{\delta}_1, \check{\delta}_1) \rightarrow (Y, \hat{\delta}_2, \check{\delta}_2)$ be a map. Then $f^\rightarrow$ is called a double fuzzy preproximal map if

$$\hat{\delta}_1(\lambda, \mu) \leq \hat{\delta}_2(f^\rightarrow(\lambda), f^\rightarrow(\mu))$$

and

$$\check{\delta}_1(\lambda, \mu) \geq \check{\delta}_2(f^\rightarrow(\lambda), f^\rightarrow(\mu))$$

for each $\lambda, \mu \in I^X$.

From $\lambda \leq f^\leftarrow f^\rightarrow(\lambda)$ and $\nu \geq f^\rightarrow f^\leftarrow(\nu)$ we obtain the following Lemma.

Lemma 3.4. Let $(X, \hat{\delta}_1, \check{\delta}_1)$ and $(Y, \hat{\delta}_2, \check{\delta}_2)$ be two double fuzzy preproximity spaces and let $f^\rightarrow: (X, \hat{\delta}_1, \check{\delta}_1) \rightarrow (Y, \hat{\delta}_2, \check{\delta}_2)$ be a map. Then $f^\rightarrow$ is a double fuzzy preproximal if and only if

$$\hat{\delta}_2(\nu, \gamma) \geq \hat{\delta}_1(f^\leftarrow(\nu), f^\leftarrow(\gamma))$$

and

$$\check{\delta}_2(\nu, \gamma) \leq \check{\delta}_1(f^\leftarrow(\nu), f^\leftarrow(\gamma))$$

for each $\nu, \gamma \in I^X$.

Clearly one can equip the set of all double fuzzy preproximities on a set X with partial ordering by $(\hat{\delta}_1, \check{\delta}_1)$ is finer than $(\hat{\delta}_2, \check{\delta}_2)$ if the identity map of X is double fuzzy preproximal map from $(X, \hat{\delta}_1, \check{\delta}_1)$ to $(X, \hat{\delta}_2, \check{\delta}_2)$, that is

$$\hat{\delta}_1(\lambda, \mu) \leq \hat{\delta}_2(\lambda, \mu)$$

and

$$\check{\delta}_1(\lambda, \mu) \geq \check{\delta}_2(\lambda, \mu)$$

for each $\lambda, \mu \in I^X$.

Definition 3.5. A double fuzzy preproximity $(\hat{\delta}, \check{\delta})$ on X is called *principal* provided that

$$\hat{\delta}(\bigvee_i \lambda_i, \mu) \leq \bigvee_i \hat{\delta}(\lambda_i, \mu)$$

and

$$\check{\delta}(\bigvee_i \lambda_i, \mu) \geq \bigwedge_i \check{\delta}(\lambda_i, \mu).$$

For each family $\{\lambda_i \mid i \in \Gamma\}$ and $\mu \in I^X$.

Theorem 3.6. Let $(X, \hat{\delta}, \check{\delta})$ be a principal double fuzzy preproximity space. Define the maps $\hat{T}_{\hat{\delta}}, \check{T}_{\check{\delta}}: I^X \rightarrow I \otimes I$ by

$$\hat{T}_{\hat{\delta}}(\lambda) = \hat{\delta}(\lambda, \lambda)'$$

and

$$\check{T}_{\check{\delta}}(\lambda) = \check{\delta}(\lambda, \lambda)'$$

for each $\lambda \in I^X$. Then $(\hat{T}_{\hat{\delta}}, \check{T}_{\check{\delta}})$ is a double fuzzy topology on X .

Proof. For $\hat{T}_{\hat{\delta}}$ we have:

$$(1) \quad \hat{T}_{\hat{\delta}}(\underline{0}) = \hat{T}_{\hat{\delta}}(\underline{1}) = 1.$$

(2)

$$\begin{aligned} \hat{T}_{\hat{\delta}}(\lambda_1, \lambda_2) &= \hat{\delta}(\lambda_1 \wedge \lambda_2, \lambda_1' \vee \lambda_2') \\ &= \hat{\delta}(\lambda_1 \wedge \lambda_2, \lambda_1') \wedge \hat{\delta}(\lambda_1 \wedge \lambda_2, \lambda_2') \\ &\geq \hat{\delta}(\lambda_1, \lambda_1') \wedge \hat{\delta}(\lambda_2, \lambda_2') \\ &= \hat{T}_{\hat{\delta}}(\lambda_1) \wedge \hat{T}_{\hat{\delta}}(\lambda_2). \end{aligned}$$

(3)

$$\begin{aligned} \hat{T}_{\hat{\delta}}(\bigvee_i \lambda_i) &= \hat{\delta}(\bigvee_i \lambda_i, \bigwedge_i \lambda_i') \geq \bigwedge_i \hat{\delta}(\lambda_i, \bigwedge_i \lambda_i') \\ &\geq \bigwedge_i \hat{\delta}(\lambda_i, \lambda_i') = \bigwedge_i \hat{T}_{\hat{\delta}}(\lambda_i). \end{aligned}$$

For $\check{T}_{\check{\delta}}$ we have,

$$(1)' \quad \check{T}_{\check{\delta}}(\underline{0}) = \check{T}_{\check{\delta}}(\underline{1}) = 0.$$

(2)'

$$\begin{aligned} \check{T}_{\check{\delta}}(\lambda_1 \wedge \lambda_2) &= \check{\delta}(\lambda_1 \wedge \lambda_2, \lambda_1' \vee \lambda_2') \\ &= \check{\delta}(\lambda_1 \wedge \lambda_2, \lambda_1') \vee \check{\delta}(\lambda_1 \wedge \lambda_2, \lambda_2') \\ &\leq \check{\delta}(\lambda_1, \lambda_1') \vee \check{\delta}(\lambda_2, \lambda_2') \\ &= \check{T}_{\check{\delta}}(\lambda_1) \vee \check{T}_{\check{\delta}}(\lambda_2). \end{aligned}$$

(3)'

$$\begin{aligned} \check{T}_{\check{\delta}}(\bigvee_i \lambda_i) &= \check{\delta}(\bigvee_i \lambda_i, \bigwedge_i \lambda_i') \leq \bigvee_i \check{\delta}(\lambda_i, \lambda_i') \\ &= \bigvee_i \check{T}_{\check{\delta}}(\lambda_i). \end{aligned}$$

□

Theorem 3.7. Let $f^{\leftarrow}: (X, \hat{\delta}_1, \check{\delta}_1) \rightarrow (Y, \hat{\delta}_2, \check{\delta}_2)$ be a double fuzzy preproximal map. Then $f^{\leftarrow}: (X, \hat{T}_{\hat{\delta}_1}, \check{T}_{\check{\delta}_1}) \rightarrow (Y, \hat{T}_{\hat{\delta}_2}, \check{T}_{\check{\delta}_2})$ is a double fuzzy continuous with respect to the induced double fuzzy topological spaces $(X, \hat{T}_{\hat{\delta}_1}, \check{T}_{\check{\delta}_1})$ and $(Y, \hat{T}_{\hat{\delta}_2}, \check{T}_{\check{\delta}_2})$.

Proof. Since

$$\begin{aligned} \hat{T}_{\hat{\delta}_1}(f^{\leftarrow}(\nu)) &= \hat{\delta}_1(f^{\leftarrow}(\nu), f^{\leftarrow}(\nu)') \leq \hat{\delta}_2(\nu, \nu')' \\ &= \hat{T}_{\hat{\delta}_2}(\nu) \end{aligned}$$

and

$$\begin{aligned} \check{T}_{\check{\delta}_1}(f^{\leftarrow}(\nu)) &= \check{\delta}_1(f^{\leftarrow}(\nu), f^{\leftarrow}(\nu)') \geq \check{\delta}_2(\nu, \nu')' \\ &= \check{T}_{\check{\delta}_2}(\nu). \end{aligned}$$

For each $\nu \in I^Y$. Thus $f^{\leftarrow}$ is a double fuzzy continuous. □

Theorem 3.8. Let $(X, \hat{\delta}, \check{\delta})$ be a double fuzzy preproximity space. Define the map $C_{\hat{\delta}, \check{\delta}}: I^X \times I \otimes I \rightarrow I^X$ by

$$C_{\hat{\delta}, \check{\delta}}(\lambda, r, s) = \bigwedge \{\nu' \mid \hat{\delta}(\nu, \lambda) < r' \text{ and } \check{\delta}(\nu, \lambda) > s'\}$$

Then we have the following properties:

$$(1) \quad C_{\hat{\delta}, \check{\delta}}(\underline{0}, r, s) = \underline{0}.$$

$$(2) \quad \lambda \leq C_{\hat{\delta}, \check{\delta}}(\lambda, r, s).$$

$$(3) \quad \text{If } \lambda_1 \leq \lambda_2, \text{ then } C_{\hat{\delta}, \check{\delta}}(\lambda_1, r, s) \leq C_{\hat{\delta}, \check{\delta}}(\lambda_2, r, s).$$

$$(4) \quad \text{If } r_1 \leq r_2 \text{ and } s_1 \geq s_2, \text{ then } C_{\hat{\delta}, \check{\delta}}(\lambda, r_1, s_1) \leq C_{\hat{\delta}, \check{\delta}}(\lambda, r_2, s_2).$$

$$(5) \quad C_{\hat{\delta}, \check{\delta}}(\lambda \vee \mu, r, s) = C_{\hat{\delta}, \check{\delta}}(\lambda, r, s) \vee C_{\hat{\delta}, \check{\delta}}(\mu, r, s).$$

Proof. (1), (2), (3), and (4) are trivial, we only prove (5):

From (4) we have

$$\mathcal{C}_{\hat{\delta}, \check{\delta}}(\lambda, r, s) \vee \mathcal{C}_{\hat{\delta}, \check{\delta}}(\mu, r, s) \leq \mathcal{C}_{\hat{\delta}, \check{\delta}}(\lambda \vee \mu, r, s).$$

Now suppose

$$\mathcal{C}_{\hat{\delta}, \check{\delta}}(\lambda \vee \mu, r, s) \not\leq \mathcal{C}_{\hat{\delta}, \check{\delta}}(\lambda, r, s) \vee \mathcal{C}_{\hat{\delta}, \check{\delta}}(\mu, r, s),$$

i.e., $\exists t \in (0, 1)$ and $x \in X$ such that

$$\mathcal{C}_{\hat{\delta}, \check{\delta}}(\lambda \vee \mu, r, s)(x) > t > \mathcal{C}_{\hat{\delta}, \check{\delta}}(\lambda, r, s)(x) \vee \mathcal{C}_{\hat{\delta}, \check{\delta}}(\mu, r, s)(x).$$

Since $\mathcal{C}_{\hat{\delta}, \check{\delta}}(\lambda, r, s)(x) < t$. By using the definition of $\mathcal{C}_{\hat{\delta}, \check{\delta}}$, $\exists \nu_1 \in I^X$ such that

$$\mathcal{C}_{\hat{\delta}, \check{\delta}}(\lambda, r, s)(x) \leq \nu_1(x)' < t.$$

and ν_1 satisfies conditions $\hat{\delta}(\nu_1, \lambda) < r'$ and $\check{\delta}(\nu_1, \lambda) > s'$.

Similarly for case $\mathcal{C}_{\hat{\delta}, \check{\delta}}(\mu, r, s)(x) < t$, $\exists \nu_2 \in I^X$ such that

$$\mathcal{C}_{\hat{\delta}, \check{\delta}}(\mu, r, s)(x) \leq \nu(x)' < t$$

and

$$\hat{\delta}(\nu_2, \mu) < r', \check{\delta}(\nu_2, \mu) > s'.$$

Thus

$$\hat{\delta}(\nu_1 \wedge \nu_2, \lambda \vee \mu) \leq \hat{\delta}(\nu_1, \lambda) \vee \hat{\delta}(\nu_2, \mu) < r'$$

and

$$\check{\delta}(\nu_1 \wedge \nu_2, \lambda \vee \mu) \geq \check{\delta}(\nu_1, \lambda) \wedge \check{\delta}(\nu_2, \mu) > s'.$$

Thus from the definition of $\mathcal{C}_{\hat{\delta}, \check{\delta}}$, we have

$$\mathcal{C}_{\hat{\delta}, \check{\delta}}(\lambda \vee \mu, r, s) \leq (\nu_1 \wedge \nu_2)(x)'$$

which is contradiction. Thus

$$\mathcal{C}_{\hat{\delta}, \check{\delta}}(\lambda \vee \mu, r, s) \leq \mathcal{C}_{\hat{\delta}, \check{\delta}}(\lambda, r, s) \vee \mathcal{C}_{\hat{\delta}, \check{\delta}}(\mu, r, s).$$

□

Theorem 3.9. Let $(X, \hat{\delta}, \check{\delta})$ be a double fuzzy preproximity space and let $\mathcal{C}_{\hat{\delta}, \check{\delta}}$ be a double fuzzy closure operator induced by $(\hat{\delta}, \check{\delta})$. Define the maps $\hat{\mathcal{T}}_{\mathcal{C}_{\hat{\delta}, \check{\delta}}}, \check{\mathcal{T}}_{\mathcal{C}_{\hat{\delta}, \check{\delta}}}: I^X \rightarrow I \otimes I$ by

$$\hat{\mathcal{T}}_{\mathcal{C}_{\hat{\delta}, \check{\delta}}}(\lambda) = \bigvee \{r \in I \mid \mathcal{C}_{\hat{\delta}, \check{\delta}}(\lambda', r, s) = \lambda'\}$$

and

$$\check{\mathcal{T}}_{\mathcal{C}_{\hat{\delta}, \check{\delta}}}(\lambda) = \bigwedge \{s \in I \mid \mathcal{C}_{\hat{\delta}, \check{\delta}}(\lambda', r, s) = \lambda'\}.$$

Then $(\hat{\mathcal{T}}_{\mathcal{C}_{\hat{\delta}, \check{\delta}}}, \check{\mathcal{T}}_{\mathcal{C}_{\hat{\delta}, \check{\delta}}})$ is a double fuzzy topology on X .

Remark 3.10. Let $(\check{\delta}, \hat{\delta})$ be a double fuzzy preproximity. We define $\hat{\delta}^{-1}(\lambda, \mu) = \hat{\delta}(\mu, \lambda)$ and $\check{\delta}^{-1}(\lambda, \mu) = \check{\delta}(\mu, \lambda)$, for each $\lambda, \mu \in I^X$. Then $(\hat{\delta}^{-1}, \check{\delta}^{-1})$ is a double fuzzy preproximity on X .

Theorem 3.11. Let $(X, \hat{\delta}_1, \check{\delta}_1)$ and $(Y, \hat{\delta}_2, \check{\delta}_2)$ be two double fuzzy preproximity spaces. If $f^\rightarrow: (X, \hat{\delta}_1, \check{\delta}_1) \rightarrow (Y, \hat{\delta}_2, \check{\delta}_2)$ is a double fuzzy preproximal map. Then:

(1) $f^\rightarrow: (X, \mathcal{C}_{\hat{\delta}_1, \check{\delta}_1}) \rightarrow (Y, \mathcal{C}_{\hat{\delta}_2, \check{\delta}_2})$ is C-map.

(2) $\mathcal{C}_{\hat{\delta}_1, \check{\delta}_1}(f^\leftarrow(\nu), r, s) \leq f^\leftarrow(\mathcal{C}_{\hat{\delta}_2, \check{\delta}_2}(\nu, r, s))$, for each $\nu \in I^Y$.

(3) $f^\rightarrow: (X, \hat{\mathcal{T}}_{\mathcal{C}_{\hat{\delta}_1, \check{\delta}_1}}, \check{\mathcal{T}}_{\mathcal{C}_{\hat{\delta}_1, \check{\delta}_1}}) \rightarrow (Y, \hat{\mathcal{T}}_{\mathcal{C}_{\hat{\delta}_2, \check{\delta}_2}}, \check{\mathcal{T}}_{\mathcal{C}_{\hat{\delta}_2, \check{\delta}_2}})$ is a double fuzzy continuous.

Proof. As to (1):

$$\begin{aligned} & f^\leftarrow(\mathcal{C}_{\hat{\delta}_2, \check{\delta}_2}(f^\rightarrow(\lambda), r, s)) \\ &= f^\leftarrow(\bigwedge \{\nu' \mid \hat{\delta}_2(\nu, f^\rightarrow(\lambda)) < r', \check{\delta}_2(\nu, f^\rightarrow(\lambda)) > s'\}) \\ &\geq \bigwedge \{f^\leftarrow(\nu)' \mid \hat{\delta}_1(f^\leftarrow(\nu), \lambda) < r', \check{\delta}_1(f^\leftarrow(\nu), \lambda) > s'\} \\ &= \mathcal{C}_{\hat{\delta}_1, \check{\delta}_1}(\lambda, r, s). \end{aligned}$$

As to (2): For each $\nu \in I^Y$, put $\lambda = f^\leftarrow(\nu)$, from (1)

$$\begin{aligned} f^\leftarrow(\mathcal{C}_{\hat{\delta}_1, \check{\delta}_1}(f^\leftarrow(\nu), r, s)) &\leq \mathcal{C}_{\hat{\delta}_2, \check{\delta}_2}(f^\leftarrow(f^\leftarrow(\nu)), r, s) \\ &\leq \mathcal{C}_{\hat{\delta}_2, \check{\delta}_2}(\nu, r, s). \end{aligned}$$

Since $f^\rightarrow f^\leftarrow(\nu) \leq \nu$. Thus

$$\mathcal{C}_{\hat{\delta}_1, \check{\delta}_1}(f^\leftarrow(\nu), r, s) \leq f^\leftarrow(\mathcal{C}_{\hat{\delta}_2, \check{\delta}_2}(\nu, r, s)),$$

for each $\nu \in I^Y$.

As to (3): Trivial. □

Theorem 3.12. Let $(X, \hat{\mathcal{T}}, \check{\mathcal{T}})$ be a double fuzzy topological space. Define the maps $\delta_{\hat{\mathcal{T}}}, \delta_{\check{\mathcal{T}}}: I^X \times I^X \rightarrow I \otimes I$ as follows:

$$\delta_{\hat{\mathcal{T}}}(\lambda, \mu) = \begin{cases} \bigwedge \{\hat{\mathcal{T}}(\gamma)' \mid \gamma \in \Phi_{\lambda, \mu}\}, & \text{if } \Phi_{\lambda, \mu} \neq \emptyset \\ 1 & \text{if } \Phi_{\lambda, \mu} = \emptyset \end{cases}$$

and

$$\delta_{\check{\mathcal{T}}}(\lambda, \mu) = \begin{cases} \bigvee \{\check{\mathcal{T}}(\gamma)' \mid \gamma \in \Phi_{\lambda, \mu}\}, & \text{if } \Phi_{\lambda, \mu} \neq \emptyset \\ 0 & \text{if } \Phi_{\lambda, \mu} = \emptyset \end{cases}$$

Such that $\Phi_{\lambda, \mu} = \{\gamma \mid \lambda \leq \gamma \leq \mu'\}$. Then $(\delta_{\hat{\mathcal{T}}}, \delta_{\check{\mathcal{T}}})$ is a double fuzzy preproximity on X .

Proof. We only prove $\delta_{\tilde{T}}$, for (DFP1),(DFP2) and (DFP3) trivial.

(DFP4). From (DFP3), we have

$$\delta_{\tilde{T}}(\nu, \lambda \vee \mu) \leq \delta_{\tilde{T}}(\nu, \lambda) \wedge \delta_{\tilde{T}}(\nu, \mu).$$

Now

$$\begin{aligned} \delta_{\tilde{T}}(\nu, \lambda) \wedge \delta_{\tilde{T}}(\nu, \mu) &= \bigvee \{ \tilde{T}(\gamma_1)' \mid \gamma_1 \in \Phi_{\nu, \lambda} \} \\ &\wedge \bigvee \{ \tilde{T}(\gamma_2)' \mid \gamma_2 \in \Phi_{\nu, \mu} \} \\ &= \bigvee \{ \tilde{T}(\gamma_1)' \wedge \tilde{T}(\gamma_2)' \mid \gamma_1 \in \Phi_{\nu, \lambda}, \gamma_2 \in \Phi_{\nu, \mu} \} \\ &= \bigvee \{ (\tilde{T}(\gamma_1) \vee \tilde{T}(\gamma_2))' \mid \gamma_1 \in \Phi_{\nu, \lambda}, \gamma_2 \in \Phi_{\nu, \mu} \} \\ &\leq \bigvee \{ (\tilde{T}(\gamma_1 \wedge \gamma_2))' \mid \gamma_1 \wedge \gamma_2 \in \Phi_{\nu, \lambda \vee \mu} \} \\ &= \delta_{\tilde{T}}(\nu, \lambda \vee \mu). \end{aligned}$$

□

Theorem 3.13. Let $(X, \hat{T}_1, \check{T}_1)$ and $(Y, \hat{T}_2, \check{T}_2)$ be two double fuzzy topological spaces and Let $f^-: (X, \hat{T}_1, \check{T}_1) \rightarrow (Y, \hat{T}_2, \check{T}_2)$ be a double fuzzy continuous map. Then $f^-: (X, \delta_{\hat{T}_1}, \delta_{\check{T}_1}) \rightarrow (Y, \delta_{\hat{T}_2}, \delta_{\check{T}_2})$ is a double fuzzy preproximal map with respect to induced double fuzzy preproximity spaces $(X, \delta_{\hat{T}_1}, \delta_{\check{T}_1})$ and $(Y, \delta_{\hat{T}_2}, \delta_{\check{T}_2})$, respectively.

Proof. For $\delta_{\hat{T}}$:

$$\begin{aligned} \delta_{\hat{T}_2}(\nu, \gamma) &= \bigwedge \{ \hat{T}_2(\zeta)' \mid \zeta \in \Phi_{\nu, \gamma} \} \\ &\geq \bigwedge \{ \hat{T}_1(f^-(\zeta))' \mid f^-(\zeta) \in \Phi_{f^-(\nu), f^-(\gamma)} \} \\ &= \delta_{\hat{T}_1}(f^-(\nu), f^-(\gamma)). \end{aligned}$$

For $\delta_{\check{T}}$:

$$\begin{aligned} \delta_{\check{T}_2}(\nu, \gamma) &= \bigvee \{ \check{T}_2(\zeta)' \mid \zeta \in \Phi_{\nu, \gamma} \} \\ &\leq \bigvee \{ \check{T}_1(f^-(\zeta))' \mid f^-(\zeta) \in \Phi_{f^-(\nu), f^-(\gamma)} \} \\ &= \delta_{\check{T}_1}(f^-(\nu), f^-(\gamma)). \end{aligned}$$

Thus f^- is a double fuzzy preproximal map. □

4. Double Fuzzy Preproximal Neighborhood

We are going to introduce the concepts of double fuzzy neighborhood and double fuzzy preproximal neighborhood.

Definition 4.1. Let X be a set. A map $\mathcal{N}: X \times I^X \times I \otimes I \rightarrow I \otimes I$ is called double fuzzy neighborhood system iff $\mathcal{N}$ satisfies the following axioms:

$$(N_0) \quad \mathcal{N}(x, \underline{1}, r, s) = 1.$$

$$(N_1) \quad \text{If } \lambda \leq \mu \Rightarrow \mathcal{N}(x, \lambda, r, s) \leq \mathcal{N}(x, \mu, r, s).$$

$$(N_2) \quad \text{If } r_1 \leq r_2, s_1 \geq s_2 \text{ then } \mathcal{N}(x, \lambda, r_2, s_2) \leq \mathcal{N}(x, \lambda, r_1, s_1).$$

$$(N_3) \quad \mathcal{N}(x, \lambda, r, s) \wedge \mathcal{N}(x, \mu, r, s) = \mathcal{N}(x, \lambda \wedge \mu, r, s).$$

$$(N_4) \quad \mathcal{N}(x, \lambda, r, s) \leq \lambda(x).$$

Theorem 4.2. Let $(X, \hat{\delta}, \check{\delta})$ be a double fuzzy preproximity space. Define a map $\mathcal{N}_{\hat{\delta}, \check{\delta}}: X \times I^X \times I \otimes I \rightarrow I \otimes I$ by :

$$\mathcal{N}_{\hat{\delta}, \check{\delta}}(x, \lambda, r, s) = \bigvee \{ \rho(x) \mid \hat{\delta}(\rho, \lambda') < r', \check{\delta}(\rho, \lambda') > s' \}.$$

Then $\mathcal{N}_{\hat{\delta}, \check{\delta}}$ is a double fuzzy neighborhood system we shall denoted it by double fuzzy preproximal neighborhood.

Proof. (N_0) Since $\hat{\delta}(\underline{1}, \underline{0}) = 0$ and $\check{\delta}(\underline{1}, \underline{0}) = 1$, $\mathcal{N}_{\hat{\delta}, \check{\delta}}(x, \underline{1}, r, s) = 1$.

(N_1) Since $\hat{\delta}(\rho, \lambda') < r'$ and $\check{\delta}(\rho, \lambda') > s'$ by (DFP2), $\rho \leq \lambda$. Thus $\mathcal{N}_{\hat{\delta}, \check{\delta}}(x, \lambda, r, s) \leq \lambda(x)$.

(N_2) For $\lambda \leq \mu$ and $r_1 \leq r_2, s_2 \leq s_1$ since

$$\hat{\delta}(\rho, \mu') \leq \hat{\delta}(\rho, \lambda') < r'_2 \leq r'_1$$

and

$$\hat{\delta}(\rho, \mu') \geq \hat{\delta}(\rho, \lambda') \geq s'_2 \geq s'_1,$$

we have

$$\mathcal{N}_{\hat{\delta}, \check{\delta}}(x, \lambda, r_1, s_1) \geq \mathcal{N}_{\hat{\delta}, \check{\delta}}(x, \lambda, r_2, s_2).$$

(N_3)

$$\begin{aligned} \mathcal{N}_{\hat{\delta}, \check{\delta}}(x, \lambda_1, r, s) \wedge \mathcal{N}_{\hat{\delta}, \check{\delta}}(x, \lambda_2, r, s) &= \\ &= \left(\bigvee \{ \rho_1 \in I^X \mid \hat{\delta}(\rho_1, \lambda'_1) < r', \check{\delta}(\rho_1, \lambda'_1) > s' \} \right) \\ &\wedge \left(\bigvee \{ \rho_2 \in I^X \mid \hat{\delta}(\rho_2, \lambda'_2) < r', \check{\delta}(\rho_2, \lambda'_2) > s' \} \right) \\ &= \left(\bigvee \{ \rho_1 \wedge \rho_2 \in L^X \mid \hat{\delta}(\rho_1, \lambda'_1) < r', \check{\delta}(\rho_2, \lambda'_2) < r' \right. \\ &\quad \left. , \hat{\delta}(\rho_1, \lambda'_1) > s', \check{\delta}(\rho_2, \lambda'_2) > s' \} \right) \\ &\leq \left(\bigvee \{ \rho_1 \wedge \rho_2 \in I^X \mid \hat{\delta}(\rho_1 \wedge \rho_2, \lambda'_1 \vee \lambda'_2) < r' \right. \\ &\quad \left. , \check{\delta}(\rho_1 \wedge \rho_2, \lambda'_1 \vee \lambda'_2) > s' \} \right) \\ &\leq \left(\bigvee \{ \mu \in L^X \mid \hat{\delta}(\mu, (\lambda_1 \wedge \lambda_2)') < r' \right. \\ &\quad \left. , \check{\delta}(\mu, \lambda'_1 \vee \lambda'_2) > s' \} \right) = \mathcal{N}_{\hat{\delta}, \check{\delta}}(x, \lambda_1 \wedge \lambda_2, r, s). \end{aligned}$$

□

As a consequence of the previous Theorem, we stat the following proposition.

Proposition 4.3. Let $\mathcal{N}_{\hat{\delta}, \check{\delta}}$ be a double fuzzy preproximal neighborhood induced by a double fuzzy preproximity $(\hat{\delta}, \check{\delta})$. Define the map $\mathcal{C}_{\mathcal{N}_{\hat{\delta}, \check{\delta}}}: I^X \times I \otimes I \rightarrow I$ by:

$$\mathcal{C}_{\mathcal{N}_{\hat{\delta}, \check{\delta}}}(\lambda, r, s)(x) = \mathcal{N}_{\hat{\delta}, \check{\delta}}(x, \lambda', r, s)'$$

is a double fuzzy closure operator on X .

Proof.(CO1) $\mathcal{C}_{\mathcal{N}_{\hat{\delta}, \check{\delta}}}(\underline{0}, r, s)(x) = \mathcal{N}_{\hat{\delta}, \check{\delta}}(x, \underline{1}, r, s)' = 0$.

(CO2) Since $\mathcal{N}_{\hat{\delta}, \check{\delta}}(x, \lambda', r, s) \leq \lambda'(x)$,

$$\lambda(x) \leq \mathcal{N}_{\hat{\delta}, \check{\delta}}(x, \lambda', r, s)' = \mathcal{C}_{\mathcal{N}_{\hat{\delta}, \check{\delta}}}(\lambda, r, s)(x)$$

for each $x \in X$. Thus $\lambda \leq \mathcal{C}_{\mathcal{N}_{\hat{\delta}, \check{\delta}}}(\lambda, r, s)$.

(CO3) If $\lambda \leq \mu$ and $r_1 \leq r_2$ and $s_1 \geq s_2$ then

$$\mathcal{N}_{\hat{\delta}, \check{\delta}}(x, \mu', r_2, s_2) \leq \mathcal{N}_{\hat{\delta}, \check{\delta}}(x, \lambda', r_1, s_1).$$

Thus

$$\mathcal{N}_{\hat{\delta}, \check{\delta}}(x, \lambda', r_1, s_1)' \leq \mathcal{N}_{\hat{\delta}, \check{\delta}}(x, \mu', r_2, s_2)',$$

i.e.,

$$\mathcal{C}_{\mathcal{N}_{\hat{\delta}, \check{\delta}}}(\lambda, r_1, s_1)(x) \leq \mathcal{C}_{\mathcal{N}_{\hat{\delta}, \check{\delta}}}(\mu, r_2, s_2)(x)$$

for each $x \in X$.

(CO4)

$$\begin{aligned} \mathcal{C}_{\mathcal{N}_{\hat{\delta}, \check{\delta}}}(\lambda \vee \mu, r, s)(x) &= \mathcal{N}_{\hat{\delta}, \check{\delta}}(x, (\lambda \vee \mu)', r, s)' \\ &= \mathcal{N}_{\hat{\delta}, \check{\delta}}(x, \lambda' \wedge \mu', r, s)' \\ &= \mathcal{N}_{\hat{\delta}, \check{\delta}}(x, \lambda', r, s)' \vee \mathcal{N}_{\hat{\delta}, \check{\delta}}(x, \mu', r, s)' \\ &= \mathcal{C}_{\mathcal{N}_{\hat{\delta}, \check{\delta}}}(\lambda, r, s)(x) \vee \mathcal{C}_{\mathcal{N}_{\hat{\delta}, \check{\delta}}}(\mu, r, s)(x). \end{aligned}$$

□

Theorem 4.4. Let $(X, \hat{\delta}, \check{\delta})$ be a double fuzzy preproximity space and let $\mathcal{C}_{\hat{\delta}, \check{\delta}}$ be a double fuzzy closure operator defined as Theorem (3.8). Then the map $\mathcal{N}_{\mathcal{C}_{\hat{\delta}, \check{\delta}}}: X \times I^X \times I \otimes I \rightarrow I \otimes I$ defined by:

$$\mathcal{N}_{\mathcal{C}_{\hat{\delta}, \check{\delta}}}(x, \lambda, r, s) = \mathcal{C}_{\hat{\delta}, \check{\delta}}(\lambda', r, s)'(x)$$

for each $x \in X$ is a double fuzzy preproximal neighborhood system.

Proof. Straightforward. □

Proposition 4.5. Let $\mathcal{N}_{\hat{\delta}, \check{\delta}}$ be a double fuzzy preproximal neighborhood and let $\mathcal{C}_{\mathcal{N}_{\hat{\delta}, \check{\delta}}}$ be a double fuzzy closure operator defined as Proposition (4.3). The functions $\hat{\mathcal{T}}_{\mathcal{C}_{\mathcal{N}_{\hat{\delta}, \check{\delta}}}}, \check{\mathcal{T}}_{\mathcal{C}_{\mathcal{N}_{\hat{\delta}, \check{\delta}}}}: I^X \rightarrow I \otimes I$ defined by:

$$\hat{\mathcal{T}}_{\mathcal{C}_{\mathcal{N}_{\hat{\delta}, \check{\delta}}}}(\lambda) = \bigvee \{r \in I \mid \mathcal{C}_{\mathcal{N}_{\hat{\delta}, \check{\delta}}}(\lambda', r, s) = \lambda'\}$$

and

$$\check{\mathcal{T}}_{\mathcal{C}_{\mathcal{N}_{\hat{\delta}, \check{\delta}}}}(\lambda) = \bigwedge \{s \in I \mid \mathcal{C}_{\mathcal{N}_{\hat{\delta}, \check{\delta}}}(\lambda', r, s) = \lambda'\}$$

is a double fuzzy topology.

References

- [1] K. Atanassove ; *Intuitionistic fuzzy sets*, Fuzzy sets and systems 20(1984), 87-90.
- [2] K. Atanassove ; *New operators defined over the intuitionistic fuzzy sets*, Fuzzy Sets and Systems, 61(2)(1993) 131-142.
- [3] C. L. Chang ; *Fuzzy topological spaces*, J. Math. Anal. Appl. 24(1968), 182-190.
- [4] D. Çoker ; *An introduction to fuzzy subspaces in intuitionistic fuzzy topological spaces*, J. Fuzzy Math. 4(1996), 749-764.
- [5] D. Çoker, M. Demirci ; *An introduction to intuitionistic fuzzy topological spaces in Šostak's sense*, BUSE-FAL, 67(1996), 67-76.
- [6] D. Çoker ; *An introduction to intuitionistic fuzzy topological spaces*, Fuzzy sets and systems, 88(1997), 81-89.
- [7] R. Badard, A. A. Ramadan and A. S. Mashhour ; *Smooth preuniform and preproximity spaces*, Fuzzy Sets and Systems, 59(1993), 95-107.
- [8] Y. C. Kim, K. C. Min ; *L-fuzzy preproximities and L-fuzzy topologies*, Information sciences, 173(2005), 93-113.
- [9] E. P. Lee, Y. B. Im ; *Mated fuzzy topological spaces*, Int. J. of Fuzz. Log. & Intil. Sys., 11(2) (2001), 161-165.
- [10] S. K. Samanta, T. K. Mondal ; *Intuitionistic gradation of openness: intuitionistic fuzzy topology*, Busefal, 73(1997), 8-17.
- [11] S. K. Samanta and T. K. Mondal ; *On intuitionistic gradation of openness*, Fuzzy Sets and Systems, 131(2002) 323-336.
- [12] S. K. Samanta ; *Fuzzy proximities and fuzzy uniformities*, Fuzzy sets and systems, vol. 70(1995).
- [13] K. C. Chattopadhyay, U. K. Mukherjee, S.K. Samanta ; *Fuzzy proximity structures and fuzzy grills*, Fuzzy Sets and Systems, 79(1996), 383-393.

- [14] K. C. Chattopadhyay, U. K. Mukherjee, S.K. Samanta ; *Product of f -proximities in fuzzy setting*, Fuzzy Sets and Systems, 121(2001), 191-201.
- [15] M. H. Ghanim, O. A. Tantawy, Fawzia M. Selim ; *Gradations of uniformity and gradation of proximity*, Fuzzy sets and systems, vol. 79 (1996), 373-382.
- [16] A. P. Šostak ; *On a fuzzy topological structure*, Suppl. Rend. Circ. Matem. Palermo, ser. II 11(1985).
- [17] U. Höhle, A. P. Sostak ; *Axiomatic foundation of fixed-basis fuzzy topology*, in: U. Hohle, S. E. Rodabaugh (Eds.), The handbooks of Fuzzy sets series, Vol.3, Kluwer academic publishers, Dordrecht, 1990.
- [18] K. Atanassov, S. Stoeva ; *Intuitionistic fuzzy sets*, Proc. of Polish Symp. on Interval and Fuzzy Mathematics, Poznan, August (1983), 23-26.
- [19] J. G. Garcia and S. E. Rodabaugh; *Order-theoretic, topological, categorical redundancies of interval-valued sets, grey sets, vague sets, interval-valued "intuitionistic" sets, "intuitionistic" fuzzy sets and topologies*, Fuzzy sets and systems, 156 (2005), 445-484.

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