-Fuzzy Extremally Disconnected Ideal Topological Spaces

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Abstract

The notion of $\star$ -Externally disconnected fuzzy ideal topological spaces is introduced and studied. Many characterizations of the space are obtained.

Key Words : r-fuzzy semi-I-open, r-fuzzy pre-I-open, r-fuzzy α -I-open and r-fuzzy β -I-open sets, fuzzy ideal continuity and fuzzy ideal α -continuity.

1. Introduction and Preliminaries

$\hat{S}$ ostak [17], introduce the fundamental concept of fuzzy topological structure as an extension of both crisp topology and Chang's fuzzy topology [2], in the sense that not only the object were fuzzified, but also the axiomatics. Chattopdhyay et al. [3,4] have redefined the similar concept. In [16] Ramadan gave a similar definition namely "Smooth fuzzy topology".

Fuzzy extremally disconnected spaces play an important role in fuzzy set-theoretical topology, studying Stone-Ćech fuzzy compactification and the Stone spaces of any complete Boolean algebra, the theory of Boolean algebra, axiomatic set theory, functional analysis, C^* -algebra, studying the space **Seq** (ζ) etc. In this paper, the class of $\star$ -Fuzzy extremally disconnected ideal topological spaces is introduced and studied. Also, properties of $\star$ -Fuzzy extremally disconnected ideal topological spaces are discussed in $\hat{S}$ ostak sense. We must point out that; the concept of fuzzy topological spaces, has been a significant concept in string theory and E-infinity theory pertaining to quantum particular physics ever since El-Naschie ([5-15]).

Throughout this paper, let X be a nonempty set $I = [0, 1]$ and $I_0 = (0, 1]$. For $\alpha \in I$, $\bar{\alpha}(x) = \alpha$ for all $x \in X$. The family of all fuzzy sets on X denoted by I^X . For two fuzzy sets we write $\lambda q \mu$ to mean that λ is quasi-coincident (q-coincident, for short) with μ , i.e, there exists at least one point $x \in X$ such that $\lambda(x) + \mu(x) > 1$. Negation of such a statement is denoted as $\lambda \bar{q} \mu$.

Definition 1.1[17]. A mapping $\tau : I^X \rightarrow I$ is called a fuzzy topology on X if it satisfies the following conditions:

$$(O1) \tau(\bar{0}) = \tau(\bar{1}) = \bar{1}.$$

$$(O2) \tau(\bigvee_{i \in \Gamma} \mu_i) \geq \bigwedge_{i \in \Gamma} \tau(\mu_i), \text{ for any } \{\mu_i\}_{i \in \Gamma} \in I^X.$$

$$(O3) \tau(\mu_1 \wedge \mu_2) \geq \tau(\mu_1) \wedge \tau(\mu_2), \text{ for any } \mu_1, \mu_2 \in I^X.$$

The pair (X, τ) is called a fuzzy topological space (for short, fts).

Definition 1.2[1]. A mapping $\mathbf{I} : I^X \rightarrow I$ is called fuzzy ideal on X iff:

$$(1) \text{ If } \lambda \leq \mu, \text{ then } \mathbf{I}(\lambda) \geq \mathbf{I}(\mu), \text{ for each } \lambda, \mu \in I^X.$$

$$(2) \text{ For each } \lambda, \mu \in I^X, \quad \mathbf{I}(\lambda \vee \mu) \geq \mathbf{I}(\lambda) \wedge \mathbf{I}(\mu) \text{ [finite additivity].}$$

Definition 1.3[1]. Let $(X, \tau, \mathbf{I})$ be a fuzzy ideal topological space. Let $\mu, \lambda \in I^X$, the r-fuzzy open local function μ_r^* of μ is defined by the union of all fuzzy points x_t such that if $\rho \in Q(x_t, r)$ and $\mathbf{I}(\lambda) \geq r$ then there is at least one $y \in X$ for which $\rho(y) + \mu(y) - 1 > \lambda(y)$.

Theorem 1.1[4]. Let (X, τ) be a fts. Then for each $r \in I_0$, $\lambda \in I^X$ we define an operator $C_\tau : I^X \times I_0 \rightarrow I^X$ as follows:

$$C_\tau(\lambda, r) = \bigwedge \{ \mu \in I^X : \lambda \leq \mu, \tau(\bar{1} - \mu) \geq r \}.$$

For $\lambda, \mu \in I^X$ and $r, s \in I_0$, the operator C_τ satisfies the following conditions:

$$(1) C_\tau(\bar{0}, r) = \bar{0}.$$

$$(2) \lambda \leq C_\tau(\lambda, r).$$

$$(3) C_\tau(\lambda, r) \vee C_\tau(\mu, r) = C_\tau(\lambda \vee \mu, r).$$

$$(4) C_\tau(\lambda, r) \leq C_\tau(\lambda, s) \text{ if } r \leq s.$$

$$(5) C_\tau(C_\tau(\lambda, r), r) = C_\tau(\lambda, r).$$

Theorem 1.2[4]. Let (X, τ) be a fts. Then for each

(4) $\Rightarrow$ (5): Let μ be **r-FS β IO** set. Then $Cl^*(\mu, r)$ is **r-FSIO** set. By (4), $Cl^*(\mu, r)$ is **r-FPIO** set. Therefore, $Cl^*(\mu, r) \leq I_\tau(Cl^*(\mu, r), r)$. Hence $\tau(Cl^*(\mu, r)) \geq r$.

(5) $\Rightarrow$ (6): Let μ be **r-FS β IO** set. By (5), we have $Cl^*(\mu, r) \leq I_\tau(Cl^*(\mu, r), r)$. Therefore, $\mu \leq Cl^*(\mu, r) \leq I_\tau(Cl^*(\mu, r), r)$. Hence, μ is **r-FPIO** set.

(6) $\Rightarrow$ (7): Let μ be **r-FSIO** set. Since a **r-FSIO** set is **r-FS β IO**, then by (6) it is μ is **r-FPIO** set. Since μ is **r-FSIO** and **r-FPIO**, therefore, μ is **r-F α IO**

(7) $\Rightarrow$ (1): Let $\tau(\mu) \geq r$, then $Cl^*(\mu, r)$ is **r-FSIO** set and by (7), $Cl^*(\mu, r)$ is **r-F α IO**. Hence

$$\begin{aligned} Cl^*(\mu, r) &\leq I_\tau(Cl^*(I_\tau(Cl^*(\mu, r), r), r), r) \\ &= I_\tau(Cl^*(\mu, r), r) \leq Cl^*(\mu, r). \end{aligned}$$

Hence, $I_\tau(Cl^*(\mu, r), r) = Cl^*(\mu, r)$. Then, $\tau(Cl^*(\mu, r)) \geq r$ and X is $\star$ -Fuzzy externally disconnected.

Theorem 2.3. Let $(X, \tau, \mathbf{I})$ be a fuzzy ideal topological space, $r \in I_0$. The following properties are equivalent:

- (1) X is $\star$ -Fuzzy extremally disconnected.
- (2) $Cl^*(\mu, r) \bar{q} C_\tau(\lambda, r)$, for every $\tau(\mu) \geq r$ and every λ is $\star$ -fuzzy open set where (λ is called $\star$ -fuzzy open set if $\underline{1} - \lambda = Cl^*(\underline{1} - \lambda, r)$) denoted by $\tau^*(\lambda) \geq r$ with $\mu \bar{q} \lambda$.
- (3) $Cl^*(I_\tau(Cl^*(\mu, r), r), r) \bar{q} C_\tau(\lambda, r)$, for every $\mu \in I^X$ and every $\tau^*(\mu) \geq r$ with $\mu \bar{q} \lambda$.

Proof. (2) $\Rightarrow$ (1): Suppose that $Cl^*(\mu, r) \bar{q} C_\tau(\lambda, r)$, for every $\tau(\mu) \geq r$ and every $\tau^*(\lambda) \geq r$ with $\mu \bar{q} \lambda$. Let $\tau(\rho) \geq r$. Since, $\tau(\rho) \geq r$ and $\tau^*(\underline{1} - Cl^*(\rho, r)) \geq r$, are $\rho \bar{q} \underline{1} - Cl^*(\rho, r)$. Therefore, by (2), we have

$$\begin{aligned} Cl^*(\rho, r) &\bar{q} C_\tau(\underline{1} - Cl^*(\rho, r), r) \\ &\Rightarrow Cl^*(\rho, r) \bar{q} \underline{1} - I_\tau(Cl^*(\rho, r), r). \end{aligned}$$

This implies that

$$Cl^*(\rho, r) \leq I_\tau(Cl^*(\rho, r), r) \leq Cl^*(\rho, r).$$

Therefore, $\tau(Cl^*(\rho, r)) \geq r$ and hence X is $\star$ -Fuzzy extremally disconnected.

(1) $\Rightarrow$ (2): For every $\tau(\mu) \geq r$ and every $\tau^*(\lambda) \geq r$. By (1) we have, $\tau(Cl^*(\mu, r)) \geq r$ and since $\tau^*(\underline{1} - Cl^*(\mu, r)) \geq r$, and $Cl^*(\mu, r) \bar{q} \underline{1} - Cl^*(\mu, r)$. Since $\tau(Cl^*(\mu, r)) \geq r$, Then,

$$C_\tau(\underline{1} - Cl^*(\mu, r), r) = \underline{1} - Cl^*(\mu, r).$$

Thus

$$Cl^*(\mu, r) \bar{q} C_\tau(\underline{1} - Cl^*(\mu, r), r).$$

(2) $\Rightarrow$ (3): Let for every $\mu \in I^X$ and every $\tau^*(\lambda) \geq r$ with $\mu \bar{q} \lambda$. Since $\tau(I_\tau(Cl^*(\mu, r), r)) \geq r$ and $Int_\tau(Cl^*(\mu, r), r) \bar{q} \lambda$, by (2),

$$Cl^*(I_\tau(Cl^*(\mu, r), r), r) \bar{q} C_\tau(\lambda, r).$$

(3) $\Rightarrow$ (2): Let for every $\tau(\mu) \geq r$ and every $\tau^*(\lambda) \geq r$ with $\mu \bar{q} \lambda$. By (4), we have $Cl^*((I_\tau(Cl^*(\mu, r)) \bar{q} C_\tau(\lambda, r))$. Since $Cl^*(\mu, r) \leq Cl^*((I_\tau(Cl^*(\mu, r))$, then

$$Cl^*(\mu, r) \bar{q} C_\tau(\lambda, r).$$

Definition 2.4. A fuzzy ideal topological space (X, τ, I) is called $\star$ -Fuzzy normal if $\lambda_1 \bar{q} \lambda_2$ with $\tau(\lambda_1) \geq r$ and $\tau^*(\lambda_2) \geq r$, implies there exist $\mu_i \in I^X$, for $i = \{1, 2\}$ with $\tau(\underline{1} - \mu_1) \geq r$, $\tau^*(\underline{1} - \mu_2) \geq r$ such that $\lambda_2 \leq \mu_1$, $\lambda_1 \leq \mu_2$ and $\mu_1 \bar{q} \mu_2$.

Theorem 2.5. Let $(X, \tau, \mathbf{I})$ be an fits. Then the following properties are equivalent:

- (1) $(X, \tau, \mathbf{I})$ is $\star$ -Fuzzy normal.
- (2) $(X, \tau, \mathbf{I})$ is $\star$ -Fuzzy extremally disconnected.

Proof. (1) $\Rightarrow$ (2): Let $(X, \tau, \mathbf{I})$ be $\star$ -Fuzzy normal and $\tau(\lambda) \geq r$, and since

$$Cl^*(\lambda, r) = Cl^*(Cl^*(\lambda, r), r).$$

Then, $\tau(\lambda) \geq r$, and $\tau^*(\underline{1} - Cl^*(\lambda, r)) \geq r$, with $\lambda \bar{q} \underline{1} - Cl^*(\lambda, r)$. Since $(X, \tau, \mathbf{I})$ is $\star$ -Fuzzy normal. This implies that there exist $\mu_i \in I^X$, for $i = \{1, 2\}$ with $\tau(\underline{1} - \mu_1) \geq r$, $\tau^*(\underline{1} - \mu_2) \geq r$ such that $\lambda \leq \mu_2$, $\underline{1} - Cl^*(\lambda, r) \leq \mu_1$ and $\mu_1 \bar{q} \mu_2$. Since

$$Cl^*(\lambda, r) \leq Cl^*(\mu_2, r) = \mu_2 \leq \underline{1} - \mu_1 \leq Cl^*(\lambda, r).$$

Then, $Cl^*(\lambda, r) = \mu_2$. Since

$$\underline{1} - Cl^*(\lambda, r) \leq \mu_1 \leq \underline{1} - \mu_2 = \underline{1} - Cl^*(\lambda, r).$$

Then, $\underline{1} - Cl^*(\lambda, r) = \mu_1$. Therefore, $\tau(Cl^*(\lambda, r)) \geq r$. Hence, $(X, \tau, \mathbf{I})$ is $\star$ -Fuzzy extremally disconnected.

(2) $\Rightarrow$ (1): Let $\lambda_1 \bar{q} \lambda_2$ with $\tau(\lambda_1) \geq r$ and $\tau^*(\lambda_2) \geq r$. Since $(X, \tau, \mathbf{I})$ is $\star$ -Fuzzy extremally disconnected, $\tau(Cl^*(\lambda_1, r)) \geq r$, then $\underline{1} - Cl^*(\lambda_1, r)$ is r -fuzzy closed set and $\tau^*(\underline{1} - Cl^*(\lambda_1, r)) \geq r$ with $\lambda_1 \leq Cl^*(\lambda_1, r)$, $\lambda_2 \leq \underline{1} - Cl^*(\lambda_1, r)$ and $\lambda_1 \bar{q} Cl^*(\lambda_1, r)$. Thus, $(X, \tau, \mathbf{I})$ is $\star$ -Fuzzy normal.

Definition 2.6. Let $(X, \tau, \mathbf{I})$ be a fuzzy ideal topological space. For $\mu \in I^X$ and $r \in I_0$.

- (1) μ is called **r-FRIO** set if $\mu = I_\tau(Cl^*(\mu, r), r)$.
- (2) μ is called **r-FRIC** set if its complement is **r-FRIO**.

Theorem 2.7. Let $(X, \tau, \mathbf{I})$ be an fits. Then the following properties are equivalent:

- (1) $(X, \tau, \mathbf{I})$ is $\star$ -Fuzzy extremally disconnected.
- (2) $\tau^*(\underline{1} - \mu) \geq r$, for each **r-FRIO** $\mu \in I^X$.
- (3) $\tau^*(\mu) \geq r$, for each **r-FRIC** $\mu \in I^X$.

Proof. (1) $\Rightarrow$ (2): For each **r-FRIO** $\mu \in I^X$. Then, $\mu = I_\tau(Cl^*(\mu, r), r)$. Since $\tau(\mu) \geq r$. By(1), we

have $\tau(Cl^*(\mu, r)) \geq r$. Thus, $\mu = I_\tau(Cl^*(\mu, r), r) = Cl^*(\mu, r)$ and hence $\tau^*(1 - \mu) \geq r$.

(2) $\Rightarrow$ (1): Suppose that $\tau^*(1 - \mu) \geq r$, for each **r-FRIO** $\mu \in I^X$. Let $\tau(\mu) \geq r$. Since $\mu = I_\tau(Cl^*(\mu, r), r)$, is **r-FRIO** then it is $\tau^*(1 - \mu) \geq r$. This implies that

$$\begin{aligned} Cl^*(\mu, r) &\leq Cl^*(I_\tau(Cl^*(\mu, r), r), r) \\ &= I_\tau(Cl^*(\mu, r), r) \leq Cl^*(\mu, r), \end{aligned}$$

Thus, $Cl^*(\mu, r) = I_\tau(Cl^*(\mu, r), r)$. Therefore, $\tau(Cl^*(\mu, r)) \geq r$, and hence $(X, \tau, \mathbf{I})$ is $\star$ -Fuzzy externally disconnected.

(2) $\Leftrightarrow$ (3): Obvious.

Remark 2.8. The union of two **r-FRIO** sets need not to be **r-FRIO**.

Theorem 2.9. If $(X, \tau, \mathbf{I})$ is a $\star$ -Fuzzy extremally disconnected ideal space and $\mu, \lambda \in I^X$, $r \in I_0$. Then the following properties are hold:

- (1) If μ and λ are **r-FRIC**, then $\mu \wedge \lambda$ is **r-FRIC**.
- (2) If μ and λ are **r-FRIO**, then $\mu \vee \lambda$ is **r-FRIO**.

Proof. (1) Let X be $\star$ -Fuzzy externally disconnected. Let μ and λ be **r-FRIC**. Since $\tau(1 - \mu) \geq r$ and $\tau(1 - \lambda) \geq r$, by Theorem 2.2, then $\tau(1 - Int^*(\mu, r)) \geq r$ and $\tau(1 - Int^*(\lambda, r)) \geq r$. This implies that

$$\begin{aligned} \mu \wedge \lambda &\leq C_\tau(Int^*(\mu, r), r) \wedge C_\tau(Int^*(\lambda, r), r) \\ &= Int^*(\mu, r) \wedge Int^*(\lambda, r) \\ &= Int^*(\mu \wedge \lambda, r) \\ &\leq C_\tau(Int^*(\mu \wedge \lambda, r), r). \end{aligned}$$

On the other hand, we have

$$\begin{aligned} C_\tau(Int^*(\mu \wedge \lambda, r), r) &= C_\tau(Int^*(\mu, r) \wedge Int^*(\lambda, r), r) \\ &\leq C_\tau(Int^*(\mu, r), r) \\ &\wedge C_\tau(Int^*(\lambda, r), r) = \mu \wedge \lambda. \end{aligned}$$

Thus, $C_\tau(Int^*(\mu \wedge \lambda, r), r) = \mu \wedge \lambda$. Therefore, $\mu \wedge \lambda$ is **r-FRIC**.

(2) It follows from (1).

Theorem 2.10. Let $(X, \tau, \mathbf{I})$ be a fuzzy ideal topological space, and $r \in I_0$. The following statements are equivalent.

- (1) X is $\star$ -Fuzzy extremally disconnected.
- (2) $\tau(Cl^*(\mu, r)) \geq r$, for every **r-FSIO** $\mu \in I^X$.
- (3) $\tau(Cl^*(\mu, r)) \geq r$, for every **r-FPIO** $\mu \in I^X$.
- (4) $\tau(Cl^*(\mu, r)) \geq r$, for every **r-FRIO** $\mu \in I^X$.

proof. (1) $\Rightarrow$ (2) and (1) $\Rightarrow$ (3). Let μ is **r-FSIO** (**r-FPIO**) set. Then μ is **r-FS β IO** and by Theorem 2.2, $\tau(Cl^*(\mu, r)) \geq r$.

(2) $\Rightarrow$ (4). Let μ is **r-FRIO** set. Then, μ is **r-FPIO** set and hence, $\tau(Cl^*(\mu, r)) \geq r$.

(3) $\Rightarrow$ (4). Let μ is **r-FRIO** set. Then, by Theorem 2.7, we have $\mu = Cl^*(\mu, r)$ and

$$\begin{aligned} \mu &= Int_\tau(Cl^*(\mu, r)) \\ &\leq Cl^*(Int_\tau(Cl^*(\mu, r), r), r) \\ &= Cl^*(Int_\tau(\mu, r), r). \end{aligned}$$

Thus, μ is **r-FSIO** set and hence, $\tau(Cl^*(\mu, r)) \geq r$.

(4) $\Rightarrow$ (1). Suppose that $\tau(Cl^*(\mu, r)) \geq r$, for every **r-FRIO** $\mu \in I^X$. Let $\tau(\mu) \geq r$. This implies that $\mu = Int_\tau(Cl^*(\mu, r), r)$, then μ is an **r-FRIO** set. Then $\tau(Cl^*(Int_\tau(Cl^*(\mu, r), r), r)) \geq r$. We have

$$\begin{aligned} Cl^*(\mu, r) &\leq Cl^*(Int_\tau(Cl^*(\mu, r), r), r) \\ &= Int_\tau(Cl^*(Int_\tau(Cl^*(\mu, r), r), r), r) \\ &= Int_\tau(Cl^*(\mu, r), r). \end{aligned}$$

Thus $\tau(Cl^*(\mu, r)) \geq r$ and hence X is $\star$ -Fuzzy extremally disconnected.

3. The fuzzy δ -I-closure

Definition 3.1. Let $(X, \tau, \mathbf{I})$ be a fuzzy ideal topological space, $\mu \in I^X$ and $x_t \in P_t(X)$.

(1) x_t is called **r-fuzzy δ -I-cluster point** of λ if for every $\mu \in Q_\tau(x_t, r)$, we have $Int_\tau(Cl^*(\mu, r), r)q\lambda$.

(2) The family of all **r-fuzzy δ -I-cluster point** of λ is called the **fuzzy δ -I-closure** of λ and is denoted $C_{\delta\mathbf{I}\tau}(\lambda, r)$. An λ is called is **r- δ -I-closed** iff $C_{\delta\mathbf{I}\tau}(\lambda, r) = \lambda$ and the complement of **r- δ -I-closed** set is called **r- δ -I-open**.

Lemma 3.2. Let $(X, \tau, \mathbf{I})$ be a fuzzy ideal topological space, if λ is **r-FS β IO** set then $C_\tau(\lambda, r) = C_{\delta\mathbf{I}\tau}(\lambda, r)$.

Proof. ($\Leftarrow$). Let $x_t \notin C_\tau(\lambda, r)$. Therefore, there exist $\mu \in Q_\tau(x_t, r)$ such that $\mu\bar{q}\lambda$. We have

$$\begin{aligned} \lambda &\leq 1 - \mu \\ &\Rightarrow Cl^*(\lambda, r) \leq Cl^*(1 - \mu, r) \leq Cl_\tau(1 - \mu, r) = 1 - \mu \\ &\Leftrightarrow 1 - Cl^*(\lambda, r) \geq \mu. \end{aligned}$$

Then $\mu\bar{q}Cl^*(\lambda, r)$. This implies that

$$Int_\tau(Cl^*(\mu, r), r)\bar{q}Cl^*(Int_\tau(Cl^*(\lambda, r), r), r).$$

Since μ is **r-FS β IO** set. Then, we have $Int_\tau(Cl^*(\lambda, r), r)\bar{q}\mu$. Thus $x_t \notin C_{\delta\mathbf{I}\tau}(\lambda, r)$. Therefore $C_\tau(\lambda, r) \geq C_{\delta\mathbf{I}\tau}(\lambda, r)$. Hence, we have $C_\tau(\lambda, r) = C_{\delta\mathbf{I}\tau}(\lambda, r)$.

($\Rightarrow$). Suppose $C_\tau(\lambda, r)C_{\delta\mathbf{I}\tau}(\lambda, r)$. Then there exists $x \in X$ and $t \in (0, 1)$ such that

$$C_\tau(\lambda, r)(x) > t > C_{\delta\mathbf{I}\tau}(\lambda, r)(x).$$

Therefore $x_t \in P_t(X)$ is not a r - δ -I-cluster point of λ . So, there exists $\mu \in Q_\tau(x_t, r)$, $Int_\tau(Cl^*(\mu, r), r) \bar{q} \lambda$. Thus

$$\lambda \leq \underline{1} - Int_\tau(Cl^*(\mu, r), r) = C_\tau(Int^*(\underline{1} - \mu, r), r).$$

Since $C_\tau(Int^*(\underline{1} - \mu, r), r)$ is r -fuzzy closed set. By definition C_τ we have

$$C_\tau(\lambda, r)(x) \leq (C_\tau(Int^*(\underline{1} - \mu, r), r))(x) < t.$$

It is a contradiction. Thus $C_\tau(\lambda, r) \leq C_{\delta I_\tau}(\lambda, r)$.

Definition 3.3. Let $(X, \tau, \mathbf{I})$ be a fuzzy ideal topological space, an fuzzy ideal $\mathbf{I}$ is called a condense if $\tau \wedge \mathbf{I} = \underline{0}$.

Lemma 3.4. Let $(X, \tau, \mathbf{I})$ be a fuzzy ideal topological space and $\mathbf{I}$ be condense. Then $C_\tau(\mu, r) = Cl^*(\mu, r)$ for every r -fuzzy semi open set $\mu \in I^X$.

Proof. Obvious.

Theorem 3.5. Let $(X, \tau, \mathbf{I})$ be a fuzzy ideal topological space and $\mathbf{I}$ be condense. If μ is a r -FS β IO set, then $Cl^*(\mu, r) = C_\tau(\mu, r) = C_{\delta I_\tau}(\mu, r)$ for every $\mu \in I^X$ and $r \in I_0$.

Proof. Let μ is a r -FS β IO set. Then

$\mu \leq Cl^*(Int_\tau(Cl^*(\mu, r), r), r)$ and $Cl^*(\mu, r) = Cl^*(Int_\tau(Cl^*(\mu, r), r), r)$, and hence, $Cl^*(\mu, r)$ is r -FSIO. Since every r -FSIO set is r -FSO, then $Cl^*(\mu, r)$ is r -FSO and by Lemma 3.4, $C_\tau(Cl^*(\mu, r), r) = Cl^*(Cl^*(\mu, r), r)$. Therefore,

$$\begin{aligned} C_\tau(\mu, r) &\leq C_\tau(Cl^*(\mu, r), r) \\ &= Cl^*(Cl^*(\mu, r), r) \\ &= Cl^*(\mu, r). \end{aligned}$$

Since $\tau \leq \tau^*$ then, $C_\tau(\mu, r) \geq Cl^*(\mu, r)$, and hence $C_\tau(\mu, r) = Cl^*(\mu, r)$. It is follows from Lemma 3.2, that

$$Cl^*(\mu, r) = C_\tau(\mu, r) = C_{\delta I_\tau}(\mu, r).$$

Theorem 3.6. Let $(X, \tau, \mathbf{I})$ be a fuzzy ideal topological space, and $\mathbf{I}$ be condense, the following properties are equivalent:

- (1) X is $\star$ -Fuzzy extremally disconnected.
- (2) The $\tau(C_{\delta I_\tau}(\mu, r)) \geq r$ for every r -FSIO set $\mu \in I^X$ and $r \in I_0$.
- (3) The $\tau(C_{\delta I_\tau}(\mu, r)) \geq r$ for every r -FS β IO set $\mu \in I^X$ and $r \in I_0$.
- (4) The $\tau(C_{\delta I_\tau}(\mu, r)) \geq r$ for every r -FPIO set $\mu \in I^X$ and $r \in I_0$.
- (5) X is Fuzzy extremally disconnected.

proof. (1) $\Rightarrow$ (2) Let X is $\star$ -Fuzzy extremally disconnected. Let μ is r -FSIO set. By Theorem 3.5, we

have $C_\tau(\mu, r) = C_{\delta I_\tau}(\mu, r)$. Since X is $\star$ -Fuzzy extremally disconnected, by Theorem 2.10 and Theorem 3.5, we have $Cl^*(\mu, r) = C_\tau(\mu, r) = C_{\delta I_\tau}(\mu, r)$. Thus, $\tau(C_{\delta I_\tau}(\mu, r)) \geq r$.

(2) $\Rightarrow$ (1) Let $\tau(\mu) \geq r$ then μ is r -FSIO set. By (2) we have, $\tau(C_{\delta I_\tau}(\mu, r)) \geq r$. then $C_{\delta I_\tau}(\mu, r)$ is r -FS β IO set and since $\mathbf{I}$ cadence by Theorem 3.5, we have

$$C_\tau(\mu, r) = Cl^*(\mu, r) = C_{\delta I_\tau}(\mu, r).$$

Therefore, $\tau(Cl^*(\mu, r)) \geq r$, and hence X is $\star$ -Fuzzy extremally disconnected.

(1) $\Rightarrow$ (3). Let μ be a r -FS β IO set. By Theorem 2.2, $\tau(Cl^*(\mu, r)) \geq r$, and hence, by Theorem 3.5, $\tau(C_{\delta I_\tau}(\mu, r)) \geq r$.

(3) $\Rightarrow$ (2) (3) $\Rightarrow$ (4). Let μ be a r -FSIO (r -FPIO) set. Since every r -FSIO (r -FPIO) set is r -FS β IO, by (3), we have $\tau(C_{\delta I_\tau}(\mu, r)) \geq r$.

(2) $\Rightarrow$ (5) (4) $\Rightarrow$ (5). Let $\tau(\mu) \geq r$. Every $\tau(\mu) \geq r$ is r -FSIO (r -FPIO). By (2)(resp (4)), we have $\tau(C_{\delta I_\tau}(\mu, r)) \geq r$, and hence, by Theorem 3.5, $\tau(C_\tau(\mu, r)) \geq r$. Therefore, X is Fuzzy extremally disconnected.

(5) $\Rightarrow$ (1). Let $\tau(\mu) \geq r$. By (5) and Theorem 3.5. $C_\tau(\mu, r) = Cl^*(\mu, r)$ and $\tau(Cl^*(\mu, r)) \geq r$, and hence X is $\star$ -Fuzzy extremally disconnected.

Definition 3.7. Let $(X, \tau, \mathbf{I})$ be an fuzzy ideal topological space, then

- (1) μ is called r -FS $\star$ IO if $\mu \leq C_\tau(Int^*(\mu, r), r)$ for each $\mu \in I^X$.
- (2) μ is called r -FS $\star$ IC if its complement is r -FS $\star$ IO.

Lemma 3.8. Let $(X, \tau, \mathbf{I})$ be an fuzzy ideal topological space, then $\mu \in I^X$ is a r -FS $\star$ IO set if and only if $C_\tau(\mu, r) = C_\tau(Int^*(\mu, r), r)$, for $r \in I_0$.

Proof. Let μ is r -FS $\star$ IO. We have $\mu \leq C_\tau(Int^*(\mu, r), r)$ and hence $C_\tau(\mu, r) \leq C_\tau(Int^*(\mu, r), r)$. Since $C_\tau(\mu, r) \geq C_\tau(Int^*(\mu, r), r)$, then,

$$C_\tau(\mu, r) = C_\tau(Int^*(\mu, r), r).$$

Conversely, Since $C_\tau(\mu, r) = C_\tau(Int^*(\mu, r), r)$, then, $\mu \leq C_\tau(\mu, r) = C_\tau(Int^*(\mu, r), r)$. Therefore, μ is r -FS $\star$ IO.

Theorem 3.9. Let $(X, \tau, \mathbf{I})$ be an fuzzy ideal topological space, then the following properties are equivalent:

- (1) X is $\star$ -Fuzzy extremally disconnected.
- (2) If μ is r -FS β IO and λ is r -FS $\star$ IO, then $Cl^*(\mu, r) \wedge C_\tau(\lambda, r) \leq C_\tau(\mu \wedge \lambda)$ for every $\mu, \lambda \in I^X$ and $r \in I_0$.
- (3) If μ is r -FSIO and λ is r -FS $\star$ IO, then $Cl^*(\mu, r) \wedge C_\tau(\lambda, r) \leq C_\tau(\mu \wedge \lambda)$ for every $\mu, \lambda \in I^X$ and $r \in I_0$.
- (4) $Cl^*(\mu, r) \bar{q} C_\tau(\lambda, r)$ for every r -FSIO set and λ is r -FS $\star$ IO set with $\mu \bar{q} \lambda$.

Proof. (1) $\Rightarrow$ (2). Let μ is r -FS β IO and λ is r -FS*IO. By Theorem 2.2, we have $\tau(Cl^*(\mu)) \geq r$, we have

$$\begin{aligned} Cl^*(\mu, r) \wedge C_\tau(\lambda, r) &= Cl^*(\mu, r) \wedge C_\tau(Int^*(\lambda, r), r) \\ &\leq C_\tau(Cl^*(\mu, r) \wedge Int^*(\lambda, r), r) \\ &\leq C_\tau(Cl^*(\mu \wedge Int^*(\lambda, r), r), r) \\ &\leq C_\tau(C_\tau(\mu \wedge Int^*(\lambda, r), r), r) \\ &= C_\tau(\mu \wedge Int^*(\lambda, r), r) \\ &\leq C_\tau(\mu \wedge \lambda, r). \end{aligned}$$

Thus, $Cl^*(\mu, r) \wedge C_\tau(\lambda, r) \leq C_\tau(\mu \wedge \lambda, r)$.

(2) $\Rightarrow$ (3). It follows from the fact that every r -FSIO is r -FS β IO.

(3) $\Rightarrow$ (4). Obvious.

(4) $\Rightarrow$ (1). Let μ is r -FSIO set. Since μ and $\underline{1} - Cl^*(\mu, r)$ are r -FSIO and r -FS*IO with $\mu \bar{q} \underline{1} - Cl^*(\mu, r)$ respectively, by (4), we have $Cl^*(\mu, r) \bar{q} C_\tau(\underline{1} - Cl^*(\mu, r), r)$. This implies that $Cl^*(\mu, r) \leq Int_\tau(Cl^*(\mu, r), r)$. Thus, $\tau(Cl^*(\mu, r)) \geq r$. Hence, by Theorem 2.10, X is $\star$ -Fuzzy extremally disconnected.

Lemma 3.10. Let $(X, \tau, \mathbf{I})$ be an fuzzy ideal topological space, if μ is a r -FS*IO set, then $C_\tau(\mu, r) = C_{\delta\mathbf{I}_\tau}(\mu, r)$ for every $\mu \in I^X$ and $r \in I_0$.

Proof. Obvious.

Theorem 3.11. Let $(X, \tau, \mathbf{I})$ be an fuzzy ideal topological space, then the following properties are equivalent:

- (1) X is $\star$ -Fuzzy extremally disconnected.
- (2) If μ is r -FS β IO and λ is r -FS*IO, then $Cl^*(\mu, r) \wedge C_{\delta\mathbf{I}_\tau}(\lambda, r) \leq C_{\delta\mathbf{I}_\tau}(\lambda, r)(\mu \wedge \lambda)$ for every $\mu \in I^X$ and $r \in I_0$.
- (3) If μ is r -FSIO and λ is r -FS*IO, then $Cl^*(\mu, r) \wedge C_{\delta\mathbf{I}_\tau}(\lambda, r) \leq C_\tau(\mu \wedge \lambda)$ for every $\mu \wedge \lambda \in I^X$ and $r \in I_0$.
- (4) $Cl^*(\mu, r) \bar{q} C_{\delta\mathbf{I}_\tau}(\lambda, r)$ for every r -FSIO set and λ is r -FS*IO set with $\mu \bar{q} \lambda$.

Proof. It follows from Theorem 3.9 and Lemma 3.10.

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