Consistency p -measure and fuzzy p -measure in decision tables

Yong Chan Kim¹ and Jung Mi Ko²

Department of Mathematics, Gangneung-Wonju National University, Gangneung, 201-702, Korea

Abstract

In this paper, we introduce the notions of inconsistency p -measure, consistency p -measure and fuzziness p -measure. We discuss various properties of them. We investigate the degree of inconsistency and roughness p -measure in a decision table.

Key words : Decision tables; Inclusion degree; (In)consistency p -measure; Fuzziness p -measure

1. Introduction and preliminaries

Rough set theory was introduced by Pawlak [4-5] to generalize the classical set theory. There has been a rapid growth in interest in rough set theory in recent years [1,2,9-14]. Its applications are decision system modeling and analysis of complex systems, fuzzy sets, neural networks, evolutionary computing, data mining and knowledge discovery, pattern recognition, etc. Recently, Qian [9-10] focused on how to measure the consistencies of a decision table and fuzziness of rough set.

In this paper, we introduce the notions of inconsistency p -measure, consistency p -measure and fuzziness p -measure which is generalized $p = 2$ in a sense Qian [9]. We discuss various properties of them. We investigate the degree of inconsistency and roughness p -measures in a decision table.

Definition 1.1. [4-8] The pair (X, AT) is called an *information system* if X is a non-empty finite set of objects and AT is a non-empty finite set of attributes. The triple (X, AT, d) is called a *decision table* if (X, AT) is an information system and d is a decision attribute set. For $a \in P \subset AT \cup d$, a map $a : X \rightarrow V_a$, where V_a is the value set of a . We define an equivalence relation $X/P = \{[x]_P \mid x \in X\}$ where

$$[x]_P = \{y \in X \mid a(x) = a(y), \forall a \in P\}.$$

Definition 1.2. [9] Let $(X, \leq)$ be a poset. A function $D : X \times X \rightarrow [0, 1]$ is called a measure of inclusion degree if it satisfies the following conditions:

- (1) $0 \leq D(a, b) \leq 1$,
- (2) if $a \leq b$, then $D(a, b) = 1$,

(3) if $c \leq a \leq b$, then $D(a, c) \geq D(b, c)$.

If we modify the condition (3) as

(3)* if $a \leq b$, then $D(a, c) \geq D(b, c)$ for all $a \in X$, D is called a S_1 -strong inclusion degree.

If D is an inclusion degree and satisfies the condition:

(4) if $b \leq c$, then $D(a, b) \leq D(a, c)$ for all $a \in X$, D is called a S_2 -strong inclusion degree.

Example 1.3. Let X be a finite set. We define functions $D_i : P(X) \times P(X) \rightarrow [0, 1]$ for $i = 1, 2, 3, 4$ as follows:

$$D_1(A, B) = \begin{cases} \frac{|A \cap B|}{|A|} & \text{if } A \neq \emptyset, \\ 1 & \text{if } A = \emptyset. \end{cases}$$

$$D_2(A, B) = \begin{cases} 1 & \text{if } A \subset B, \\ \frac{|B^c|}{|X|} & \text{if } A \not\subset B. \end{cases}$$

$$D_3(A, B) = \begin{cases} 1 & \text{if } A \subset B, \\ \frac{|A|}{|X|} & \text{if } A \not\subset B. \end{cases}$$

$$D_4(A, B) = \frac{|A^c \cup B|}{|X|}.$$

(1) D_1 is a S_2 -strong inclusion degree but not a S_1 -strong inclusion degree because

$$D_1(B, C) = \frac{|B \cap C|}{|B|} = \frac{2}{3} \not\geq D_1(A, C) = \frac{|A \cap C|}{|A|} = \frac{1}{2}$$

where $A = \{2, 5\}$, $B = \{2, 3, 5\}$, $C = \{1, 2, 3, 4\}$.

(2) D_2 is a S_1 -strong inclusion degree but not a S_2 -strong inclusion degree because, $X = \{1, 2, 3, 4, 5\}$,

$$D_2(A, B) = \frac{|B^c|}{|X|} = \frac{3}{5} \not\geq D_2(A, C) = \frac{|C^c|}{|X|} = \frac{2}{5}$$

where $A = \{1, 2, 3, 4\}$, $B = \{1, 2\}$, $C = \{1, 2, 3\}$.

(3) D_3 is a S_2 -strong inclusion degree but not a S_1 -strong inclusion degree because, $X = \{1, 2, 3, 4, 5\}$,

$$D_3(A, C) = \frac{|A|}{|X|} = \frac{2}{5} \not\geq D_3(B, C) = \frac{|B|}{|X|} = \frac{3}{5}$$

where $A = \{1, 2\}$, $B = \{1, 2, 3\}$, $C = \{1\}$.

(4) D_4 is a S_1 and S_2 -strong inclusion degree.

2. Consistency p-measure and fuzzy p-measure in decision tables

Let $F(X) = \{A : X \rightarrow [0, 1] \text{ is function}\}$ be given. For $A \in F(X)$, $x \in X$, $\delta_A(x) = A(x)$.

Definition 2.1. [9] A function $e : F(X) \rightarrow [0, 1]$ is called an *entropy* on $F(X)$ if it satisfies the following conditions:

- (1) $e(A) = 0$ iff $A \in P(X)$;
- (2) $e(A) = \bigvee_{B \in F(X)} e(B)$ iff $A = \underline{0.5}$;
- (3) if $A \leq B$ for $A \geq \frac{1}{2}$, then $e(B) \leq e(A)$;
- (4) if $B \leq A$ for $A \leq \frac{1}{2}$, then $e(B) \leq e(A)$;
- (5) $e(A) = e(A^c)$, for $A \in F(X)$.

Definition 2.2. Let (X, AT, d) be a decision table. $E \in X/AT$ an equivalence class and $X/d = \{[x]_d \mid x \in X\}$.

(1) A inconsistency p -measure of E with respect to d is defined as:

$$M_p(\delta_E) = \frac{1}{|X|} \sum_{x \in X} (-2^p |\delta_E(x) - \frac{1}{2}|^p + 1) \quad (p > 0)$$

where $\delta_E(x) = \frac{|E \cap [x]_d|}{|E|}$.

(2) A consistency p -measure of E with respect to d is defined as:

$$C_p(\delta_E) = 1 - \frac{1}{|X|} \sum_{x \in X} (-2^p |\delta_E(x) - \frac{1}{2}|^p + 1) \quad (p > 0).$$

(3) A consistency p -measure of AT with respect to d is defined as:

$$C(AT, d) = \sum_{j=1}^m \frac{|E_j|}{|X|} \left(1 - M_p(\delta_{E_j}) \right) \quad (p > 0)$$

where $X/AT = \{E_j \mid j = 1, \dots, m\}$.

If $A \subset B$, A is *consistent* with respect to B . If $\delta_E(x) = 1$, then E is consistent with respect to $[x]_d$, that is, $E \subset [x]_d$.

Let $F(X)$ be the class of all fuzzy sets of X . For $A \in F(X)$, $x \in X$, $\delta_A(x) = A(x)$.

Remark 2.3. In above definition, $p = 2$ we obtain definitions in a sense Quan [9] as follows:

(1) An inconsistency 2-measure of E with respect to d is defined as:

$$M_2(\delta_E) = \frac{4}{|X|} \sum_{x \in X} \delta_E(x)(1 - \delta_E(x))$$

(2) A consistency 2-measure of E with respect to d is defined as:

$$C(\delta_E) = 1 - \frac{4}{|X|} \sum_{x \in X} \delta_E(x)(1 - \delta_E(x))$$

(3) A consistency 2-measure of AT with respect to d is defined as:

$$C(AT, d) = \sum_{j=1}^m \frac{|E_j|}{|X|} \left(1 - \frac{4}{|X|} \sum_{x \in X} \delta_{E_j}(x)(1 - \delta_{E_j}(x)) \right)$$

Theorem 2.4. An inconsistency p -measure $M_p : F(X) \rightarrow [0, 1]$ is an entropy on $F(X)$.

Proof. (1) If $A \in P(X)$, then $\delta_A(x) = A(x) = 0$ or $\delta_A(x) = A(x) = 1$. Since $-2^p |\delta_A(x) - \frac{1}{2}|^p + 1 = 0$, then $M_p(A) = 0$.

Let $M_p(A) = \frac{1}{|X|} \sum_{x \in X} (-2^p |\delta_A(x) - \frac{1}{2}|^p + 1) = \frac{1}{|X|} \sum_{x \in X} (-2^p |A(x) - \frac{1}{2}|^p + 1) = 0$. Then $A(x_1) = 1$; i.e., $x \in A$ or $A(x) = 0$; i.e., $x \notin A$. Hence $A \in P(X)$.

(2) Since $0 \leq \delta_A(x) = A(x) \leq 1$, we have

$$M_p(A) = \bigvee_{B \in F(X)} \frac{1}{|X|} \sum_{x \in X} (-2^p |\delta_B(x) - \frac{1}{2}|^p + 1) = 1$$

Thus, $\delta_B(x) = B(x) = \frac{1}{2}$ for all $x \in X$. Conversely, $M_p(\underline{0.5}) = \bigvee_{B \in F(X)} M_p(B) = 1$.

(3) If $\delta_A(x) \leq \delta_B(x)$ for $\delta_A(x) \geq \frac{1}{2}$, then M_p is a decreasing function. Hence $M(B) \leq M(A)$.

(4) If $\delta_A(x) \leq \delta_B(x)$ for $\delta_B(x) \leq \frac{1}{2}$, then M_p is an increasing function. Hence $M_p(A) \leq M_p(B)$.

(5)

$$\begin{aligned} M_p(A^c) &= \frac{1}{|X|} \sum_{x \in X} (-2^p |\delta_{A^c}(x) - \frac{1}{2}|^p + 1) \\ &= \frac{1}{|X|} \sum_{x \in X} (-2^p |A^c(x) - \frac{1}{2}|^p + 1) \\ &= \frac{1}{|X|} \sum_{x \in X} (-2^p |1 - A(x) - \frac{1}{2}|^p + 1) \\ &= \frac{1}{|X|} \sum_{x \in X} (-2^p |A(x) - \frac{1}{2}|^p + 1) \\ &= M_p(A). \end{aligned}$$

□

Theorem 2.5. Let (X, AT, d) be a decision table. If A is a consistence set, then an inconsistency p -measure $M_p(\delta_A) = 0$.

Proof. Let $A \in X/AT$ and $X/d = \{[x]_d \mid x \in X\}$. Since A is a consistence set, then, for $x \in A$, $A \subset [x]_d$. Hence

$\delta_A(x) = \frac{|A \cap [x]_d|}{|A|} = 1$. For $x \in X - A$, $A \cap [x]_d = \emptyset$.

Hence $\delta_A(x) = \frac{|A \cap [x]_d|}{|A|} = 0$. Therefore

$$M_p(\delta_A) = \frac{1}{|X|} \sum_{x \in X} (-2^p |\delta_A(x) - \frac{1}{2}|^p + 1) = 0.$$

□

Theorem 2.6. Let (X, AT, d) be a consistent decision table. Then $C_p(AT, d) = 1$.

Proof. Let $X/AT = \{X_1, \dots, X_m\}$ and $X/d = \{[x]_d \mid x \in X\}$. Since (X, AT, d) is consistent, every $X_i \in X/AT$ is consistence, $X_i \subset [x]_d$. Hence $\delta_{X_i} = \frac{|X_i \cap [x]_d|}{|X_i|} = 1$. For $x \in X - X_i$, $X_i \cap [x]_d = \emptyset$. Thus, $\delta_{X_i} = \frac{|X_i \cap [x]_d|}{|X_i|} = 0$. Therefore, for $x \in X$, we obtain $-2^p |\delta_{X_j}(x) - \frac{1}{2}|^p + 1 = 0$. Hence

$$\begin{aligned} C_p(AT, d) &= \sum_{j=1}^m \frac{|X_j|}{|X|} (1 - M_p(\delta_{X_j})) \\ &= \sum_{j=1}^m \frac{|X_j|}{|X|} \left(1 - \frac{1}{|X|} \sum_{x \in X} (-2^p |\delta_{X_j}(x) - \frac{1}{2}|^p + 1)\right) \\ &= \sum_{j=1}^m \frac{|X_j|}{|X|} = 1 \end{aligned}$$

□

Theorem 2.7. Let (X, AT) be an information system. Then

$$D(P, Q) = \sum_{A \in X/P} \frac{|A|}{|X|} \left(1 - \frac{1}{|X|} \sum_{x \in X} (-2^p |\delta_A(x) - \frac{1}{2}|^p + 1)\right)$$

where $\delta_A(x) = \frac{|A \cap [x]_Q|}{|A|}$ with $\delta_A(x) \geq \frac{1}{2}$, is an inclusion degree on $(P(AT), \leq)$.

Proof. (1) Let $P, Q \in P(AT)$. We have

$$\begin{aligned} D(P, Q) &= \sum_{A \in X/P} \frac{|A|}{|X|} \left(1 - \frac{1}{|X|} \sum_{x \in X} (-2^p |\delta_A(x) - \frac{1}{2}|^p + 1)\right) \\ &= \sum_{A \in X/P} \frac{|A|}{|X|} \left(\frac{1}{|X|} \sum_{x \in X} (2^p |\delta_A(x) - \frac{1}{2}|^p)\right) \end{aligned}$$

Since $1 \geq \delta_A(x) \geq \frac{1}{2}$, we have $0 \leq D(P, Q) \leq 1$.

(2) Let $P \leq Q$. Then, for $A \in X/P$, there exists $B \in X/Q$ such that $A \subset B$. For $x \in X$, if $x \in B$, then $\delta_A(x) = \frac{|A \cap [x]_Q|}{|A|} = \frac{|A|}{|A|} = 1$. If $x \notin B$, then $\delta_A(x) = \frac{|A \cap [x]_Q|}{|A|} = \frac{|\emptyset|}{|A|} = 0$. Hence $D(P, Q) = \sum_{A \in X/P} \frac{|A|}{|X|} = 1$.

(3) Let $P \leq Q \leq R$. Then, for $[x]_P \in X/P$, there exist $[x]_Q \in X/Q$ and $[x]_R \in X/R$ such that $[x]_P \subset [x]_Q \subset [x]_R$.

$[x]_R$. Since $\delta_A(x) \geq \frac{1}{2}$, we have

$$\begin{aligned} D(P, R) &= \sum_{A \in X/P} \frac{|A|}{|X|} \left(\frac{1}{|X|} \sum_{x \in X} (2^p |\delta_A(x) - \frac{1}{2}|^p)\right) \\ &= \sum_{A \in X/P} \frac{|A|}{|X|} \left(\frac{1}{|X|} \sum_{x \in X} (2^p |\frac{|A \cap [x]_P|}{|A|} - \frac{1}{2}|^p)\right) \\ &= \sum_{A \in X/P} \frac{|A|}{|X|} \left(\frac{1}{|X|} \sum_{x \in X} (2^p |\frac{|[x]_P|}{|A|} - \frac{1}{2}|^p)\right) \\ &\leq \sum_{A \in X/P} \frac{|A|}{|X|} \left(\frac{1}{|X|} \sum_{x \in X} (2^p |\frac{|[x]_Q|}{|A|} - \frac{1}{2}|^p)\right) \\ &= D(Q, R) \end{aligned}$$

□

Example 2.8. Let (X, AT, d) be a decision table as follows:

car	Price	Mileage	Size	MaxSpeed	d
x_1	High	Low	Full	Low	Good
x_2	Low	High	Full	Low	Good
x_3	Low	Low	Compact	Low	Poor
x_4	High	High	Full	High	Good
x_5	High	High	Full	High	Excel
x_6	Low	High	Full	Low	Good

Then $X/AT = \{E_1 = \{x_1\}, E_2 = \{x_2, x_6\}, E_3 = \{x_3\}, E_4 = \{x_4, x_5\}\}$

$$X/d = \{\{x_1, x_2, x_4, x_6\}, \{x_3\}, \{x_5\}\}.$$

Furthermore, we have

$$\begin{aligned} \delta_{E_1}(x_1) &= \delta_{E_1}(x_2) = \delta_{E_1}(x_4) = \delta_{E_1}(x_6) = 1 \\ \delta_{E_1}(x_3) &= \delta_{E_1}(x_5) = 0 \\ \delta_{E_2}(x_1) &= \delta_{E_2}(x_2) = \delta_{E_2}(x_4) = \delta_{E_2}(x_6) = 1 \\ \delta_{E_2}(x_3) &= \delta_{E_2}(x_5) = 0 \\ \delta_{E_3}(x_1) &= \delta_{E_3}(x_2) = \delta_{E_3}(x_4) = \delta_{E_3}(x_6) = 0 \\ \delta_{E_3}(x_3) &= \delta_{E_3}(x_5) = 1 \\ \delta_{E_4}(x_1) &= \delta_{E_4}(x_2) = \delta_{E_4}(x_4) = \delta_{E_4}(x_6) = \frac{1}{2} \\ \delta_{E_4}(x_3) &= \frac{1}{2}, \delta_{E_4}(x_5) = 0 \end{aligned}$$

(1) Since $E_1, E_2 \subset \{x_1, x_2, x_4, x_6\} \in X/d$ and $E_3 \subset \{x_3\} \in X/d$, then E_1, E_2 and E_3 are consistence sets, then an inconsistency p -measure $M_p(\delta_{E_i}) = 0$ for $i = 1, 2, 3$.

$$M_p(\delta_{E_4}) = \frac{1}{|X|} \sum_{i=1}^6 (-2^p |\delta_{E_4}(x_i) - \frac{1}{2}|^p + 1) = \frac{5}{6},$$

$$C_p(\delta_{E_4}) = 1 - \frac{1}{|X|} \sum_{i=1}^6 (-2^p |\delta_{E_4}(x_i) - \frac{1}{2}|^p + 1) = \frac{1}{6},$$

$$C_p(AT, d) = \sum_{j=1}^4 \frac{|E_j|}{6} (1 - M_p(\delta_{E_j})) = \frac{13}{18}.$$

(2) Let $A \in F(X)$ as

$$A(x_1) = 0.7, A(x_2) = 0.9, A(x_3) = 0.5,$$

$$A(x_4) = 0.2, A(x_5) = 0.4, A(x_6) = 0.1.$$

we have

$$\begin{aligned} M_p(A) &= \frac{1}{6} \sum_{i=1}^6 (-2^p |\delta_A(x_i) - \frac{1}{2}|^p + 1) \\ &= 1 - \frac{1}{6} 2^p ((0.2)^p + 2(0.4)^p + (0.3)^p + (0.1)^p). \end{aligned}$$

(3) We obtain equivalence relations for Price and Mileage, respectively as follows;

$$X/P = \{A_1 = \{x_1, x_4, x_5\}, A_2 = \{x_2, x_3, x_6\}\}$$

$$X/M = \{B_1 = \{x_1, x_3\}, B_2 = \{x_2, x_4, x_5, x_6\}\},$$

$$\begin{aligned} \delta_{A_1}(x_1) &= \frac{1}{3}, \delta_{A_1}(x_2) = \frac{2}{3}, \delta_{A_1}(x_3) = \frac{1}{3} \\ \delta_{A_1}(x_4) &= \frac{2}{3}, \delta_{A_1}(x_5) = \frac{2}{3}, \delta_{A_1}(x_6) = 0 \\ \delta_{A_2}(x_1) &= \frac{1}{3}, \delta_{A_2}(x_2) = \frac{2}{3}, \delta_{A_2}(x_3) = \frac{1}{3} \\ \delta_{A_2}(x_4) &= \frac{2}{3}, \delta_{A_2}(x_5) = \frac{2}{3}, \delta_{A_2}(x_6) = \frac{2}{3}. \end{aligned}$$

We can obtain the inclusion degrees of $D(P, M)$ and $D(P, M)$ as follows;

$$\begin{aligned} D(P, M) &= \sum_{A \in X/P} \frac{|A|}{|X|} \left(1 - \frac{1}{|X|} \sum_{x \in X} (1 - 2^p |\delta_A(x) - \frac{1}{2}|^p)\right) \\ &= \frac{3}{6} \left(1 - \frac{5}{6} \left(1 - \frac{1}{3^p}\right)\right) + \frac{3}{6} \left(1 - \left(1 - \frac{1}{3^p}\right)\right) \\ &= \frac{1}{12} - \frac{1}{12} \left(\frac{1}{3^p}\right) \end{aligned}$$

$$\delta_{B_1}(x_i) = \delta_{B_2}(x_i) = \frac{1}{2}, \quad \forall i \in \{1, 2, 3, 4, 5, 6\}$$

$$\begin{aligned} D(M, P) &= \sum_{A \in X/M} \frac{|A|}{|X|} \left(1 - \frac{1}{|X|} \sum_{x \in X} (1 - 2^p |\delta_A(x) - \frac{1}{2}|^p)\right) \\ &= \frac{2}{6} \left(1 - \frac{6}{6}\right) + \frac{4}{6} \left(1 - \frac{6}{6}\right) = 0 \end{aligned}$$

We interpret that the attribute of Mileage does not include that of Price. Similarly, we can obtain the inclusion degrees between attributes.

Definition 2.9. Let (X, AT, d) be a decision table and $Y \subset X$.

(1) A fuzziness p -measure of the rough set Y is defined as:

$$M_p(\mu_Y) = \frac{1}{|X|} \sum_{x \in X} (-2^p |\mu_Y(x) - \frac{1}{2}|^p + 1) \quad (p > 0)$$

where $\mu_Y(x) = \frac{|Y \cap [x]_{AT}|}{|[x]_{AT}|}$.

(2) If $X/d = \{Z_1, \dots, Z_n\}$ is an equivalence relation of decision, a fuzziness p -measure of a rough decision is defined as:

$$M_p(\mu_d) = \frac{1}{|X|} \sum_{x \in X} (-2^p |\mu_d(x) - \frac{1}{2}|^p + 1) \quad (p > 0)$$

where $\mu_d(x) = \frac{|Z_i \cap [x]_{AT}|}{|[x]_{AT}|}$ for $x \in Z_i$.

Theorem 2.10. (1) For a crisp set χ_A , $M(\chi_A) = 0$.

(2) $M(A^c) = M(A)$ for $A \in F(X)$.

(3) If $X/d = \{Z_1, \dots, Z_n\}$ is an equivalence relation of decision and χ_A a crisp set, a fuzziness p -measure $M(\mu_d) = 0$.

Proof. (1) If $A \in P(X)$, then $\mu_A(x) = A(x) = 0$ or $\mu_A(x) = A(x) = 1$. Thus, $M(\chi_A) = 0$.

(2)

$$\begin{aligned} M_p(A^c) &= \frac{1}{|X|} \sum_{x \in X} (-2^p |\mu_{A^c}(x) - \frac{1}{2}|^p + 1) \\ &= \frac{1}{|X|} \sum_{x \in X} (-2^p |A^c(x) - \frac{1}{2}|^p + 1) \\ &= \frac{1}{|X|} \sum_{x \in X} (-2^p |1 - A(x) - \frac{1}{2}|^p + 1) \\ &= \frac{1}{|X|} \sum_{x \in X} (-2^p |\mu_A(x) - \frac{1}{2}|^p + 1) \\ &= M_p(A). \end{aligned}$$

(3) It is easy from (1). $\square$

Theorem 2.11. Let (X, AT) be a information system. Then

$$D_2(P, Q) = 1 - \frac{1}{|X|} \sum_{x \in X} (-2^p |\mu_Q(x) - \frac{1}{2}|^p + 1) \quad (p > 0)$$

where $\mu_Q(x) = \frac{|Y_i \cap [x]_P|}{|[x]_P|}$ with $\mu_Q(x) \geq \frac{1}{2}$ and $x \in Y_i \in X/Q$. Then $D_2(P, Q)$ is a S_2 -inclusion degree on $(P(AT), \leq)$.

Proof. (1) Let $P, Q \in P(AT)$. We have

$$\begin{aligned} D_2(P, Q) &= 1 - \frac{1}{|X|} \sum_{x \in X} (-2^p |\mu_Q(x) - \frac{1}{2}|^p + 1) \\ &= \frac{1}{|X|} \sum_{x \in X} 2^p |\mu_Q(x) - \frac{1}{2}|^p \end{aligned}$$

Since $1 \geq \mu_Q(x) \geq \frac{1}{2}$, we have $0 \leq D_2(P, Q) \leq 1$.

(2) Let $P \leq Q$. Then, for $A \in X/P$, there exists $B \in X/Q$ such that $A \subset B$. For $x \in Y_i$ and $Y_i \in X/Q$, then $\mu_Q(x) = \frac{|Y_i \cap [x]_P|}{|[x]_P|} = \frac{|[x]_P|}{|[x]_P|} = 1$. Hence

$$D_2(P, Q) = \frac{1}{|X|} \sum_{x \in X} 2^p |\mu_Q(x) - \frac{1}{2}|^p = 1.$$

(3) Let $P \leq Q \leq R$. Then, for $[x]_P \in X/P$, there exist $[x]_Q \in X/Q$ and $[x]_R \in X/Q$ such that $[x]_P \subset [x]_Q \subset [x]_R$. Since $\mu_A(x) \geq \frac{1}{2}$, we have

$$\begin{aligned} D_2(P, R) &= \frac{1}{|X|} \sum_{i=1}^{|X|} \sum_{A \in X/P} 2^p |\mu_A(x) - \frac{1}{2}|^p \\ &= \frac{1}{|X|} \sum_{i=1}^{|X|} \sum_{A \in X/P} 2^p \frac{|A \cap [x]_R|}{|[x]_R|} - \frac{1}{2}|^p \\ &\leq \frac{1}{|X|} \sum_{i=1}^{|X|} \sum_{B \in X/Q} 2^p \frac{|B \cap [x]_R|}{|[x]_R|} - \frac{1}{2}|^p \\ &= \frac{1}{|X|} \sum_{i=1}^{|X|} \sum_{B \in X/Q} 2^p |\mu_B(x) - \frac{1}{2}|^p \\ &= D_2(Q, R) \end{aligned}$$

$\square$

Example 2.12. Let (X, AT, d) be a decision table as in Example 2.8. Since

$$\mu_d(x_4) = \frac{|\{x_1, x_2, x_4, x_6\} \cap \{x_4, x_5\}|}{|\{x_4, x_5\}|} = \frac{1}{2},$$

we have

$$\mu_d(x_1) = \mu_d(x_2) = \mu_d(x_3) = \mu_d(x_6) = 1, \quad \mu_d(x_5) = \frac{1}{2}$$

$$M_p(\mu_d) = \frac{1}{|X|} \sum_{i=1}^6 (-2^p |\mu_d(x_i) - \frac{1}{2}|^p + 1) = \frac{1}{3}.$$

(1) The values of consistency and fuzziness p -measures induced by Price.

$$X/P = \{\{x_1, x_4, x_5\}, \{x_2, x_3, x_6\}\}$$

$$X/d = \{\{x_1, x_2, x_4, x_6\}, \{x_3\}, \{x_5\}\}$$

$$\begin{aligned} (A = \{x_1, x_4, x_5\}, \delta_A(x) = \frac{A \cap [x]_d}{A}) \\ M_p(\delta_A) &= \frac{1}{|X|} \sum_{x \in X} (-2^p |\delta_A(x) - \frac{1}{2}|^p + 1) \\ &= \frac{1}{6} \left(3 - 2^p (2(\frac{1}{6})^p + (\frac{1}{2})^p) \right) \\ (B = \{x_2, x_3, x_6\}, \delta_B(x) = \frac{B \cap [x]_d}{B}) \\ M_p(\delta_B) &= \frac{1}{|X|} \sum_{x \in X} (-2^p |\delta_B(x) - \frac{1}{2}|^p + 1) \\ &= \frac{1}{6} \left(3 - 2^p (2(\frac{1}{6})^p + (\frac{1}{2})^p) \right) \\ C_p(P, d) &= \sum_{A \in X/P} \frac{|A|}{|X|} \left(1 - M_p(\delta_A) \right) \\ &= \frac{1}{2} \left(1 - M_p(\delta_A) \right) + \frac{1}{2} \left(1 - M_p(\delta_B) \right) \\ &= \frac{1}{2} + \frac{1}{6} \left(2^p (2(\frac{1}{6})^p + (\frac{1}{2})^p) \right). \\ \mu_d(x_1) &= \mu_d(x_2) = \mu_d(x_4) = \mu_d(x_6) = \frac{2}{3} \\ \mu_d(x_3) &= \mu_d(x_5) = \frac{1}{3} \\ M_p(\mu_d) &= \frac{1}{|X|} \sum_{i=1}^6 (-2^p |\mu_d(x_i) - \frac{1}{2}|^p + 1) \\ &= 1 - (\frac{2}{3})^p. \end{aligned}$$

(2) The values of consistency and fuzziness p -measures induced by Mileage.

$$X/M = \{\{x_1, x_3\}, \{x_2, x_4, x_5, x_6\}\}$$

$$\begin{aligned} (A = \{x_1, x_3\}, \delta_A(x) = \frac{A \cap [x]_d}{A}) \\ M_p(\delta_A) &= \frac{1}{|X|} \sum_{x \in X} (-2^p |\delta_A(x) - \frac{1}{2}|^p + 1) \\ &= \frac{2}{3} \\ (B = \{x_2, x_4, x_5, x_6\}, \delta_B(x) = \frac{B \cap [x]_d}{B}) \\ M_p(\delta_B) &= \frac{1}{|X|} \sum_{x \in X} (-2^p |\delta_B(x) - \frac{1}{2}|^p + 1) \\ &= \frac{2}{3} (1 - (\frac{1}{2})^p) \\ C_p(M, d) &= \sum_{A \in X/P} \frac{|A|}{|X|} \left(1 - M_p(\delta_A) \right) \\ &= \frac{1}{3} \left(1 - M_p(\delta_A) \right) + \frac{2}{3} \left(1 - M_p(\delta_B) \right) \\ &= \frac{1}{3} + \frac{4}{9} (\frac{1}{2})^p. \\ \mu_d(x_1) &= \mu_d(x_3) = \frac{1}{2}, \mu_d(x_5) = \frac{1}{4} \\ \mu_d(x_2) &= \mu_d(x_4) = \mu_d(x_6) = \frac{3}{4} \\ M_p(\mu_d) &= \frac{1}{|X|} \sum_{i=1}^6 (-2^p |\mu_d(x_i) - \frac{1}{2}|^p + 1) \\ &= 1 - \frac{2}{3} (\frac{1}{2})^p. \end{aligned}$$

(3) The values of consistency and fuzziness p -measures induced by Size.

$$X/S = \{\{x_1, x_2, x_4, x_5, x_6\}, \{x_3\}\}$$

$$\begin{aligned} (A = \{x_1, x_2, x_4, x_5, x_6\}, \delta_A(x) = \frac{A \cap [x]_d}{A}) \\ M_p(\delta_A) &= \frac{5}{6} (1 - \frac{4}{5}^p) \\ (B = \{x_3\}, \delta_B(x) = \frac{B \cap [x]_d}{B}) \\ M_p(\delta_B) &= 0 \\ C_p(S, d) &= \frac{5}{6} \left(1 - M_p(\delta_A) \right) + \frac{1}{6} \left(1 - M_p(\delta_B) \right) \\ &= \frac{11}{36} + \frac{25}{36} (\frac{4}{5})^p. \\ \mu_d(x_1) &= \mu_d(x_2) = \mu_d(x_4) = \mu_d(x_6) = \frac{4}{5} \\ \mu_d(x_3) &= 1, \mu_d(x_5) = \frac{1}{5} \\ M_p(\mu_d) &= \frac{1}{|X|} \sum_{i=1}^6 (-2^p |\mu_d(x_i) - \frac{1}{2}|^p + 1) \\ &= \frac{5}{6} (1 - \frac{4}{5}^p). \end{aligned}$$

(4) The values of consistency and fuzziness p -measures induced by Max-speed.

$$X/E = \{\{x_1, x_2, x_3, x_6\}, \{x_4, x_5\}\}$$

$$\begin{aligned} (A = \{x_1, x_2, x_3, x_6\}, \delta_A(x) = \frac{A \cap [x]_d}{A}) \\ M_p(\delta_A) &= \frac{2}{3} (1 - (\frac{1}{2})^p) \\ (B = \{x_4, x_5\}, \delta_B(x) = \frac{B \cap [x]_d}{B}) \\ M_p(\delta_B) &= \frac{5}{6} \\ C_p(E, d) &= \frac{2}{3} \left(1 - M_p(\delta_A) \right) + \frac{1}{3} \left(1 - M_p(\delta_B) \right) \\ &= \frac{1}{6} + \frac{4}{9} (\frac{1}{2})^p. \\ \mu_d(x_1) &= \mu_d(x_2) = \mu_d(x_4) = \mu_d(x_6) = \frac{3}{4} \\ \mu_d(x_3) &= \frac{1}{4}, \mu_d(x_5) = \frac{1}{2} \\ M_p(\mu_d) &= \frac{1}{|X|} \sum_{i=1}^6 (-2^p |\mu_d(x_i) - \frac{1}{2}|^p + 1) \\ &= \frac{5}{6} (1 - \frac{1}{2}^p). \end{aligned}$$

car	$C_p(, d)$	$M_p(\mu_d)$
X/P	$\frac{1}{2} + \frac{1}{6} \left(2^p (2(\frac{1}{6})^p + (\frac{1}{2})^p) \right)$	$1 - (\frac{2}{3})^p$
X/M	$\frac{1}{3} + \frac{4}{9} (\frac{1}{2})^p$	$1 - \frac{2}{3} (\frac{1}{2})^p$
X/S	$\frac{11}{36} + \frac{25}{36} (\frac{4}{5})^p$	$\frac{5}{6} (1 - \frac{4}{5}^p)$
X/E	$\frac{1}{6} + \frac{4}{9} (\frac{1}{2})^p$	$\frac{5}{6} (1 - \frac{1}{2}^p)$

(5) Since

$$X/P = \{A_1 = \{x_1, x_4, x_5\}, A_2 = \{x_2, x_3, x_6\}\}$$

$$X/M = \{B_1 = \{x_1, x_3\}, B_2 = \{x_2, x_4, x_5, x_6\}\}$$

$$\begin{aligned} \mu_M(x_1) &= \frac{1}{3}, \mu_M(x_2) = \frac{2}{3}, \mu_M(x_3) = \frac{1}{3} \\ \mu_M(x_4) &= \frac{2}{3}, \mu_M(x_5) = \frac{2}{3}, \mu_M(x_6) = \frac{2}{3}. \end{aligned}$$

$$\begin{aligned} D_2(P, M) &= 1 - \frac{1}{|X|} \sum_{x \in X} (1 - 2^p |\mu_M(x) - \frac{1}{2}|^p) \\ &= \frac{1}{3^p}. \end{aligned}$$

$$\mu_P(x_i) = \frac{1}{2}, \forall i \in \{1, 2, 3, 4, 5, 6\}$$

$$D_2(M, P) = 1 - \frac{1}{|X|} \sum_{x \in X} (1 - 2^p |\mu_M(x) - \frac{1}{2}|^p) = 0.$$

Similarly, we can obtain the inclusion degrees between attributes.

References

- [1] Z. Bonikowski, E. Bryniarski, U. Wybraniec-Skardowska, "Extensions and intensions in rough set theory," *Information Sciences*, vol. 107, pp. 149-167, 1998.
- [2] D. Dubois, H. Prade, "Rough fuzzy sets and fuzzy rough sets," *Internat. J. Gen. Systems*, vol. 17, no 2-3, pp. 191-209, 1990.
- [3] M. Kryszkiewicz, "Rough set approach to incomplete information systems," *Information Sciences*, vol. 112, pp. 39-49, 1998.
- [4] Z. Pawlak, "Rough sets," *Internat. J. Comput. Inform. Sci.*, vol. 11, pp. 341-356, 1982.
- [5] Z. Pawlak, *Rough sets: Theoretical Aspects of Reasoning about Data, System Theory, Knowledge Engineering and Problem Solving*, Kluwer Academic Publishers, Dordrecht, The Netherlands, 1991.
- [6] Z. Pawlak, A. Skowron, "Rudiments of rough sets," *Information Science*, vol.177, pp. 3-27, 2007.
- [7] Z. Pawlak, A. Skowron, "Rough sets," *Information Science*, vol.177, pp. 28-40, 2007.
- [8] Z. Pawlak, A. Skowron, "Rough sets and Boolean reasoning," *Information Science*, vol.177, pp. 41-73, 2007.
- [9] Y. Qian, J. Liang, C. Dang, "Consistency measures, inclusion degree and fuzzy measure in decision tables," *Fuzzy Sets and Systems*, vol.159, pp. 2353-2377, 2008.
- [10] Y. Qian, C. Dang, J. Liang, H. Zhang, J. Ma, "On the evaluation of the decision performance of incomplete decision table," *Data and Knowledge Engineering*, vol.65, no 3, pp. 373-400, 2008.
- [11] Y.Y. Yao, "Constructive and algebraic methods of theory of rough sets," *Information Sciences*, vol. 109, pp.21-47, 1998.
- [12] Y.Y. Yao, "Relational interpretations of neighborhood operators and rough set approximation operators," *Information Sciences*, vol. 111, pp. 239-259, 1998.
- [13] W.X. Zhang, J.S. Mi, "Incomplete Information system and its optimal selections," *Computers and Mathematical with Applications*, vol.48, pp.691-698, 2004.
- [14] W.X. Zhang, J.S. Mi, W.Z. Wu, "Approaches to Knowledge reductions in inconsistency systems," *Int. J. of intelligent Systems*, vol. 18, pp. 989-1000, 2003.

저 자 소 개

Yong Chan Kim

He received the M.S and Ph.D. degrees in Department of Mathematics from Yonsei University, in 1984 and 1991, respectively. From 1991 to present, he is a professor in Department of Mathematics, Gangneung-Wonju University. His research interests are fuzzy logic and fuzzy topology.

Jung Mi Ko

She received the M.S and Ph.D. degrees in Department of Mathematics from Yonsei University, in 1983 and 1988, respectively. From 1988 to present, she is a professor in Department of Mathematics, Gangneung-Wonju University. Her research interests are fuzzy logic.